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Oscillatory properties Test for Even-Order Differential Equations of Neutral type

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Abstract. This paper presents a study on the oscillatory behavior of solutions to even-order neutral differential equations involving p-Laplacian-like operator. We obtain oscillation criteria using techniques from first-order delay differential equations, Riccati technique and integral averages technique. The results of this work contribute to a deeper understanding of even-order differential equations and their connections to various branches of mathematics and practical sciences. The findings emphasize the importance of continued research in this area.

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1. Introduction

Differential equations are characterized by many important advantages that contribute to many practical applications in this life. It is involved in the aviation industry, especially in controlling vibrational motion, in medicine and in civil engineering in building bridges; see [2, 3, 5–7] .

The p -Laplace equations have some significant applications in elasticity theory and continuum mechanics. The oscillation theory of equations has undergone many research contributions by many researchers, especially the study of approximate and oscillatory behavior; see [1, 4, 8, 10, 10, 11, 16, 17].
The aim of this work is to study the oscillatory properties of the solutions of even-order delay differential equations

$$\left(a(\iota)\, w^{(\beta-1)}(\iota)\right)' + \sum_{i=1}^{r} b_i(\iota)\, \varphi\left(\xi\left(z_i(\iota)\right)\right) = 0, \tag{1}$$

where $\beta \geq 2$ and

$$w(\iota) = |\xi(\iota)|^{p-2}\, \xi(\iota) + \varsigma(\iota)\, \xi(\gamma(\iota)) \tag{2}$$

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also

$$\int_{\iota_0}^{\infty} \frac{1}{a(s)} \mathrm{d}s = \infty. \tag{3}$$

We also suppose the following conditions:

$$\begin{cases} (H_1) a, \varsigma \in C([\iota_0, \infty), [0, \infty)), \ b_i \in C([\iota_0, \infty), \mathbb{R}^+), a(\iota) > 0, a'(\iota) \geq 0, 0 \leq \varsigma(\iota) < 1, \\ (H_2) \gamma \in C([\iota_0, \infty), (0, \infty)), \ \gamma(\iota) \leq \iota, \ \lim_{\iota \to \infty} \gamma(\iota) = \infty; z_i \in C([\iota_0, \infty), \mathbb{R}), \ i = 1, 2, ..., r \\ (H_3) \varphi \in C(\mathbb{R}, \mathbb{R}), \varphi(\xi) \geq |\xi|^{p-2} \xi \text{ for } \xi \neq 0, z_i(\iota) \leq \iota, z_i'(\iota) > 0, \lim_{\iota \to \infty} z_i(\iota) = \infty. \\ (H_4) \beta \ \text{ and } p \text{ are positive integers, } \beta \text{ is even, } p > 1. \end{cases}$$

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Definition 1. *A solution of (1) is said oscillatory if it has arbitrarily large zeros on* $[\iota_\xi, \infty)$, *and otherwise is called to be nonoscillatory.*

Definition 2. *Eq. (1) is called to be oscillatory if all its solutions are oscillatory.*

Definition 3. *[14] Let*

$$L_0 = \{(\iota, s) : \iota > s > \iota_0\} \ \text{ and } L = \{(\iota, s) : \iota \geq s \geq \iota_0\}.$$

A function $W \in C(L, \mathbb{R})$ *is said to belong to the function class* ς, *written by* $W \in \varsigma$, *if*

(i) $W(\iota, s) > 0$ *on* L_0 *and* $W(\iota, s) = 0$ *for* $\iota \geq \iota_0$ *with* $(\iota, s) \notin L_0$;

$W(\iota, s)$ has a continuous and nonpositive partial derivative $\partial W / \partial s$ on L_0 and $\varsigma_i \in C(L_0, \mathbb{R})$ such that

$$\frac{\partial W(\iota, s)}{\partial s} = -g(\iota, s) \sqrt{W(\iota, s)}.$$

In recent times, some conditions and properties have been found for delay and neutral differential equations of different orders; see [9, 11–15, 18].

