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On the Diophantine Equations $a^x + b^y + c^z = w^2$

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Abstract. Over the past decade, exponential Diophantine equations of the form $a^x + b^y = w^n$ have been studied as if they were a phenomenon. In particular, numerous articles have focused on the cases where $n = 2$ or $n = 4$ and $2 \leq a, b \leq 200$. However, these articles are primarily concerned with determining whether the left-hand side of the equation needs to consist of more than two exponentials. Therefore, in this article, we investigate the exponential Diophantine equation in the form $a^x + b^y + c^z = w^2$, using only elementary tools related to modulo concepts. We present three theorems in which the variables a, b and c vary under certain conditions, and three additional theorems where the variable c is fixed at 7. Furthermore, if we restrict our parameters a, b and c to $2 \leq a \leq b \leq c \leq 20$, then 1,330 equations have been considered. Our results confirm that 135 of these equations have been fully clarified.

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1. Introduction

Many mathematicians have proposed generalized forms of the Diophantine equation in various ways. One example is the exponential Diophantine equation $3^x + 4^y = 5^z$, which has no solutions for any natural numbers x, y, z except when $x = y = z = 2$. This was proven in 1956 by W. Sierpinski, [17]. In the same year, L. Jamanowicz published an article that seemed to follow in W. Sierpinski's footsteps by selecting other equations such as $5^x + 12^y = 13^z, 7^x + 24^y = 25^z, 9^x + 40^y = 41^z$, and $11^x + 60^y = 61^z$[see 13]. Since then, many mathematicians have explored variations by changing the base values a, b, and c in the exponential Diophantine equation $a^x + b^y = c^z$. For example, in 2005, D. Acu [1] considered three cases: Case 1 where $a = b = c = p$, Case 2 where $a = b = p$ and $c = 2p$, and the last case where $a = p, b = q$, and $c = pq$, with p and q being prime numbers. Furthermore, he proposed the alternative form $2^x + 5^y = z^2$ in 2007, and noted that Catalan's conjecture was an important tool for finding solutions,[see 2]. It's also worth noting that the exponential term c^z was interchanged with the polynomial term z^2.

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Since 2007, hundreds of articles inspired by the equation $2^x + 5^y = z^2$ have been published. Many were authored by B. Sroysang, as seen in references [18–20], among others listed in [5, 9, 16, 22]. In addition, there are exponential Diophantine equations similar to $2^x + 5^y = z^2$ but with more than three variables. Some of these are showcased in Table 1. Ever since D. Acu introduced the equation $2^x + 5^y = z^2$ in 2007, researchers have explored alternative forms by altering the base numbers 2 and 5. They have also tried to generalize the equation, creating new forms and investigating their solutions. Interestingly, there are few equations like $3^x + 5^y + 7^z = w^2$ that consider three base numbers for the exponential terms. This particular equation was established by J. B. Bacani and J. F. T. Rabago, and serves as the inspiration for this article. Here, we focus on the form $a^x + b^y + c^z = w^2$ under certain conditions for a, b, c, x, y, z, and w, using only elementary tools related to modulo concepts. Most equations in Table 1 have more than three variables, but they still involve only two base numbers. Even the equation $p^x + (p+1)^y + (p+2)^z = M^2$, cited in [6], appears to have three exponential terms, but its parameters x, y, and z are restricted to the set $\{1, 2, 3\}$.

Table 1: Example of the exponential Diophantine equations during the past ten years which were considered more than three variables.

Ref.	Author	Equation
[3]	J.B. Bacani and J.F.T. Rabago	$3^x + 5^y + 7^z = w^2$
[4]	J.B. Bacani and J.F.T. Rabago	$p^x + q^y = z^2$
[6]	Nechemia Burshtein	$p^x + (p+1)^y + (p+2)^z = M^2$
[7]	Nechemia Burshtein	$p^3 + q^y = z^3$
[10]	R. Dokchan and A. Pakapongpun	$p^x + (p+20)^y = z^2$
[14]	K. Laipaporn, S. Wananiyakul and P. Khachorncharoenkul	$3^x + p5^y = z^2$
[21]	S. Subburam	$la^x + mb^y = nc^z$
[23]	A. Suvarnamani	$p^x + (p+1)^y = z^2$

2. Main Theorem

Our main results are divided into two groups. In the first group, we focus on the equation $a^x+b^y+c^z = w^2$ with the variable c fixed at 7, while varying (a, b) in specific cases. These cases consider all elements in the set A, which includes $(3, 4), (9, 4), (3, 16), (9, 16), (4, 6), (16, 6), (9, 10)$, and $(6, 10)$. This is discussed in Theorems 1–4 and Corollaries 1–2. The second group consists of Theorems 5–7 and Corollary 3, where all variables a, b, and c are allowed to vary under certain conditions. Additionally, we introduce an auxiliary result, referred to as 'Result 1,' which is utilized in sections 1 and 2.

