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Cubic Trigonometric B-spline Method for Solving a Linear System of Second Order Boundary Value Problems

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Abstract. This paper introduces a novel trigonometric B-spline collocation method for solving a specific class of second-order boundary value problems. The study showcases the method's practicality and effectiveness through various numerical examples. Furthermore, it evaluates the technique's performance by calculating maximum errors for different step sizes in the spatial domain. The paper also conducts a comparative analysis with alternative methods, demonstrating the superior accuracy of the trigonometric B-spline approach.

2020 Mathematics Subject Classifications: 34A30, 65L10

Key Words and Phrases: Linear system of second order boundary value problems, boundary value problems, cubic trigonometric B-spline basis functions.

1. Introduction

System of second order ordinary differential equation associated with boundary conditions arises in several applications and have been discussed in [1–22]. Issue of existence of solution to such system has been discussed in [1–3]. The standard numerical approaches to solve second order boundary value problems are to use the shooting and finite difference methods. In this paper, we focus on the use of cubic trigonometric B-spline method (CTBSM). Consider the following system of second order boundary value problems:

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_2(x)u(x) + a_3(x)v''(x) + a_4(x)v'(x) + a_5(x)v(x) = f_1(x) \\ v''(x) + b_1(x)v'(x) + b_2(x)v(x) + b_3(x)u''(x) + b_4(x)u'(x) + b_5(x)u(x) = f_2(x). \end{cases} \tag{1}$$

The following boundary conditions are associated with the system

$$u(a) = u(b) = 0, v(a) = v(b) = 0, \tag{2}$$

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where $a \leq x \leq b$, $f_1(x)$ and $f_2(x)$ are continuous functions, $a_i(x)$ and $b_i(x)$, for $i = 1, 2, 3, 4, 5$, are sufficiently smooth real-valued functions of x.

Many approximate techniques have been suggested to solve linear and nonlinear systems of second order BVPs. In [4], Saadatmandi et al. introduced a homotopy perturbation approach aimed at addressing nonlinear second-order boundary value problems. This method generates solutions in the form of convergent series with readily computable or minor perturbations. Ogunlaran and Ademola [5] introduced the Laplace homotopy analysis method as an alternative for solving the same system. This technique obtains solutions without resorting to discretization or imposing restrictive assumptions. The resulting solution manifests in the form of a rapidly convergent series. Lu [6] introduced a variational iteration approach to solve systems analogous to problem (1)-(2). [7, 8] presented a method to obtain the analytical and approximate solutions in the form of series in the reproducing kernal space for linear and nonlinear system of second order BVPs. In [9, 10], Dehghan, Saadatmandi, and El-Gamel utilized a numerical approach employing the Sinc-collocation method for solving second-order nonlinear systems. This method is acknowledged as effective for problems with potential singularities, infinite domains, or boundary layers. Furthermore, it is applied to streamline the computation of solutions for the system represented by equations (1)-(2) into a set of algebraic equations. Local radial basis function based differential quadrature collocation method [11] is also suggested to solve system of second order BVPs, this method is easy to implementation and the results is more efficient. Saadatmandi and J. Askari [12] solved similar systems using the Chebyshev finite element method. Moreover, the continuous genetic algorithm, as described in reference [13], has proven to be successful in addressing our problem. Additionally, a comprehensive analysis of the convergence and sensitivity of genetic operators and control parameters for the algorithm has been conducted. Finally, [14–16] are also proposed different methods to solve the system (1)-(2). The aim of this research is to use CTBSM for solving linear system of second order BVPs. The system had already been treated using spline collocation approach [17], cubic B-spline scaling functions [18], B-spline method [19], Extended cubic B-spline method [20], Hybrid cubic B-spline method [21]. CTBSM was developed to solve dynamical systems in [22], the results are promising. In our present work, a new trigonometric B-spline technique is described and presented for solving a linear system of second order BVPs. The technique is based on the cubic trigonometric B-spline functions. CTBSM is used as an interpolating function in the space dimension. The regular system is attempted using a trigonometric B-spline collocation approach constructed over uniform mesh. The efficiency and applicability of the technique are demonstrated by applying the scheme to solve several examples. The numerical results demonstrate that this method is superior as it yields more accurate solutions.
The outline of this study is as follows: In section 2, the CTBSM is utilized as an interpolating function in the space dimension for solving a linear system of second order BVPs. A numerical solution of linear system of second order BVPs is presented in section 3. Numerical results are also considered in section 4 to show the achievable of the proposed method. Finally, the concluding remarks of this study are in section 5.