Agarwal et al. [2] used the Riccati method to obtain conditions for the oscillation of equation

$$\left[\left| \xi^{(\beta-1)}(\iota) \right|^{\alpha-1} \xi^{(\beta-1)}(\iota) \right]' + b(\iota) |\xi(\gamma(\iota))|^{\alpha-1} \xi(\gamma(\iota)) = 0.$$

Elabbasy et al. [8] used the comparison method to obtain comparison for the oscillation of equation

$$\left[a(\iota) \left| \left(\xi^{(\beta-1)}(\iota) \right) \right|^{p-2} \xi^{(\beta-1)}(\iota) \right]' + b(\iota) \varphi(\xi(\gamma(\iota))) = 0, \ p > 1,$$

under

$$\int_{\iota_0}^{\infty} \frac{1}{a^{1/(p-1)}(s)} ds = \infty$$

Bazighifan et al. in [13] considered the equation

$$\left(a(\iota)\varphi\left(\xi^{(\beta-1)}(\iota)\right)\right)' + b(\iota)\varphi(\xi(\gamma(\iota))) = 0, \tag{4}$$

where $\varphi(s) = |s|^{p-2}s$ and obtained properties for oscillation of (4).

In [18]. Zhang et al. studied the oscillation of the solutions of equation

$$\left(a(\iota)\left(\xi^{(\beta-1)}(\iota)\right)^{\alpha}\right)' + b(\iota)\xi^{\gamma}(\gamma(\iota)) = 0,$$

under $\int_{\iota_0}^{\infty} a^{-1/\alpha}(s)\,\mathrm{d}s < \infty$.

In our current research, we applied three techniques with some auxiliary lemmas to obtain several conditions and properties of the approximate and oscillatory behavior of the studied equation. These techniques are the comparison method, Riccati method and integral averages method.

2. Oscillation results

Now, we present some the lemmas:

Lemma 1. *[7] If $w \in C^{\beta}([\iota_0,\infty),(0,\infty))$ and $w^{(\beta-1)}(\iota)w^{(\beta)}(\iota) \le 0$ for $\iota \ge \iota_0$, then for every $\nu \in (0,1)$ there exists a constant $j > 0$ such that*

$$|w(\nu\iota)| \ge j\iota^{\beta-1}\left|w^{(\beta-1)}(\iota)\right|,$$

for all large t.

Lemma 2. *[4] Let $w \in C^{\beta}([\iota_0,\infty),(0,\infty))$ and $w^{(\beta-1)}(\iota)w^{(\beta)}(\iota) \le 0$. If $\lim_{\iota\to\infty} w(\iota) \ne 0$, then for every $\mu \in (0,1)$ there exists a $\iota_\mu \ge \iota_0$ such that*

$$|w(\iota)| \ge \frac{\mu}{(\beta-1)!}\iota^{\beta-1}\left|w^{(\beta-1)}(\iota)\right|,$$

for all $\iota \ge \iota_\mu$.

Lemma 3. *[3] Let $w(\iota)$ be an β times differentiable function on $[\iota_0,\infty)$ of constant sign and $w^{(\beta)}(\iota) \ne 0$ on $[\iota_0,\infty)$ which satisfies $w(\iota)w^{(\beta)}(\iota) \le 0$. Then,*

(I) there exists a $\iota_1 \ge \iota_0$ such that the functions $w^{(i)}(\iota)$, $i = 1,2,...,\beta-1$ are of constant sign on $[\iota_0,\infty)$;

(II) there exists a number $l \in \{1,3,5,...,\beta-1\}$ when β is even, $l \in \{0,2,4, ..., \beta-1\}$ when β is odd, such that, for $\iota \ge \iota_1$,

$$w(\iota)w^{(i)}(\iota) > 0,$$

for all $i = 0,1,...,l$ and

$$(-1)^{\beta+i+1}w(\iota)w^{(i)}(\iota) > 0,$$

for all $i = l+1,...,\beta$.

Lemma 4. *Let $\xi(\iota)$ is an eventually positive solution of equation (1). Then*

$$w(\iota) > 0,\ w'(\iota) > 0,\ w^{(\beta-1)}(\iota) \geq 0 \quad and \quad w^{(\beta)}(\iota) \leq 0, \tag{5}$$

for $\iota \geq \iota_2$.

