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Solutions of some quadratic Diophantine equations

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Abstract. Let $P(t)_i^{\pm} = t^{2k} \pm it^m$ be a non square polynomial and $Q(t)_i^{\pm} = 4k^2t^{4k-2} + i^2m^2t^{2m-2} \pm 4imkt^{2k+m-2} - 4t^{2k} \mp 4it^m - 1$ be a polynomial, such that $k \geq 2m$ and $i \in \{1, 2\}$. In this paper, we consider the number of integer solutions of Diophantine equation

$$E \; : \; x^2 - P(t)_i^{\pm}y^2 - 2P'(t)_i^{\pm}x + 4P(t)_i^{\pm}y + Q(t)_i^{\pm} = 0.$$

We extend a previous results given by A. Tekcan and A. Chandoul et al. We also derive some recurrence relations on the integer solutions of a Pell equation.

2020 Mathematics Subject Classifications: 11D45, 11Y65

Key Words and Phrases: Diophantine equation, Pell's equation, Continued fraction, Quadratic residue

1. Introduction

Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial with integer coefficients in one or more variables. A Diophantine equation is an algebraic equation

$$f(x_1, x_2, \ldots, x_n) = 0$$

for which integer solutions are sought.
The problem to be solved is to determine whether or not a given Diophantine equation has solutions in the domain of integer numbers.
In the case where the Diophantine equation is solvable, there are some natural questions:
$*$) Is the number of solutions finite or infinite ?
$**$) Is it possible to determine all solutions ?
In 1900, Hilbert [4] asked for general algorithm to determine, in a finite number of steps, the solvability of any given Diophantine equation. In other words, he asked if there are any universal method of solving all Diophantine equations.
Unfortunately, it was proven by Matyasevich, in 1970, that this problem is unsolvable [3].

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The absence of a general algorithm was not by itself obstacle to involve more than technique in solving Diophantine equations. In fact, Diophantine equations can be very creative and mathematiciens usually have to exhibit creativity to solve these questions.
One of the best-known techniques is that one based on reduction of the Diophantine equation of arbitrary size with many arbitrary unknowns to another equation having a fixed degree and fixed number of unknowns.
Another one of the most common techniques used to examine Diophantine equations problem is that based on considering residues by checking certain common modulos on each term of the equation, one can either arrive at a contradiction to prove that there's no solution, or to find the unique solutions that satisfy the equation. This technique assumes basic knowledge of modular arithmetic as well as important notions and theorem like the quadratic residues modulo a prime number p and Euler theorem.
Recently, there are a number of paper have been written and published by A. Tekcan using the techniques mentioned above.
This paper offers an extension of one of the results given by A. Tekcan [2] and A. Chandoul et al. [1].
In [2], Tekcan consider the number of integer solutions of Diophantine equation $E: \; x^2 - (t^2-t)y^2 - (4t-2)x + (4t^2-4t)y = 0$ over $\mathbb{Z}$, where $t \geq 2$. Then, we assume that the Diophantine equation E can be extended to the form

$$E: \; x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + (P'(t))^2 - 4P(t) - 1 = 0$$

where $P(t)$ be a non-square polynomial.
Few later years, Chandoul et al. [1] considered the number of integer solutions of Diophantine equation $E_1: \; x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + P'(t)^2 - 4P(t) - 1 = 0$. They derived some recurrence relations on the integer solutions (x_n, y_n) of E_1 and giving a nice generaliations of previous results given by Tekcan [2]. These extensions allows us to solve many types of such equations. We also derive some recurrence relations on the integer solutions of a Pell equation.
Another advantage of our results is that the procedure can be implemented by computer, which allows us to obtain all the solutions after the insertion of the coefficients and the verification of the conditions of the method.