Lemma 1. *For any non-negative integers w and z, the equation $5 + 7^z = w^2$ has no solution.*

Proof. It is clear that w has to be even. So, we have that z is odd because of

$$0 \equiv w^2 \equiv 5 + 7^z \equiv 1 + (-1)^z \equiv \begin{cases} 0 \pmod 4 & \textit{if } z \textit{ is odd,} \\ 2 \pmod 4 & \textit{if } z \textit{ is even.} \end{cases}$$

If $z = 1$ then w is not integer, so we can let $z = 2k + 1$ for some integer $k \geq 1$. Then the eqaution $5 + 7^z = w^2$ becomes $(w-2)(w+2) = (7+1)(7^{2k} - 7^{2k-1} + \cdots - 7 + 1)$ So $8|(w-2)(w+2)$. Since $2|(w-2)$, $2|(w+2)$, and $w + 2 = (w-2) + 4$, we conclude that $4|(w-2)$ and $4|(w+2)$. This implies that $0 \equiv w^2 - 4 \equiv 7^{2k+1} + 1 \equiv 8 \pmod{16}$, so it is a contratiction. Hence the equation $5 + 7^z = w^2$ has no solution.

With the previous lemma, we are ready to solve the following equation.

2.1. The exponential Diophantine equation $a^x + b^y + 7^z = w^2$

First, we note from [3], J. B. Bacani and J. F. T. Rabago that they focused on the equation $3^x + 5^y + 7^z = w^2$. In case $x = y = 0$, the equation becomes to $2 + 7^z = w^2$ that already considered that why we examined our theorems in this section over the set $U = \mathbb{N}_0^4 - \{(0, 0, z, w) | z, w \in \mathbb{N}_0\}$ where $\mathbb{N}_0$ is the set of all non-negative integers. From now on, it causes us to investigate x and y are unequal to zero simultaneously. Then we have the results of the exponential Diophantine equation $a^x + b^y + 7^z = w^2$ where a and b are positive integers and the variables (x, y, z, w) are elements in U as follows:

Theorem 1. *The equation*

$$3^x + 4^y + 7^z = w^2 \tag{1}$$

has no solution for any $(x, y, z, w) \in U$.

Proof. The proof of the theorem is separated into 3 cases.

Case 1 $z = 0$. *Then the equation (1) becomes*

$$3^x + 4^y + 1 = w^2. \tag{2}$$

We note that

$$3^x + 4^y + 1 \equiv \begin{cases} 2 \pmod 3 & \textit{for } x \geq 1 \textit{ and } y \geq 0 \\ 2 \pmod 4 & \textit{for } x = 0 \textit{ and } y \geq 1 \end{cases} \tag{3}$$

Since $w^2 \equiv 0, 1 \pmod 3$ and $w^2 \equiv 0, 1 \pmod 4$, we conclude that $3^x + 4^y + 1 = w^2$ has no solution for all $x, y, w \in \mathbb{N}_0$.

Case 2 $z = 1$. *For any $x \geq 1$ and $y \geq 0$, we see that $3^x + 4^y + 7 \equiv 2 \pmod 3$. Again, $w^2 \equiv 0, 1 \pmod 3$ and this fact forces the equation $3^x + 4^y + 7 = w^2$ has no solution. Now, we remain to consider $x = 0$ and $y \geq 1$ and then the equation (1) is in the form $8 = (w - 2^y)(w + 2^y)$. So, we can let $w - 2^y = 2^u$ and $w + 2^y = 2^{3-u}$ where $u = 0$ or $u = 1$. Since $2w = \begin{cases} 9 & \textit{if } u = 0 \\ 6 & \textit{if } u = 1 \end{cases}$ and w has to be integer, we have $w = 3$ and $y = 0$. It contradics to $y \geq 1$.*

Case 3 $z \geq 2$. *First, we can see that* $3^x + 4^y + 7^z \equiv 2 \pmod 3$ *for any* $x \geq 1$ *and* $y \geq 0$, *but* $w^2 \not\equiv 2 \pmod 3$, *so the equation* (1) *has no solution. Next, we consider* $x = 0$ *and* $y = 1$. *By lemma 1, we know that the equation* (1) *has no solution. Finally, we assume that* $x = 0$ *and* $y \geq 2$. *So* w *is even. It implies that* $0 \equiv w^2 \equiv 1 + 4^y + 7^z \equiv 1 + (-1)^z \pmod 4$ *and then* z *has to be odd. Again* $w^2 \equiv 1 + 4^y + 7^z \equiv 1 + 0 + 7 \equiv 8 \pmod{16}$ *but we have the fact that* $w^2 \equiv 0, 1, 4, 9 \pmod{16}$ *which is a contradiction. Hence* $3^x + 4^y + 7^z = w^2$ *has no solution for any* $(x, y, z, w) \in U$.

Corollary 1. *Each of the following equations:*

$$9^x + 4^y + 7^z = w^2, \tag{4}$$

$$3^x + 16^y + 7^z = w^2 \tag{5}$$

and

$$9^x + 16^y + 7^z = w^2 \tag{6}$$

have no solution for any $(x, y, z, w) \in U$.

Proof. Suppose that $(x_0, y_0, z_0, w_0) \in U$ *is a solution of the equation* (4). *Then* $w_0^2 = 3^{2x_0} + 4^{y_0} + 7^{z_0}$, *i.e.* $(2x_0, y_0, z_0, w_0)$ *is a solution of the equation* (1). *It contradicts to 1. With the same trace of the proof of equation* (4), *we can conclude that equation* (5) *and equation* (6) *have no solution for any* $(x, y, z, w) \in U$.

Theorem 2. *The equation*

$$4^x + 6^y + 7^z = w^2 \tag{7}$$

has no solution for all $(x, y, z, w) \in U$.