2. Cubic Trigonometric B-Spline Method

In this section, the CTBSM is applied for the numerical solution of second order BVPs (1)-(2). Let $a = x_0 < x_1 < ... < x_n = b$ be the mesh point in the interval $[a, b]$ such that $x_i = a + ih, \ \ i = 0, 1, 2, ..., n$ with $h = \frac{b-a}{n}$.
Let the approximate solution $U(x)$ and $V(x)$ to the exact solution $u(x)$ and $v(x)$ at the point x_i respectively, and can be defined as:

$$\begin{cases} U(x) = \displaystyle\sum_{i=-3}^{n-1} \alpha_i T_i^4(x), \ x \in [x_0, x_n], \ \alpha_i \in \mathbb{R}, \\ V(x) = \displaystyle\sum_{i=-3}^{n-1} \beta_i T_i^4(x), \ x \in [x_0, x_n], \ \beta_i \in \mathbb{R}. \end{cases} \tag{3}$$

where α_i, β_i are real coefficient to be determined for the approximated solutions $U(x)$ and $V(x)$ respectively and $T_i(x)$ are trigonometric B-spline basis functions which have C^2 continuity at each knots and defined over the mesh by [21–23]

$$T_i^4(x) = \frac{1}{\omega} \begin{cases} p^3(x_i), & x \in [x_i, x_{i+1}], \\ p(x_i)(p(x_i)q(x_{i+2}) + q(x_{i+3})p(x_{i+1})) + q(x_{i+4})p^2(x_{i+1}), & x \in [x_{i+1}, x_{i+2}], \\ q(x_{i+4})(p(x_{i+1})q(x_{i+3}) + q(x_{i+4})p(x_{i+2})) + p(x_i)q^2(x_{i+3}), & x \in [x_{i+2}, x_{i+3}], \\ q^3(x_{i+4}), & x \in [x_{i+3}, x_{i+4}]. \end{cases} \tag{4}$$

where, $\ \ p(x_i) = \sin(\frac{x-x_i}{2}), \ \ q(x_i) = \sin(\frac{x_i-x}{2}), \ \ \omega = \sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})$.
For the numerical solution of proposed problem, the values of $T_i(x)$ and its derivatives $T_i^{'}(x), T_i^{''}(x)$ at nodal points are need and recorded in Table. 1.

Table 1: Coefficient of $T_i(x)$, $T_i^{'}(x)$ and $T_i^{''}(x)$

x	x_{i-1}	x_i	x_{i+1}
$T_i(x)$	$m_1(x)$	$m_2(x)$	$m_1(x)$
$T_i^{'}(x)$	$m_3(x)$	0	$m_4(x)$
$T_i^{''}(x)$	$m_5(x)$	$m_6(x)$	$m_5(x)$

where

$$m_1(x) = \frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}, \quad m_2(x) = \frac{2}{1 + 2\cos(h)}, \quad m_3(x) = \frac{3}{4\sin(\frac{3h}{2})}, \quad m_4(x) = \frac{-3}{4\sin(\frac{3h}{2})},$$

$$m_5(x) = \frac{3(1 + 3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2}) + \cos(\frac{3h}{2}))}, \qquad m_6(x) = \frac{-3\cot^2(\frac{h}{2})}{2 + 4\cos(h)}.$$