2. Main results

Let $P(t) = t^{2k} \pm it^m$ and $Q(t) = 4k^2t^{4k-2} + i^2m^2t^{2m-2} \pm 4imkt^{2k+m-2} - 4t^{2k} \mp 4it^m - 1$, where $k \geq 2m$ and $i \in \{1, 2\}$. We consider the equation

$$E: \; x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + Q(t) = 0 \tag{1}$$

Theorem 1. *Let $P(t)_i^{\pm} = t^{2k} \pm it^m$, then the continued fraction of $\sqrt{P(t)_i^{\pm}}$ is given as follow ;*

1) $\sqrt{P(t)_1^+} = \begin{cases} \left[t^k; \overline{2}\right], & if\ t = 1 \\ \left[t^k; \overline{2t^{k-m}, 2t^k}\right], & if\ t \geq 2 \end{cases}$

2) $\sqrt{P(t)_1^-} = \left[t^k; \overline{1, 2t^{k-m} - 2, 1, 2t^k - 2}\right],\ t \geq 2$

3) $\sqrt{P(t)_2^+} = \left[t^k; \overline{t^{k-m}, 2t^k}\right]$

4) $\sqrt{P(t)_2^+} = \left[t^k - 1; \overline{1, t^{k-1} - 2, 1, 2t^{k-1} - 2}\right],\ if\ t \geq 3$

Proof. We have ;

$$\begin{aligned}
\sqrt{P(t)_1^+} = \sqrt{t^{2k} + t^m} &= t^k + \sqrt{t^{2k} + t^m} - t^k \\
&= t^k + \cfrac{1}{\cfrac{\sqrt{t^{2k} + t^m} + t^k}{t^m}} \\
&= t^k + \cfrac{1}{2t^{k-m} + \cfrac{\sqrt{t^{2k} + t^m} - t^k}{t^m}} \\
&= t^k + \cfrac{1}{2t^{k-m} + \cfrac{1}{\sqrt{t^{2k} + t^m} + t^k}} \\
&= t^k + \cfrac{1}{2t^{k-m} + \cfrac{1}{2t^k + \sqrt{t^{2k} + t^m} - t^k}}
\end{aligned}$$

Hence, $\sqrt{P(t)_1^+} = \begin{cases} \left[t^k; \overline{2}\right], & \text{if } t = 1 \\ \left[t^k; \overline{2t^{k-m}, 2t^k}\right], & \text{if } t \geq 2 \end{cases}$

Similarly, one can find the requered form of the continued fractions.

Theorem 2. *Let $P(t) = t^{2k} \pm it^m$, where $k \geq 2m \neq 0$ and $i \in \{1, 2\}$ be a non square polynomial and let the Diophantine equation*

$$E \ :\ x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + (P'(t))^2 - 4P(t) - 1 = 0.$$

Then
(1) The fundamental (minimal) solution of E is $(x_1, y_1) = (u_1 + 2kt^{2k-1} \pm imt^{m-1}, v_1 + 2)$
(2) Define the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n + 2kt^{2k-1} \pm imt^{m-1}, v_n + 2)\}$, then (x_n, y_n) is a solution of E. So it has infinitely many integer solutions $(x_n, y_n) \in \mathbb{Z} \times \mathbb{Z}$.

(3) The solutions (x_n, y_n) satisfy the recurrence relations

$$\begin{cases} x_k = u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 + 2kt^{2k-1} \pm imt^{m-1}) - 2\alpha + 2kt^{2k-1} \pm imt^{m-1} \\ \\ y_k = v_1x_{k-1} + (a_0v_1 + \beta)y_{n-1} - v_1(2a_0 + 2kt^{2k-1} \pm imt^{m-1}) - 2\beta + 2 \end{cases}$$

for $k \geq 2$.

Theorem 3. *Let $P(t) = t^{2k} + i$, where $k \neq 0$ and $i \in \{-2, -1, 1, 2\}$ be a non square polynomial and let the Diophantine equation*

$$E \;:\; x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + (P'(t))^2 - 4P(t) - 1 = 0.$$

Then
(1) The fundamental (minimal) solution of E is $(x_1, y_1) = (u_1 + 2kt^{2k-1} \pm imt^{m-1}, v_1 + 2)$
(2) Define the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n + 2kt^{2k-1} \pm imt^{m-1}, v_n + 2)\}$, then (x_n, y_n) is a solution of E. So it has infinitely many integer solutions $(x_n, y_n) \in \mathbb{Z} \times \mathbb{Z}$.
(3) The solutions (x_n, y_n) satisfy the recurrence relations

$$\begin{cases} x_k = u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 + 2kt^{2k-1} \pm imt^{m-1}) - 2\alpha + 2kt^{2k-1} \pm imt^{m-1} \\ \\ y_k = v_1x_{k-1} + (a_0v_1 + \beta)y_{n-1} - v_1(2a_0 + 2kt^{2k-1} \pm imt^{m-1}) - 2\beta + 2 \end{cases}$$

if $i \neq 0$ for $k \geq 2$.