Proof. Suppose $\xi(\iota)$ is an eventually positive solution of (1). Then $w(\iota) > 0$ and

$$\left(aw^{(\beta-1)}\right)'(\iota) = -\sum_{i=1}^{r} b_i(\iota)\,\varphi(\xi(z_i(\iota))) \leq 0. \tag{6}$$

Which means that $a(\iota)\,w^{(\beta-1)}(\iota)$ is decreasing and $w^{(\beta-1)}(\iota)$ is eventually of one sign. We claim that $w^{(\beta-1)}(\iota) \geq 0$. Otherwise, if there exists a $\iota_2 \geq \iota_1$ such that $w^{(\beta-1)}(\iota) < 0$ for $\iota \geq \iota_2$, and

$$\left(aw^{(\beta-1)}\right)(\iota) \leq \left(aw^{(\beta-1)}\right)(\iota_2) = -L,$$

where $L > 0$. Integrating the above inequality from ι_2 to ι we find

$$w^{(\beta-2)}(\iota) \leq w^{(\beta-2)}(\iota_2) - L\int_{\iota_2}^{\iota} \frac{1}{a(s)}\mathrm{d}s.$$

Letting $\iota \to \infty$, we have $\lim_{\iota\to\infty} w^{(\beta-2)}(\iota) = -\infty$, which contradicts the fact that $w(\iota) > 0$. Hence, we obtain $w^{(\beta-1)}(\iota) \geq 0$ for $\iota \geq \iota_1$.
From Eq. (1), we get

$$\left(aw^{(\beta)}\right)(\iota) = -\left(a'w^{(\beta-1)}\right)(\iota) - \sum_{i=1}^{r} b_i(\iota)\,\varphi(\xi(z_i(\iota))) \leq 0,$$

this implies that $w^{(\beta)}(\iota) \leq 0$, $\iota \geq \iota_1$. From Lemma 3, we find (5) hold. The proof is complete.

Theorem 1. *Let*

$$\xi'(\iota) + \widehat{K}(\iota)\,\xi(z(\iota)) = 0, \tag{7}$$

is oscillatory, where

$$\begin{aligned} \widehat{K}(\iota) \ &: \ = \frac{\mu z^{\beta-1}(\iota)}{(\beta-1)!a(z(\iota))}K(\iota), \\ K(\iota) \ &: \ = \sum_{i=1}^{r} b_i(\iota)(1-\varsigma(z(\iota))), \end{aligned}$$

then Eq. (1) is oscillatory.

Proof. Let (1) has a nonoscillatory solution. From Lemma 4, we find (5) holds. From

$$w(\iota) = |\xi(\iota)|^{p-2}\xi(\iota) + \varsigma(\iota)\xi(\gamma(\iota)),$$

we see that

$$\begin{aligned} \xi^{p-1}(\iota) &= w(\iota) - \varsigma(\iota)\xi(\gamma(\iota)) \geq w(\iota) - \varsigma(\iota)w(\gamma(\iota)) \geq w(\iota) - \varsigma(\iota)w(\iota) \\ &\geq (1-\varsigma(\iota))w(\iota) \end{aligned}$$

and so

$$\xi^{p-1}(z_i(\iota)) \geq w(z_i(\iota))(1-\varsigma(z_i(\iota))). \tag{8}$$

From (8), we get

$$\varphi(\xi(z_i(\iota))) \geq w(z_i(\iota))(1-\varsigma(z_i(\iota))). \tag{9}$$

From (1) and (9), we see

$$\begin{aligned} \left(aw^{(\beta-1)}\right)'(\iota) &\leq -\sum_{i=1}^{r} b_i(\iota)w(z_i(\iota))(1-\varsigma(z_i(\iota))) \\ &\leq -w(z(\iota))\sum_{i=1}^{r} b_i(\iota)(1-\varsigma(z_i(\iota))) \\ &= -K(\iota)w(z(\iota)) \end{aligned} \tag{10}$$

In view of Lemma 2, we obtain

$$w(\iota) \geq \frac{\mu}{(\beta-1)!}\iota^{\beta-1}w^{(\beta-1)}(\iota),$$

for all $\iota \geq \iota_2 \geq \max\{\iota_1, \iota_\mu\}$. Thus, by using (10), we find

$$\left(a(\iota)w^{(\beta-1)}(\iota)\right)' + \frac{\mu z^{\beta-1}(\iota)K(\iota)}{(\beta-1)!a(z(\iota))}\left(a(z(\iota))w^{(\beta-1)}(z(\iota))\right) \leq 0.$$

Therefore, we get $\xi(\iota) = a(\iota)w^{(\beta-1)}(\iota)$ is a positive solution of the differential inequality

$$\xi'(\iota) + \widehat{K}(\iota)\xi(z(\iota)) \leq 0.$$

From [14, Corollary 1], we see that (7) also has a positive solution, a contradiction. This completes the proof.