From equation (3) and (4), the values of $U(x)$, $U^{'}(x)$, $U^{''}(x)$, $V(x)$, $V^{'}(x)$, and $V^{''}(x)$ at the knots x_i are determined in the terms of α_i and β_i as

$$\begin{cases} U(x_i) = m_1(x)\alpha_{i-3} + m_2(x)\alpha_{i-2} + m_1(x)\alpha_{i-1}, \\ U^{'}(x_i) = m_3(x)\alpha_{i-3} + m_4(x)\alpha_{i-1}, \\ U^{''}(x_i) = m_5(x)\alpha_{i-3} + m_6(x)\alpha_{i-2} + m_5(x)\alpha_{i-1}. \end{cases} \tag{5}$$

and

$$\begin{cases} V(x_i) = m_1(x)\beta_{i-3} + m_2(x)\beta_{i-2} + m_1(x)\beta_{i-1}, \\ V^{'}(x_i) = m_3(x)\beta_{i-3} + m_4(x)\beta_{i-1}, \\ V^{''}(x_i) = m_5(x)\beta_{i-3} + m_6(x)\beta_{i-2} + m_5(x)\beta_{i-1}. \end{cases} \tag{6}$$

3. Solution of System of Second Order Boundary Value Problems

In this section, a numerical solution of a class of system of second order BVPs (1)-(2) is obtained using collocation approach based on cubic trigonometric basis functions. It is necessary to satisfy the proposed system of second order differential equation (1)-(2) by putting the approximate (3) in (1) at $x = x_i$, it follows as:

$$\begin{aligned} &\sum_{i=-3}^{n-1} \alpha_i[T_i^{''}(x_i) + a_1(x_i)T_i^{'}(x_i) + a_2(x_i)T_i(x_i)] + \\ &\sum_{i=-3}^{n-1} \beta_i[a_3(x_i)T_i^{''}(x_i) + a_4(x_i)T_i^{'}(x_i) + a_5(x_i)T_i(x_i)] = f_1(x_i), \ i = 0, 1, ..., n \end{aligned} \tag{7}$$

$$\begin{aligned} &\sum_{i=-3}^{n-1} \beta_i[T_i^{''}(x_i) + b_1(x_i)T_i^{'}(x_i) + b_2(x_i)T_i(x_i)] + \\ &\sum_{i=-3}^{n-1} \alpha_i[b_3(x_i)T_i^{''}(x_i) + b_4(x_i)T_i^{'}(x_i) + b_5(x_i)T_i(x_i)] = f_2(x_i), \ i = 0, 1, ..., n \end{aligned} \tag{8}$$

The boundary conditions (2) can be written as:

$$\sum_{i=-3}^{n-1} \alpha_i T_i(x_i) = 0, \qquad x = 0, n \tag{9}$$

$$\sum_{i=-3}^{n-1} \beta_i T_i(x_i) = 0, \qquad x = 0, n \tag{10}$$

The values and derivatives of spline functions at knots x_i are determined using Table 1 and putting in (7)-(10). A system of equations with unknown $\alpha_{-3}, \alpha_{-2}, \alpha_{-1}, ..., \alpha_{n-1}, \beta_{-3}, \beta_{-2}, \beta_{-1}, ..., \beta_{n-1}$

of order $(2n+3)\times(2n+3)$ is obtained and can be solved by Thompson algorithm [24] for the trigonometric spline solution of proposed problem. This system can be written in the matrix-vector form as follows

$$AB = C \tag{11}$$

where $B = [\alpha_{-3}, \alpha_{-2}, ..., \alpha_{n-1}, \beta_{-3}, \beta_{-2}, ..., \beta_{n-1}]^T$, $C = [0, f_1(x_0), ..., f_1(x_n), 0, 0, f_2(x_0), ..., f_2(x_n), 0]^T$, and A is a $2(n+3)\times 2(n+3)$ matrix given by