Here, we show that: if P is a non perfect square polynomial, then (1) has an infinitude of integer solutions. In this case we find a closed expression (x_n, y_n), the general positive integer solution, by an original method.
Note that the resolution of E in its present form is difficult, that is, we can not determine how many solutions E has and what they are. So, we have to transform E into a Pell equation which can be easily solved. To get this let

$$T \;:\; \begin{cases} x = u + P'(t) = u + 2kt^{2k-1} \pm imt^{m-1} \\ y = v + 2 \end{cases} \tag{2}$$

we get, $T(E) := \widetilde{E}$, such that

$$\begin{aligned} \widetilde{E} :\;\; & (u + 2kt^{2k-1} \pm imt^{m-1})^2 - (t^{2k} \pm it^m)(v + 2)^2 - (4kt^{2k-1} \pm 2imt^{m-1}) \\ & (u + 2kt^{2k-1} \pm imt^{m-1}) + (4t^{2k} \pm 4it^m)(v + 2) + 4k^2t^{4k-2} + i^2m^2t^{2m-2} \\ & \pm 4imkt^{2k+m-2} - 4t^{2k} \mp 4it^m - 1 \end{aligned}$$

Then, the equation (1) becomes

$$\widetilde{E} \;:\; u^2 - (t^{2k} \pm it^m)v^2 = 1 \tag{3}$$

which is a Pell equation.
It is known that the above Pell equation is always solvable. Its solutions are related to the continued fraction expansion of $\sqrt{P(t)}$.
We will be concerned with the continued fraction expansions of $\sqrt{P(t)}$, where $P(t)$ is a non-square. In fact, this continued fractions have a very interesting form, which is summarized in the next theorem.

Theorem 4. *Let $P(t)$ be a prime. Then $\sqrt{P(t)} = [a_0; a_1, a_2, \cdots, a_l, 2a_0]$, where the repeating portion, excluding the last term, is symmetric upon reversal, and the central term may appear either once or twice.*

Theorem 5. *Let $\sqrt{P(t)} = \left[a_0; \overline{a_1, a_2, \cdots, a_l, 2a_0}\right]$ denote the continued fraction expansion of period lenght l, where $P(t)$ be a non-square polynomial.*

Let $\dfrac{p_n}{q_n}$ be the nth convergent of $\sqrt{P(t)}$. Then

(1) The fundamental solution of the Pell equation $\widetilde{E}$ in (3) is (u_1, v_1), such that

$$\begin{cases} u_1 = p_{l-1} \\ v_1 = q_{l-1} \end{cases}, \text{ if } l \text{ is even,} \quad \text{and} \quad \begin{cases} u_1 = p_{2l-1} \\ v_1 = q_{2l-1} \end{cases}, \text{ if } l \text{ is even}$$

Set $\{(u_k, v_k)\} = \{(p_{kl-1}, q_{kl-1})\}$ where

$$\frac{p_{kl-1}}{q_{kl-1}} = \left[a_0;\ \underbrace{a_1, a_2, \cdots, a_l,}_{l-1} 2a_0,\ \underbrace{a_1, a_2, \cdots, a_l, 2a_0, a_1, a_2, \cdots, a_l}_{(k-1)l-1}\right], \text{ if } l \text{ is even.}$$

$$\text{And } \frac{p_{2kl-1}}{q_{2kl-1}} = \left[a_0; \underbrace{a_1, \cdots, a_l, 2a_0, a_1, \cdots, a_l,}_{2l-1} 2a_0, \underbrace{a_1, \cdots, a_l, 2a_0, \cdots, a_1, \cdots, a_l,}_{(2k-2)l-1}\right],$$

if l is odd. Then (u_k, v_k) is a solution of $\widetilde{E}$.