Proof. The proof of 2 follows from the footprint of 1 by using modulo $3, 4$ *and* 16 *but dividing the value* z *to two cases.*

Case 1 $z = 0$. *Since* $w^2 \not\equiv 2 \pmod 3$ *but* $4^x + 6^y + 1 \equiv 2 \pmod 3$ *for all* $x \geq 0$ *and* $y \geq 1$, *the equation* (7) *has no solution. So, we remain to proof the case* $x \geq 1$ *and* $y = 0$, *the equation* (7) *becomes to* $4^x + 2 = w^2$. *By the fact that* $x \geq 1$, *we have* w *is even and then* $w^2 \equiv 0 \pmod 4$ *that contradicts to* $w^2 \equiv 4^x + 2 \equiv 2 \pmod 4$.

Case 2 $z \geq 1$. *Again* $w^2 \not\equiv 2 \pmod 3$ *and* $4^x + 6^y + 7^z \equiv 2 \pmod 3$ *for any* $x \geq 0$ *and* $y \geq 1$. *So the equation* (7) *has no solution. Next, we have to consider only subcase* $x \geq 1$ *and* $y = 0$. *If* $x = 1$ *then the equation* (7) *has also no solution by 1. Now, we focus on the equation* $4^x + 1 + 7^z = w^2$ *for* $x \geq 2$. *Since* $w^2 \equiv 0, 1, 4, 9 \pmod{16}$ *and*

$$4^x + 1 + 7^z \equiv \begin{cases} 2 \pmod{16} & \text{if } z \text{ is even,} \\ 8 \pmod{16} & \text{if } z \text{ is odd.} \end{cases}$$

we have $4^x + 1 + 7^z \not\equiv w^2 \pmod{16}$. *Thus the equation* (7) *has no solution.*

Corollary 2. *The equation*

$$16^x + 6^y + 7^z = w^2 \tag{8}$$

has no solution for any $(x, y, z, w) \in U$.

Proof. *With the same trace of 1, we can conclude that equation* (8) *has no solution by using 2.*

Theorem 3. *The equation*

$$9^x + 10^y + 7^z = w^2 \tag{9}$$

has no solution for any $(x, y, z, w) \in U - T$, *where* $T = \{(0, 3, z, w)|z \geq 2$ *and* z *is odd*$\}$.

Proof. *For proving this theorem, we still use modulo* 3 *and* 16 *that play the main role to verify the existence of its solution and also use modulo* 5 *and* 10 *in some subcaes of the variable* z.

Case 1 $z = 0$. *Note that*

$$9^x + 10^y + 1 \equiv \begin{cases} 2 \pmod{3} & \text{if } x \geq 1 \text{ and } y \geq 0, \\ 2 \pmod{10} & \text{if } x = 0 \text{ and } y \geq 1, \end{cases}$$

and we know that neither $w^2 \not\equiv 2 \pmod{3}$ *nor* $w^2 \not\equiv 2 \pmod{10}$. *Then it is clear that the equation* (9) *has no solution on* U *if* $z = 0$.

Case 2 $z = 1$. *With the same fashion in Case 1, we note that*

$$9^x + 10^y + 7 \equiv \begin{cases} 2 \pmod{3} & \text{if } x \geq 1 \text{ and } y \geq 0, \\ 3 \pmod{5} & \text{if } x = 0 \text{ and } y \geq 1. \end{cases}$$

Since $w^2 \equiv 2 \pmod{3}$ *and* $w^2 \not\equiv 3 \pmod{5}$ *the equation* $9^x + 10^y + 7 = w^2$ *has no solution on* U *if* $z = 1$.

Case 3 $z \geq 2$. *If* $x \geq 1$ *and* $y \geq 0$ *then it is easy to see that* $9^x + 10^y + 7^z = w^2$ *has no solution by examing with modulo* 3. *Now, we remain to investigate the last subcase* $x = 0$ *and* $y \geq 1$. *Then the equation* (9) *becomes*

$$1 + 10^y + 7^z = w^2. \tag{10}$$

Note that

$$1 + 7^z \equiv \begin{cases} 2 \pmod{16} & \text{if } z \text{ is even}, \\ 8 \pmod{16} & \text{if } z \text{ is odd}, \end{cases}$$

and

$$10^y \equiv \begin{cases} 10 \pmod{16} & \text{if } y = 1, \\ 4 \pmod{16} & \text{if } y = 2, \\ 8 \pmod{16} & \text{if } y = 3, \\ 0 \pmod{16} & \text{if } y \geq 4. \end{cases}$$

So

$$1 + 10^y + 7^z \equiv \begin{cases} 2, 6, 10, 12 \pmod{16} & \text{if } z \text{ is even and } y \geq 1, \\ 2, 8, 12 \pmod{16} & \text{if } z \text{ is odd, } y \geq 1 \text{ and } y \neq 3, \end{cases}$$

but $w^2 \equiv 0, 1, 4, 9 \pmod{16}$*, this leads us to complete the proof that the equation* (9) *has no solution on* $U-T$ *if* $z \geq 2$*. From here we obtain the equation* $9^x+10^y+7^z = w^2$ *has no solution for any* $(x, y, z, w) \in U - T$*.*