Using Theorem 2.1.1 in [18], we get the following corollary.

Corollary 1. *If*

$$\liminf_{\iota\to\infty}\int_{z(\iota)}^{\iota}\frac{z^{\beta-1}(s)}{a(z(s))}K(s)\,\mathrm{d}s > \frac{(\beta-1)!}{\mu\mathrm{e}}, \text{for } \mu \in (0,1),$$

then (1) is oscillatory.

Theorem 2. *If*

$$\int_{\iota_0}^{\infty}\left(\hbar(u)K(u)-\frac{1}{4\nu}\left(\frac{\hbar'(u)}{\hbar(u)}\right)^2 b(u)\right)\mathrm{d}u=\infty, \textit{for } \nu\in(0,1), j>0, \tag{11}$$

then (1) is oscillatory, where $\hbar\in C^1([\iota_0,\infty),\mathbb{R}^+)$ *and*

$$b(\iota):=\frac{a(\iota)\hbar(\iota)}{jz^{\beta-2}(\iota)z'(\iota)}.$$

Proof. Let (1) has a nonoscillatory solution. As in the proof of Theorem 1, we arrive at (10). From Lemma 1 with $\xi=w'$, there exists a $j>0$ and $z(\iota)\le\iota$ such that

$$\begin{aligned} w'(\nu z(\iota)) &\ge jz^{\beta-2}(\iota)w^{(\beta-1)}(z(\iota))\\ &\ge jz^{\beta-2}(\iota)w^{(\beta-1)}(\iota). \end{aligned}\tag{12}$$

Define

$$\psi(\iota):=\hbar(\iota)\frac{a(\iota)w^{(\beta-1)}(\iota)}{w(\nu z(\iota))}>0,$$

we have

$$\psi'(\iota)=\frac{\hbar'(\iota)}{\hbar(\iota)}\psi(\iota)+\hbar(\iota)\frac{\left(a(\iota)w^{(\beta-1)}(\iota)\right)'}{w(\nu z(\iota))}-\nu\hbar(\iota)\frac{a(\iota)w^{(\beta-1)}(\iota)w'(\nu z(\iota))z'(\iota)}{(w(\nu z(\iota)))^2}.$$

From (10), we obtain

$$\psi'(\iota)\le\frac{\hbar'(\iota)}{\hbar(\iota)}\psi(\iota)-\hbar(\iota)K(\iota)-\nu\frac{w'(z(\iota))z'(\iota)}{w(\nu z(\iota))}\psi(\iota).$$

By using (12), we have

$$\begin{aligned} \psi'(\iota) &\le \frac{\hbar'(\iota)}{\hbar(\iota)}\psi(\iota)-\hbar(\iota)K(\iota)-\nu\frac{jz^{\beta-2}(\iota)w^{(\beta-1)}(\iota)z'(\iota)}{w(\nu z(\iota))}\psi(\iota)\\ &\le \frac{\hbar'(\iota)}{\hbar(\iota)}\psi(\iota)-\hbar(\iota)K(\iota)-\nu\frac{jz^{\beta-2}(\iota)z'(\iota)}{a(\iota)\hbar(\iota)}\frac{\hbar(\iota)a(\iota)w^{(\beta-1)}(\iota)}{w(\nu z(\iota))}\psi(\iota)\\ &\le \frac{\hbar'(\iota)}{\hbar(\iota)}\psi(\iota)-\hbar(\iota)K(\iota)-\frac{\nu}{b(\iota)}\psi^2(\iota). \end{aligned}\tag{13}$$