(2) The consecutive solutions (u_{k-1}, v_{k-1}) and (u_k, v_k) the Pell equation $\widetilde{E}$ in (3) satisfy

$$\begin{cases} u_k = u_1 u_{k-1} + (a_0 u_1 + \alpha) v_{k-1} \\ v_k = v_1 u_{k-1} + (a_0 v_1 + \beta) v_{k-1} \end{cases}, \text{ forall } k \geq 2, \quad \text{if } l \text{ is even}$$

where $\alpha = x_{l-2}$ and $\beta = x_{l-2}$.
and

$$\begin{cases} u_k = u_1 u_{k-1} + (a_0 u_1 + \eta) v_{k-1} \\ v_k = v_1 u_{k-1} + (a_0 v_1 + \delta) v_{k-1} \end{cases}, \text{ forall } k \geq 2, \quad \text{if } l \text{ is odd}$$

where $\eta = x_{2l-2}$ *and* $\delta = x_{2l-2}$.

To prove this theorem, we need the following Lemma

Lemma 1. *Let* $\sqrt{P(t)} = \left[a_0; \overline{a_1, a_2, \cdots, a_l, 2a_0}\right]$ *denote the continued fraction expansion of period lenght* l. *Then*

$$\begin{cases} a_0 x_{kl-1} + x_{kl-2} = P(t) y_{kl-1} \\ a_0 y_{kl-1} + y_{kl-2} = x_{kl-1} \end{cases}$$

for all $k \geq 2$.

Proof. (Lemma 1)
We have $\sqrt{P(t)} = \left[a_0; \overline{a_1, a_2, \cdots, a_1, 2a_0}\right]$.

Thus, we may write $\sqrt{P(t)} = \left[a_0; a_1, a_2, \cdots, a_{kl-1}, a_0 + \sqrt{P(t)}\right]$, then $\sqrt{P(t)} = \dfrac{(a_0 + \sqrt{P(t)})x_{kl-1} + x_{kl-2}}{(a_0 + \sqrt{P(t)})y_{kl-1} + y_{kl-2}}$, which gives rise to the equation

$$P(t) y_{kl-1} + \sqrt{P(t)}(a_0 y_{kl-1} + y_{kl-2}) = (a_0 x_{kl-1} + x_{kl-2}) + \sqrt{P(t)} x_{kl-1}.$$

Which yields, $a_0 x_{kl-1} + x_{kl-2} = P(t) y_{kl-1}$ and $a_0 y_{kl-1} + y_{kl-2} = x_{kl-1}$.

Proof. (Theorem 4)
(1) We prove the theorem only for even number l. It is easily seen that $x_{kl-1}^2 - P(t) y_{kl-1}^2 = x_{kl-1} y_{kl-1} - y_{kl-1} x_{kl-2}$. Then $x_{kl-1}^2 - P(t) y_{kl-1}^2 = (-1)^{kl}$. Thus, if l is even $x_{kl-1}^2 - P(t) y_{kl-1}^2 = 1$ which yields (u_k, v_k) are solutions of $\widetilde{E}$ for all $k \geq 1$ and (u_1, v_1) is the fundamental solution.
We can also prove it using the method of mathematical induction. In fact, if l is even, we have

$$
\begin{aligned}
\frac{u_k}{v_k}=\frac{x_{kl-1}}{y_{kl-1}} &= \left[a_0;\underbrace{a_1,a_2,\cdots,a_1}_{l-1},2a_0,\underbrace{a_1,a_2,\cdots,a_1,2a_0,\cdots,a_1,a_2,\cdots,a_1}_{(k-1)l-1}\right]\\
&= \left[a_0;\underbrace{a_1,a_2,\cdots,a_1,}_{l-1}\ a_0+a_0,\underbrace{a_1,a_2,\cdots,a_1,2a_0,\cdots,a_1,\cdots,a_1}_{(k-1)l-1}\right]\\
&= \left[a_0;\underbrace{a_1,a_2,\cdots,a_1,}_{l-1}\ a_0+\frac{x_{(k-1)l-1}}{y_{(k-1)l-1}}\right] .\\
&= \frac{\left(a_0+\frac{x_{(k-1)l-1}}{y_{(k-1)l-1}}\right)x_{l-1}+x_{l-2}}{\left(a_0+\frac{x_{(k-1)l-1}}{y_{(k-1)l-1}}\right)y_{l-1}+y_{l-2}}\\
&= \frac{a_0y_{(k-1)l-1}x_{l-1}+x_{(k-1)l-1}x_{l-1}+y_{(k-1)l-1}x_{l-2}}{a_0y_{(k-1)l-1}y_{l-1}+x_{(k-1)l-1}y_{l-1}+y_{(k-1)l-1}y_{l-2}}
\end{aligned}
$$