Theorem 4. *The equation*

$$6^x + 10^y + 7^z = w^2 \tag{11}$$

has no solution for any $(x, y, z, w) \in U - T$ *where* $T = \{(0, 3, z, w) | z \geq 2 \text{ and } z \text{ is odd}\}$*.*

Proof. *Note that, if* $x \geq 1$ *and* $y \geq 0$ *then* $6^x + 10^y + 7^z \equiv 2 \pmod{3}$ *for all* $z \geq 0$*. From this fact and the fact that* $w^2 \not\equiv 2 \pmod{3}$*, we conclude that the equation* (11) *has no solution for* $x \neq 0$*. Now, it leads us to consider the equation* (11) *in the form* $1 + 10^y + 7^z = w^2$ *for any* $y \geq 1, z$ *and* w *are non-negative. For the case* $z = 0$ *or* $z = 1$*, we get* $1 + 10^y + 7^z \equiv 2$ *or* $3 \pmod{5}$*. but* $w^2 \not\equiv 2, 3 \pmod{5}$*. So, we remain to examine the equation* $1 + 10^y + 7^z = w^2$ *in the case of* $y \geq 1$*,* $z \geq 2$ *and* $w \geq 0$*.That is the same equation* (10) *in 3 and we immediately conclude that the equation* (11) *has no solution for all* $(x, y, z, w) \in U - T$*.*

2.2. The exponential Diophantine equation $a^x + b^y + c^z = w^2$

In this section, we present our results and discussion for $a^x + b^y + c^z = w^2$, where all bases of the exponential are in terms of variables except 7 and 3.

Theorem 5. *The Diophantine equation* $a^x + b^y + c^z = w^2$ *has no solution when* a, b *and* c *satisfy one of the following conditions:*

(i) $a, b, c \equiv 1 \pmod{4}$.

(ii) $a, b, c \equiv 1 \pmod{5}$.

(iii) $a \equiv 0 \pmod{4}$ *and* $b, c \equiv 1 \pmod{4}$.

(iv) $a \equiv 0 \pmod{5}$ *and* $b, c \equiv 1 \pmod{5}$.

(v) $a \equiv 3 \pmod{8}$ *and* $b, c \equiv 1 \pmod{8}$.

(vi) $a \equiv 1 \pmod{8}$ *and* $b, c \equiv 3 \pmod{8}$.

Proof.

(i) *Suppose that* $a, b, c \equiv 1 \pmod{4}$*. For any nonnegative integers* x, y, z *and* w*, it is obvious that* $a^x + b^y + c^z \equiv 3 \pmod{4}$*, but* $w^2 \equiv 0, 1 \pmod{4}$*. This means that* $a^x + b^y + c^z \not\equiv w^2 \pmod{4}$*. Hence, the Diophantine equation has no solution.*

(ii) *Suppose that* $a, b, c \equiv 1 \pmod 5$. *Given that* $w^2 \equiv 0, 1, 4 \pmod 5$ *and by the same trace of (i), we can conclude that the Diophantine equation has no solution.*

(iii) *Suppose that* $a \equiv 0 \pmod 4$ *and* $b, c \equiv 1 \pmod 4$. *For any nonnegative integers* x, y *and* z, *we know that*

$$a^x \equiv \begin{cases} 1 \pmod 4 & \text{if } x = 0 \\ 0 \pmod 4 & \text{if } x \geq 1, \end{cases}$$

and $b^y, c^z \equiv 1 \pmod 4$. *Thus,*

$$a^x + b^y + c^z \equiv \begin{cases} 3 \pmod 4 & \text{if } x = 0 \\ 2 \pmod 4 & \text{if } x \geq 1, \end{cases}$$

for any nonnegative integers x, y *and* z. *Since* $w^2 \equiv 0, 1 \pmod 4$, *the Diophantine equation has no solution.*

(iv) *By the same trace of (i) and (ii), we can conclude that the Diophantine equation has no solution.*

(v) *Suppose that* $a \equiv 3 \pmod 8$ *and* $b, c \equiv 1 \pmod 8$. *For any nonnegative integers* x, y *and* z, *we can see that*

$$a^x \equiv \begin{cases} 1 \pmod 8 & \text{if } x \text{ is even} \\ 3 \pmod 8 & \text{if } x \text{ is odd}, \end{cases}$$

and $b^y, c^z \equiv 1 \pmod 8$. *Thus,*

$$a^x + b^y + c^z \equiv \begin{cases} 3 \pmod 8 & \text{if } x \text{ is even} \\ 5 \pmod 8 & \text{if } x \text{ is odd}, \end{cases}$$

for any nonnegative integers x, y *and* z. *Since* $w^2 \equiv 0, 1, 4 \pmod 8$, *the Diophantine equation has no solution.*

(vi) *Suppose that* $a \equiv 1 \pmod 8$ *and* $b, c \equiv 3 \pmod 8$. *First, we note that*

$$b^y \equiv \begin{cases} 1 \pmod 8 & \text{if } x \text{ is even} \\ 3 \pmod 8 & \text{if } x \text{ is odd}, \end{cases}$$

$$c^z \equiv \begin{cases} 1 \pmod 8 & \text{if } x \text{ is even} \\ 3 \pmod 8 & \text{if } x \text{ is odd}, \end{cases}$$

$a^x \equiv 1 \pmod 8$ *and* $w^2 \equiv 0, 1, 4 \pmod 8$ *for any nonnegative integers* w, x, y *and* z.