Using the inequality

$$\xi w-uw^{\frac{a+1}{a}}\le\frac{a^a}{(a+1)^{a+1}}\frac{\xi^{a+1}}{u^a},$$

with $\xi=\hbar'/\hbar$, $u=\nu jz^{\beta-2}(\iota)z'(\iota)/(a(\iota)\hbar(\iota))$ and $w=\psi(\iota)$, we find

$$\psi'(\iota)\le-\hbar(\iota)K(\iota)+\frac{1}{4\nu}\left(\frac{\hbar'(\iota)}{\hbar(\iota)}\right)^2\frac{a(\iota)\hbar(\iota)}{jz^{\beta-2}(\iota)z'(\iota)}.\tag{14}$$

Integrating (14) from ι_1 to ι we find

$$\begin{aligned}\int_{\iota_1}^{\iota}\left(\hbar(u)K(u)-\frac{1}{4\nu}\left(\frac{\hbar'(u)}{\hbar(u)}\right)^2 b(u)\right)\mathrm{d}u &\leq \psi(\iota_1)-\psi(\iota)\\ &\leq \psi(\iota_1),\end{aligned}$$

which contradicts (11). This completes the proof.

Theorem 3. *If $\hbar\in C^1([\iota_0,\infty),\mathbb{R}^+)$ such that*

$$\limsup_{\iota\to\infty}\frac{1}{W(\iota,\iota_0)}\int_{\iota_0}^{\iota}W(\iota,u)\left(\hbar(u)K(u)-\frac{1}{4\nu}b(u)\varsigma^2(\iota,u)\right)\mathrm{d}u=\infty, \tag{15}$$

where

$$\varsigma(\iota,s)=\frac{\hbar'(s)}{\hbar(s)}-\frac{g(\iota,s)}{\sqrt{W(\iota,s)}},$$

then Eq.(1) is oscillatory.

Proof. Multiplying (13) by $W(\iota,s)$ and integrating both sides from ι_2 to ι, we obtain

$$\begin{aligned}\int_{\iota_2}^{\iota}W(\iota,u)\hbar(u)K(u)\,\mathrm{d}u &\leq -\int_{\iota_2}^{\iota}W(\iota,u)\psi'(u)\,\mathrm{d}u-\int_{\iota_2}^{\iota}W(\iota,u)\frac{\nu}{b(u)}\psi^2(u)\,\mathrm{d}u\\ &\quad+\int_{\iota_2}^{\iota}W(\iota,u)\frac{\hbar'(u)}{\hbar(u)}\psi(u)\,\mathrm{d}u\\ &\leq W(\iota,\iota_2)\psi(\iota_2)-\int_{\iota_2}^{\iota}W(\iota,u)\frac{\nu}{b(u)}\psi^2(u)\,\mathrm{d}u\\ &\quad+\int_{\iota_2}^{\iota}W(\iota,u)\psi(u)\varsigma(\iota,u)\,\mathrm{d}u\end{aligned}$$

which implies that

$$\begin{aligned}\int_{\iota_2}^{\iota}W(\iota,u)\hbar(u)K(u)\,\mathrm{d}u &\leq W(\iota,\iota_2)\psi(\iota_2)\\ &\quad-\int_{\iota_2}^{\iota}W(\iota,u)\frac{\nu}{b(u)}\left(\psi^2(u)-\frac{b(u)}{\nu}\varsigma(\iota,u)\psi(u)\right)\mathrm{d}u\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{W(\iota,\iota_2)}\int_{\iota_2}^{\iota}W(\iota,u)\left(\hbar(u)K(u)-\frac{1}{4\nu}b(u)\varsigma^2(\iota,u)\right)\mathrm{d}u\\ &\quad\leq\psi(\iota_2)-\frac{1}{W(\iota,\iota_2)}\int_{\iota_2}^{\iota}W(\iota,u)\frac{\nu}{b(u)}\left(\psi(u)-\frac{1}{2\nu}b(u)\varsigma(\iota,u)\right)^2\mathrm{d}u,\end{aligned}$$

which implies

$$\limsup_{\iota\to\infty}\frac{1}{W(\iota,\iota_2)}\int_{\iota_2}^{\iota}W(\iota,u)\left(\hbar(u)K(u)-\frac{1}{4\nu}b(u)\varsigma^2(\iota,u)\right)\mathrm{d}u\le\psi(\iota_2).$$

From (15), we have a contradiction. This completes the proof.