Then

$$
\begin{aligned}
u_k^2-P(t)v_k^2 &= (a_0y_{(k-1)l-1}x_{l-1}+x_{(k-1)l-1}x_{l-1}+y_{(k-1)l-1}x_{l-2})^2\\
&\quad -P(t)(a_0y_{(k-1)l-1}y_{l-1}+x_{(k-1)l-1}y_{l-1}+y_{(k-1)l-1}y_{l-2})^2\\
&= (u_1u_{k-1}+(a_0u_1+\alpha)v_{k-1})^2-P(t)(v_1u_{k-1}+(a_0v_1+\beta)v_{k-1})^2\\
&= u_1^2u_{k-1}^2+2u_1(a_0u_1+\alpha)u_{k-1}v_{k-1}+(a_0u_1+\alpha)^2v_{k-1}^2\\
&\quad -P(t)v_1^2u_{k-1}^2-2P(t)(a_0v_1+\beta)v_1u_{k-1}v_{k-1}-P(t)(a_0v_1+\beta)^2v_{k-1}^2\\
&= (u_1^2-P(t)v_1^2)u_{k-1}^2-\left[(P(t)(a_0v_1+\beta)^2-(a_0u_1+\alpha)^2\right]v_{k-1}^2\\
&\quad +2\left[u_1(a_0u_1+\alpha)-P(t)v_1(a_0v_1+\beta)\right]u_{k-1}v_{k-1}
\end{aligned}
$$

Using the above lemma, we have $(P(t)(a_0v_1+\beta)^2-(a_0u_1+\alpha)^2 = P(t)u_1^2-P(t)^2v_1^2 = P(t)(u_1^2-P(t)v_1^2) = P(t)$ and $u_1(a_0u_1+\alpha)-P(t)v_1(a_0v_1+\beta)=0$. Hence, we conclude that

$$
u_k^2-P(t)v_k^2=u_{k-1}^2-P(t)v_{k-1}^2=1
$$

So (u_k,v_k) is also solution of $\widetilde{E}$. Completing the proof.
(2) This assertion is clear by the above.

As we reported above, the Diophantine equation E could be transformed into the Diophantine equation $\widetilde{E}$ via the transformation T. Also, we showed that $x=u+P'(t)$ and $y=v+2$. So, we can retransfer all results from $\widetilde{E}$ to E by applying the inverse of T. Thus, we can give the following main theorem

Theorem 6. *Let D be the Diophantine equation in (1). Then*
(1) The fundamental (minimal) solution of E is $(x_1,y_1)=(u_1+P'(t),v_1+2)$

(2) Define the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n + P'(t), v_n + 2)\}$, *where* $\{(x_n, y_n)\}$ *defined in (3). Then* (x_n, y_n) *is a solution of* E. *So it has infinitely many integer solutions* $(x_n, y_n) \in \mathbb{Z} \times \mathbb{Z}$.
(3) The solutions (x_n, y_n) *satisfy the recurrence relations*

$$\begin{cases} x_k = u_1 x_{k-1} + (a_0 u_1 + \alpha) y_{n-1} - u_1(2a_0 + P'(t)) - 2\alpha + P'(t) \\ \\ y_k = v_1 x_{k-1} + (a_0 v_1 + \beta) y_{n-1} - v_1(2a_0 + P'(t)) - 2\beta + 2 \end{cases} , \text{ if } l \text{ is even}$$

for $k \geq 2$, *and*

$$\begin{cases} x_k = u_1 x_{k-1} + (a_0 u_1 + \eta) y_{n-1} - u_1(2a_0 + P'(t)) - 2\eta + P'(t) \\ \\ y_k = v_1 x_{k-1} + (a_0 v_1 + \delta) y_{n-1} - v_1(2a_0 + P'(t)) - 2\delta + 2 \end{cases} , \text{ if } l \text{ is odd}$$

for $k \geq 2$.