Next, we separate the proof into four cases.

Case 1. *Here, $y = 0$ and $z = 0$. Since $a^x + b^y + c^z \equiv 3 \pmod 8$ and $w^2 \not\equiv 3 \pmod 8$, the Diophantine equation has no solution.*

Case 2. *Here, $y = 0$ and $z > 0$. Since*

$$a^x + 1 + c^z \equiv \begin{cases} 3 \pmod 8 & \text{if } x \text{ is even} \\ 5 \pmod 8 & \text{if } x \text{ is odd,} \end{cases}$$

and $w^2 \not\equiv 3, 5 \pmod 8$, the Diophantine equation has no solution.

Case 3. *Here, $y > 0$ and $z = 0$. The proof of this case is the same as in Case 2.*

Case 4. *Here, $y > 0$ and $z > 0$. Then,*

$$b^y + c^z \equiv \begin{cases} 6 \pmod 8 & \text{if } y \text{ and } z \text{ are odd,} \\ 2 \pmod 8 & \text{if } y \text{ and } z \text{ are even,} \\ 4 \pmod 8 & \text{if otherwise.} \end{cases}$$

However, $w^2 \equiv 0, 1, 4 \pmod 8$, so, the Diophantine equation has no solution.

Theorem 6. *If $a + 2$ is a perfect square number such that $a \equiv 2 \pmod{36}$ and $b, c \equiv 9 \pmod{36}$, then the solutions of Diophantine equation $a^x + b^y + c^z = w^2$ are $(x, y, z, w) = (1, 0, 0, \sqrt{a+2})$.*

Proof. *Note that $a \equiv 2 \pmod 4, b, c \equiv 1 \pmod 4, a \equiv 2 \pmod 9$ and $b, c \equiv 0 \pmod 9$. Since $1 + b^y + c^z \equiv 3 \pmod 4$ and $w^2 \equiv 0, 1 \pmod 4$, equation $a^x + b^y + c^z = w^2$ has no solution for the case $x = 0$. If $x \geq 2$, then $a^x + b^y + c^z \equiv 2 \pmod 4$, which contradicts $w^2 \equiv 0, 1 \pmod 4$. Thus, in this case, $x \geq 2$, and there is no solution. We now consider that for $x = 1, y$ and z are nonzero at the same time. Because*

$$a + b^y + c^z \equiv \begin{cases} 2 \pmod 9 & \text{if } y, z \geq 1, \\ 3 \pmod 9 & \text{if either } y \text{ or } z \text{ is zero,} \end{cases}$$

and $w^2 \equiv 0, 1, 4, 7 \pmod 9$, a solution still does not exist. For the last case, that is, $x = 1, y = z = 0$, we see that $a^x + b^y + c^z = a + 2$ is a perfect square number. Hence, the solution of equation $a^x + b^y + c^z = w^2$ is $(1, 0, 0, \sqrt{a+2})$.

Theorem 7. *If $b, c \equiv 5 \pmod{20}$, then the solution of the Diophantine equation $2^x + b^y + c^z = w^2$ is $(x, y, z, w) = (1, 0, 0, 2)$.*

Proof. *First, we note that $b, c \equiv 1 \pmod 4$ and $b, c \equiv 0 \pmod 5$ since $b, c \equiv 5 \pmod{20}$. If $x = 0$, then we odd value, which leads to $w^2 \equiv 1 \pmod 4$. However, $1 + b^y + c^z \equiv 3 \pmod 4$, contradicts the equation $2^x + b^y + c^z = w^2$, which has a solution in this case. Moreover, for the case $x \geq 2$, there is no solution for nonnegative integer (x, y, z, w) because of $2^x + b^y + c^z \equiv 2 \pmod 4$ and $w^2 \equiv 0, 1 \pmod 4$. Now, it remains to consider only the case $x = 1$. The result of*

$$2 + b^y + c^z \equiv \begin{cases} 3 \pmod 5 & \text{if } y \text{ or } z \text{ are zero,} \\ 2 \pmod 5 & \text{if } y \text{ or } z \text{ are positive,} \end{cases}$$

and $w^2 \equiv 0, 1, 4 \pmod 5$*, we conclude that* $2+b^y+c^z = w^2$ *has no solution for* $y, z, w \in \mathbb{N}_0$*, and* y *and* z *are not all zero simultaneously. Hence,* $2^x+b^y+c^z = w^2$ *has only one solution* $(x, y, z, w) = (1, 0, 0, 2)$.

Corollary 3. *For any nonnegative integers* a, b, c, x, y, z *and* w*, if* $a + 2$ *is a perfect square number, such that* $a \equiv 2 \pmod{20}$ *and* $b, c \equiv 5 \pmod{20}$*, then the solutions of the Diophantine equation* $a^x + b^y + c^z = w^2$ *are* $(x, y, z, w) = (1, 0, 0, \sqrt{a+2})$.

Proof. The proof of corollary 3 follows from the same manner as 7.