Corollary 2. *Let*

$$0<\inf_{s\ge\iota}\left(\liminf_{\iota\to\infty}\frac{W(\iota,s)}{W(\iota,\iota_0)}\right)\le\infty$$

and

$$\limsup_{\iota\to\infty}\frac{1}{W(\iota,\iota_0)}\int_{\iota_0}^{\iota}W(\iota,u)\,b(u)\,\varsigma^2(\iota,u)\,\mathrm{d}u<\infty.$$

If

$$\limsup_{\iota\to\infty}\int_{\iota_0}^{\iota}\frac{\varsigma^2(s)}{b(s)}\mathrm{d}s=\infty$$

for $\varsigma\in C([\iota_0,\infty),\mathbb{R})$ *and* $\varsigma(\iota)=\max\{\varsigma(\iota),0\}$, *also,*

$$\limsup_{\iota\to\infty}\frac{1}{W(\iota,\iota_0)}\int_{\iota_0}^{\iota}W(\iota,u)\left(\hbar(u)K(u)-\frac{1}{4\nu}b(u)\varsigma^2(\iota,u)\right)\mathrm{d}u\ge\sup_{\iota\ge\iota_0}\varsigma(\iota),$$

then (1) is oscillatory.

Example 1. *Consider the second-order equation:*

$$\left[\iota\left(\xi(\iota)+\frac{1}{2}\xi\left(\frac{\iota}{3}\right)\right)'\right]'+\frac{b_0}{\iota}\left(\xi^2+\xi\right)\left(\frac{\iota}{2}\right)=0,\iota\ge1, \tag{16}$$

where $b_0>0$ *is a constant. Let* $\beta=p=2,\ a(\iota)=\iota,\ \varsigma(\iota)=1/2,\ \gamma(\iota)=\iota/3,\ b(\iota)=b_0/\iota,\ z(\iota)=\iota/2,\ \varphi(\xi)=\xi^2+\xi.$
Now, we see that

$$K(\iota)=b(\iota)(1-\varsigma(z(\iota)))=\frac{b_0}{2\iota}$$

and

$$b(\iota)=\frac{a(\iota)\hbar(\iota)}{jz^{\beta-2}(\iota)z'(\iota)}=\frac{2\iota^2}{j}.$$

If we set $\hbar=\iota$ *then any for constants* $j>0,\ 0<\nu<1$

$$\begin{aligned}&\int_{\iota_0}^{\infty}\left(\hbar(u)K(u)-\frac{1}{4\nu}\left(\frac{\hbar'(u)}{\hbar(u)}\right)^2b(u)\right)\mathrm{d}u,\\ =&\int_{\iota_0}^{\infty}\left(\frac{b_0}{2}-\frac{1}{2\nu j}\right)\mathrm{d}u\\ =&\ \infty\qquad \textit{if } b_0>1.\end{aligned}$$

From Theorem 2, every solution of equation (16) is oscillatory if $b_0>1$.

Example 2. *Consider the fourth-order equation:*

$$\left[\iota w'''(\iota)\right]' + \frac{b}{\iota}\xi\left(\frac{\iota}{3}\right) = 0, \iota \geq 1, \tag{17}$$

where $w(\iota) = \xi(\iota) + \frac{1}{3}\xi\left(\frac{\iota}{2}\right)$ *and* $b > 0$ *is a constant. Let* $\beta = 4,\ p = 2,\ a(\iota) = \iota,\ \varsigma(\iota) = 1/3,\ \gamma(\iota) = \iota/2,\ b(\iota) = b_0/\iota,\ z(\iota) = \iota/3,\ \varphi(\xi) = \xi.$
Thus, we see that

$$\int^{\infty} a^{-1}(\iota)\, d\iota = \infty.$$

By Theorem 3, every solution of equation (17) is oscillatory.

3. Conclusion

In conclusion, this study aimed at investigating the oscillatory properties of solutions to even-order differential equations with a p-Laplacian. The findings of this paper contribute to the understanding of the asymptotic and oscillatory behavior of such equations and provide new oscillation criteria through the use of comparison methods with first-order differential equations, Riccati technique and integral averages technique. This work highlights the relevance of the theory of fourth-order differential equations to various fields of mathematics and practical sciences, emphasizing the importance of continued research in this area.

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