As an application, we can give the following examples:

Example 1. *Let* $P(t) = t^4 + 4t^3 + 6t^2 + 4t + 2$, *Then*

$$\sqrt{P(t)} = \left[t^2 + 2t + 1; \overline{2t^2 + 4t + 2}\right].$$

So, $(u_1, v_1) = (2t^4 + 8t^3 + 4t^2 + 3, 2t^2 + 4t + 2)$ *is the fundamental solution of*

$$\widetilde{E}_1 \ : \ u^2 - (t^4 + 4t^3 + 2t^2 + 2)v^2 = 1$$

and the other solutions are given by

$$\begin{cases} u_k = (2t^4 + 8t^3 + 4t^2 + 3)u_{k-1} + (2t^6 + 12t^5 + 30t^4 + 40t^3 + 32t^2 + 16t + 4)v_{k-1} \\ \\ v_k = (2t^2 + 4t + 2)u_{k-1} + (2t^4 + 8t^3 + 4t^2 + 3)v_{k-1} \end{cases}$$

For $k \geq 2$.

Morover, let $n = t^2 + 2t + 1$, *then* $P(t)$ *become* $D(n) = n^2 + 1$. *Then*

$$\sqrt{D(n)} = \left[n; \overline{2n}\right].$$

So, $(u_1, v_1) = (2n^2 + 1, 2n)$ *is the fundamental solution of*

$$\widetilde{E}_1 \ : \ u^2 - (n^2 + 1)v^2 = 1$$

and the other solutions are given by

$$\begin{cases} u_k = (2n^2+1)u_{k-1} + (2n^3+2n)v_{k-1} \\ v_k = 2nu_{k-1} + (2n^2+1)v_{k-1} \end{cases}$$

For $k \geq 2$.
Then the fundamental solution of

$$E_1 \ : x^2 - (n^2+1)y^2 - 4nx + (4n^2+4)y - 2 = 0$$

is $(x_1, y_1) = (2n^2+2n+1, 2n+2)$ *and the other solutions are given, for* $k \geq 2$, *by*

$$\begin{cases} x_k = (2n^2+1)x_{k-1} + (2n^3+2n)y_{k-1} - 8n^3 - 4n \\ y_k = 2nx_{k-1} + (2n^2+1)y_{k-1} - 8n^2 + 2n - 2. \end{cases}$$

Further, for $t = 1$, $P(t) = 17$. *Then*

$$\sqrt{P(t)} = \left[4; \overline{8}\right].$$

So, $(u_1, v_1) = (33, 8)$ *is the fundamental solution of*

$$\widetilde{E}_1 \ : \ u^2 - 17v^2 = 1$$

and the other solutions are given by

$$\begin{cases} u_k = 33u_{k-1} + 136v_{k-1} \\ v_k = 8u_{k-1} + 33v_{k-1} \end{cases}$$

For $k \geq 2$.

Then the fundamental solution of

$$E_1 \ : x^2 - 17y^2 - 64x + 68y + 955 = 0$$

is $(x_1, y_1) = (65, 10)$ *and the other solutions are given, for* $k \geq 2$, *by*

$$\begin{cases} x_k = 33x_{k-1} + 136y_{k-1} - 1296 \\ y_k = 8x_{k-1} + 33y_{k-1} - 320. \end{cases}$$

Example 2. *In this example, we consider the number of integer solutions of the Diophantine equation*

$$E \ : x^2 - (t^2+t)y^2 - (4t+2)x + (4t^2+4t)y = 0$$

We have $P(t) = t^2 + t$, *thus* $P'(t) = 2t + 1$ *and the continued fraction expansion of* $\sqrt{P(t)}$ *is*

$$\sqrt{P(t)} = \left[t; \overline{2, 2t}\right]$$

which yields, $\dfrac{u_1}{v_1} = [t; 2] = \dfrac{2t+1}{2}$. *Then the fundamental solution of* E *is* $(x_1, y_1) = (2t + 1 + P'(t), 2 + 2) = (4t + 2, 4)$ *and the other solutions are given by*

$$\begin{cases} x_k = (2t+1)x_{k-1} + (2t^2 + 2t)y_{k-1} - 8t^2 - 6t \\ y_k = 2x_{k-1} + (2t+1)y_{k-1} - 8t - 2 \end{cases} \quad \textit{for,} \;\; k \geq 2$$

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