Code Listing 1: Python source code.

```
import math
import collections
def is_perfact_square(n):
  i = 1
  while i<=math.floor(n**0.5):
    if i*i==n: return True
    i = i+1
  return False

def check_theorem(a,b,c):
  theorem_dict = {
    (3,4,7):"Theoremm 1",
    (9,4,7):"Corollary 1",
    (3,16,7):"Corollary 1",
    (9,16,7):"Corollary 1",
    (4,6,7):"Theoremm 2",
    (16,6,7):"Corollary 2",
    (9,10,7):"Theoremm 3",
    (6,10,7):"Theoremm 4"
  }
  list_of_thm = []
  if (a,b,c) in theorem_dict.keys(): list_of_thm.append(theorem_dict[(a,b,c
                                      )])
  if b%4==1 and c%4==1:
    if a%4==1:  list_of_thm.append("Theoremm 5(i)")
    elif a%4==0:    list_of_thm.append("Theoremm 5(iii)")
  if b%5==1 and c%5==1:
    if a%5==1:  list_of_thm.append("Theoremm 5(ii)")
    elif a%5==0:    list_of_thm.append("Theoremm 5(iv)")
  if a%8==3 and b%8==1 and c%8==1: list_of_thm.append("Theoremm 5(v)")
  if a%8==1 and b%8==3 and c%8==3: list_of_thm.append("Theoremm 5(vi)")
  if a==2 and b%20==5 and c%20==5: list_of_thm.append("Theoremm 7")
  if is_perfact_square(a+2):
    if a%36==2 and b%36==9 and c%36==9: list_of_thm.append("TTheoremm 6")
    if a%20==2 and b%20==5 and c%20==5: list_of_thm.append("Corollary 3")
  return list_of_thm

if __name__ == "__main__":
  my_dict = dict()
  base_min, base_max = 2, 20
```

```
for a in range(base_min, base_max + 1):
  for b in range(base_min, base_max + 1):
    for c in range(base_min, base_max + 1):
      if len(check_theorem(a,b,c))>0:
        L = [a,b,c]
        L.sort()
        my_dict[tuple(L)] = check_theorem(a,b,c)
ordered_my_dict = collections.OrderedDict(sorted(my_dict.items()))
print(f"No.\tCase\tReference of Theorem")
k = 1
for abc in ordered_my_dict:
  print(f"{k}\t{abc}\t {','.join(ordered_my_dict[abc])}")
  k=k+1
```

2.3. Conclusion

In summary, this article began by extending the number of exponential bases in Diophantine equations from two—represented as $a^x + b^y = z^2$—to three, represented as $a^x + b^y + c^z = w^2$. Most of our results focus on situations where the equation $a^x + b^y + c^z = w^2$ has no solution. Some of these cases relate to conditions where (a, b) is in the set containing pairs like $(3, 4), (9, 4), (3, 16), (9, 16), (4, 6), (16, 6), (9, 10)$, and $(6, 10)$, along with c being equal to 7. Other results are categorized as follows:

(i) $a, b, c \equiv 1 \pmod 4$.

(ii) $a, b, c \equiv 1 \pmod 5$.

(iii) $a \equiv 0 \pmod 4$ and $b, c \equiv 1 \pmod 4$.

(iv) $a \equiv 0 \pmod 5$ and $b, c \equiv 1 \pmod 5$.

(v) $a \equiv 3 \pmod 8$ and $b, c \equiv 1 \pmod 8$.

(vi) $a \equiv 1 \pmod 8$ and $b, c \equiv 3 \pmod 8$.

Some equations of the form $a^x + b^y + c^z = w^2$ do have solutions, as discussed in Theorem 6, 7, and Corollary 3. Our research yielded 135 equations that have been clarified from a total of 1,330 equations, assuming we limit all variables a, b, and c to range from 2 to 20. We used Python code, as shown in Code Listing 1, to count the number of equations that satisfy our theorems and corollaries. Specifically, we offer code that counts all equations of the form $a^x + b^y + c^z = w^2$ that are related to our results in the theorems and corollaries. The accompanying table lists the exponential Diophantine equations with variables a, b, and c limited to the range from 2 to 20. For future work, we aim to combine this research with a new form of Diophantine equation, as seen in reference [8], or we will expand our research using other techniques, such as transforming the equation into an elliptic curve[12], extending into the quadratic field [see details 11], or using Legendre symbols with related theorems,[see also 15, 24].

Table 2: The list of the equations $a^x + b^y + c^z = w^2$ where $2 \leq a \leq b \leq c \leq 20$ and all variables a, b and c are satisfied our results.

No.	$a^x + b^y + c^z = w^2$	Result	Ref. of Theorem
1	$2^x + 5^y + 5^z = w^2$	$(1, 0, 0, 2)$	7, Corollary 3
2	$2^x + 9^y + 9^z = w^2$	$(1, 0, 0, 2)$	6
3	$3^x + 3^y + 9^z = w^2$	No solution	5(vi)
4	$3^x + 3^y + 17^z = w^2$	No solution	5(vi)
5	$3^x + 4^y + 7^z = w^2$	No solution	1
6	$3^x + 7^y + 16^z = w^2$	No solution	Corollary 1
7	$3^x + 9^y + 9^z = w^2$	No solution	5(v)
8	$3^x + 9^y + 11^z = w^2$	No solution	5(vi)
9	$3^x + 9^y + 17^z = w^2$	No solution	5(v)
10	$3^x + 9^y + 19^z = w^2$	No solution	5(vi)
11	$3^x + 11^y + 17^z = w^2$	No solution	5(vi)
12	$3^x + 17^y + 17^z = w^2$	No solution	5(v)
13	$3^x + 17^y + 19^z = w^2$	No solution	5(vi)
14	$4^x + 5^y + 5^z = w^2$	No solution	5(iii)
15	$4^x + 5^y + 9^z = w^2$	No solution	5(iii)
16	$4^x + 5^y + 13^z = w^2$	No solution	5(iii)
17	$4^x + 5^y + 17^z = w^2$	No solution	5(iii)
18	$4^x + 6^y + 7^z = w^2$	No solution	2
19	$4^x + 7^y + 9^z = w^2$	No solution	Corollary 1
20	$4^x + 9^y + 9^z = w^2$	No solution	5(iii)
21	$4^x + 9^y + 13^z = w^2$	No solution	5(iii)
22	$4^x + 9^y + 17^z = w^2$	No solution	5(iii)
23	$4^x + 13^y + 13^z = w^2$	No solution	5(iii)
24	$4^x + 13^y + 17^z = w^2$	No solution	5(iii)
25	$4^x + 17^y + 17^z = w^2$	No solution	5(iii)
26	$5^x + 5^y + 5^z = w^2$	No solution	5(i)
27	$5^x + 5^y + 8^z = w^2$	No solution	5(iii)
28	$5^x + 5^y + 9^z = w^2$	No solution	5(i)
29	$5^x + 5^y + 12^z = w^2$	No solution	5(iii)
30	$5^x + 5^y + 13^z = w^2$	No solution	5(i)
31	$5^x + 5^y + 16^z = w^2$	No solution	5(iii)
32	$5^x + 5^y + 17^z = w^2$	No solution	5(i)
33	$5^x + 5^y + 20^z = w^2$	No solution	5(iii)
34	$5^x + 6^y + 6^z = w^2$	No solution	5(iv)
35	$5^x + 6^y + 11^z = w^2$	No solution	5(iv)

Table 3: The list of the equations $a^x + b^y + c^z = w^2$ where $2 \leq a \leq b \leq c \leq 20$ and all variables a, b and c are satisfied our results.

No.	$a^x + b^y + c^z = w^2$	Result	Ref. of Theorem
36	$5^x + 6^y + 16^z = w^2$	No solution	5(iv)
37	$5^x + 8^y + 9^z = w^2$	No solution	5(iii)
38	$5^x + 8^y + 13^z = w^2$	No solution	5(iii)
39	$5^x + 8^y + 17^z = w^2$	No solution	5(iii)
40	$5^x + 9^y + 9^z = w^2$	No solution	5(i)
41	$5^x + 9^y + 12^z = w^2$	No solution	5(iii)
42	$5^x + 9^y + 13^z = w^2$	No solution	5(i)
43	$5^x + 9^y + 16^z = w^2$	No solution	5(iii)
44	$5^x + 9^y + 17^z = w^2$	No solution	5(i)
45	$5^x + 9^y + 20^z = w^2$	No solution	5(iii)
46	$5^x + 11^y + 11^z = w^2$	No solution	5(iv)
47	$5^x + 11^y + 16^z = w^2$	No solution	5(iv)
48	$5^x + 12^y + 13^z = w^2$	No solution	5(iii)
49	$5^x + 12^y + 17^z = w^2$	No solution	5(iii)
50	$5^x + 13^y + 13^z = w^2$	No solution	5(i)
51	$5^x + 13^y + 16^z = w^2$	No solution	5(iii)
52	$5^x + 13^y + 17^z = w^2$	No solution	5(i)
53	$5^x + 13^y + 20^z = w^2$	No solution	5(iii)
54	$5^x + 16^y + 16^z = w^2$	No solution	5(iv)
55	$5^x + 16^y + 17^z = w^2$	No solution	5(iii)
56	$5^x + 17^y + 17^z = w^2$	No solution	5(i)
57	$5^x + 17^y + 20^z = w^2$	No solution	5(iii)
58	$6^x + 6^y + 6^z = w^2$	No solution	5(ii)
59	$6^x + 6^y + 10^z = w^2$	No solution	5(iv)
60	$6^x + 6^y + 11^z = w^2$	No solution	5(ii)
61	$6^x + 6^y + 15^z = w^2$	No solution	5(iv)
62	$6^x + 6^y + 16^z = w^2$	No solution	5(ii)
63	$6^x + 6^y + 20^z = w^2$	No solution	5(iv)
64	$6^x + 7^y + 10^z = w^2$	No solution	4
65	$6^x + 7^y + 16^z = w^2$	No solution	Corollary 2
66	$6^x + 10^y + 11^z = w^2$	No solution	5(iv)
67	$6^x + 10^y + 16^z = w^2$	No solution	5(iv)
68	$6^x + 11^y + 11^z = w^2$	No solution	5(ii)
69	$6^x + 11^y + 15^z = w^2$	No solution	5(iv)
70	$6^x + 11^y + 16^z = w^2$	No solution	5(ii)
71	$6^x + 11^y + 20^z = w^2$	No solution	5(iv)
72	$6^x + 15^y + 16^z = w^2$	No solution	5(iv)
73	$6^x + 16^y + 16^z = w^2$	No solution	5(ii)

Table 4: The list of the equations $a^x + b^y + c^z = w^2$ where $2 \le a \le b \le c \le 20$ and all variables a, b and c are satisfied our results.

No.	$a^x + b^y + c^z = w^2$	Result	Ref. of Theorem
74	$6^x + 16^y + 20^z = w^2$	No solution	5(iv)
75	$7^x + 9^y + 10^z = w^2$	No solution	3
76	$7^x + 9^y + 16^z = w^2$	No solution	Corollary 1
77	$8^x + 9^y + 9^z = w^2$	No solution	5(iii)
78	$8^x + 9^y + 13^z = w^2$	No solution	5(iii)
79	$8^x + 9^y + 17^z = w^2$	No solution	5(iii)
80	$8^x + 13^y + 13^z = w^2$	No solution	5(iii)
81	$8^x + 13^y + 17^z = w^2$	No solution	5(iii)
82	$8^x + 17^y + 17^z = w^2$	No solution	5(iii)
83	$9^x + 9^y + 9^z = w^2$	No solution	5(i)
84	$9^x + 9^y + 11^z = w^2$	No solution	5(v)
85	$9^x + 9^y + 12^z = w^2$	No solution	5(iii)
86	$9^x + 9^y + 13^z = w^2$	No solution	5(i)
87	$9^x + 9^y + 16^z = w^2$	No solution	5(iii)
88	$9^x + 9^y + 17^z = w^2$	No solution	5(i)
89	$9^x + 9^y + 19^z = w^2$	No solution	5(v)
90	$9^x + 9^y + 20^z = w^2$	No solution	5(iii)
91	$9^x + 11^y + 11^z = w^2$	No solution	5(vi)
92	$9^x + 11^y + 17^z = w^2$	No solution	5(v)
93	$9^x + 11^y + 19^z = w^2$	No solution	5(vi)
94	$9^x + 12^y + 13^z = w^2$	No solution	5(iii)
95	$9^x + 12^y + 17^z = w^2$	No solution	5(iii)
96	$9^x + 13^y + 13^z = w^2$	No solution	5(i)
97	$9^x + 13^y + 16^z = w^2$	No solution	5(iii)
98	$9^x + 13^y + 17^z = w^2$	No solution	5(i)
99	$9^x + 13^y + 20^z = w^2$	No solution	5(iii)
100	$9^x + 16^y + 17^z = w^2$	No solution	5(iii)
101	$9^x + 17^y + 17^z = w^2$	No solution	5(i)
102	$9^x + 17^y + 19^z = w^2$	No solution	5(v)
103	$9^x + 17^y + 20^z = w^2$	No solution	5(iii)
104	$9^x + 19^y + 19^z = w^2$	No solution	5(vi)
105	$10^x + 11^y + 11^z = w^2$	No solution	5(iv)
106	$10^x + 11^y + 16^z = w^2$	No solution	5(iv)
107	$10^x + 16^y + 16^z = w^2$	No solution	5(iv)
108	$11^x + 11^y + 11^z = w^2$	No solution	5(ii)
109	$11^x + 11^y + 15^z = w^2$	No solution	5(iv)
110	$11^x + 11^y + 16^z = w^2$	No solution	5(ii)

Table 5: The list of the equations $a^x + b^y + c^z = w^2$ where $2 \leq a \leq b \leq c \leq 20$ and all variables a, b and c are satisfied our results.

No.	$a^x + b^y + c^z = w^2$	Result	Ref. of Theorem
111	$11^x + 11^y + 17^z = w^2$	No solution	5(vi)
112	$11^x + 11^y + 20^z = w^2$	No solution	5(iv)
113	$11^x + 15^y + 16^z = w^2$	No solution	5(iv)
114	$11^x + 16^y + 16^z = w^2$	No solution	5(ii)
115	$11^x + 16^y + 20^z = w^2$	No solution	5(iv)
116	$11^x + 17^y + 17^z = w^2$	No solution	5(v)
117	$11^x + 17^y + 19^z = w^2$	No solution	5(vi)
118	$12^x + 13^y + 13^z = w^2$	No solution	5(iii)
119	$12^x + 13^y + 17^z = w^2$	No solution	5(iii)
120	$12^x + 17^y + 17^z = w^2$	No solution	5(iii)
121	$13^x + 13^y + 13^z = w^2$	No solution	5(i)
122	$13^x + 13^y + 16^z = w^2$	No solution	5(iii)
123	$13^x + 13^y + 17^z = w^2$	No solution	5(i)
124	$13^x + 13^y + 20^z = w^2$	No solution	5(iii)
125	$13^x + 16^y + 17^z = w^2$	No solution	5(iii)
126	$13^x + 17^y + 17^z = w^2$	No solution	5(i)
127	$13^x + 17^y + 20^z = w^2$	No solution	5(iii)
128	$15^x + 16^y + 16^z = w^2$	No solution	5(iv)
129	$16^x + 16^y + 16^z = w^2$	No solution	5(ii)
130	$16^x + 16^y + 20^z = w^2$	No solution	5(iv)
131	$16^x + 17^y + 17^z = w^2$	No solution	5(iii)
132	$17^x + 17^y + 17^z = w^2$	No solution	5(i)
133	$17^x + 17^y + 19^z = w^2$	No solution	5(v)
134	$17^x + 17^y + 20^z = w^2$	No solution	5(iii)
135	$17^x + 19^y + 19^z = w^2$	No solution	5(vi)

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