$$A = \begin{pmatrix} M_1 & | & M_2 \\ \cdots & \cdots & \cdots \\ M_4 & | & M_3 \end{pmatrix}.$$

where $M_K, K = 1, 2, 3, 4$ are four submatrices, can be defined as:

$$M_1 = \begin{pmatrix} m_1(x) & m_2(x) & m_1(x) & 0 & \cdots & 0 & 0 \\ \epsilon_1(x_0) & \rho_1(x_0) & \tau_1(x_0) & 0 & \cdots & 0 & 0 \\ 0 & \epsilon_1(x_1) & \rho_1(x_1) & \tau_1(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \epsilon_1(x_n) & \rho_1(x_n) & \tau_1(x_n) \\ . & . & . & . & m_1(x) & m_2(x) & m_1(x) \end{pmatrix}_{(n+3)\times(n+3)}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \epsilon_2(x_0) & \rho_2(x_0) & \tau_2(x_0) & 0 & \cdots & 0 & 0 \\ 0 & \epsilon_2(x_1) & \rho_2(x_1) & \tau_2(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \epsilon_2(x_n) & \rho_2(x_n) & \tau_2(x_n) \\ . & . & . & . & 0 & 0 & 0 \end{pmatrix}_{(n+3)\times(n+3)}$$

$$M_3 = \begin{pmatrix} m_1(x) & m_2(x) & m_1(x) & 0 & \cdots & 0 & 0 \\ \epsilon_3(x_0) & \rho_3(x_0) & \tau_3(x_0) & 0 & \cdots & 0 & 0 \\ 0 & \epsilon_3(x_1) & \rho_3(x_1) & \tau_3(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \epsilon_3(x_n) & \rho_3(x_n) & \tau_3(x_n) \\ . & . & . & . & m_1(x) & m_2(x) & m_1(x) \end{pmatrix}_{(n+3)\times(n+3)}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \epsilon_4(x_0) & \rho_4(x_0) & \tau_4(x_0) & 0 & \cdots & 0 & 0 \\ 0 & \epsilon_4(x_1) & \rho_4(x_1) & \tau_4(x_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \epsilon_4(x_n) & \rho_4(x_n) & \tau_4(x_n) \\ . & . & . & . & 0 & 0 & 0 \end{pmatrix}_{(n+3)\times(n+3)}$$

The input of M_1, M_2, M_3, and M_4 are given below for $i = 0, 1, ..., n$.

$$\epsilon_1(x_i) = \frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + a_1(x_i)\frac{-3}{4\sin(\frac{3h}{2})} + a_2(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}$$

$$\begin{aligned}
\rho_1(x_i) &= \frac{-3\cot^2(\frac{h}{2})}{2+4\cos(h)} + a_1(x_i)(0) + a_2(x_i)\frac{2}{1+2\cos(h)}\\
\tau_1(x_i) &= \frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + a_1(x_i)\frac{3}{4\sin(\frac{3h}{2})} + a_2(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}\\
\epsilon_2(x_i) &= a_3(x_i)\frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + a_4(x_i)\frac{-3}{4\sin(\frac{3h}{2})} + a_5(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}\\
\rho_2(x_i) &= a_3(x_i)\frac{-3\cot^2(\frac{h}{2})}{2+4\cos(h)} + a_4(x_i)(0) + a_5(x_i)\frac{2}{1+2\cos(h)}\\
\tau_2(x_i) &= a_3(x_i)\frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + a_4(x_i)\frac{3}{4\sin(\frac{3h}{2})} + a_5(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}\\
\epsilon_3(x_i) &= \frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + b_1(x_i)\frac{-3}{4\sin(\frac{3h}{2})} + b_2(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}\\
\rho_3(x_i) &= \frac{-3\cot^2(\frac{h}{2})}{2+4\cos(h)} + b_1(x_i)(0) + b_2(x_i)\frac{2}{1+2\cos(h)}\\
\tau_3(x_i) &= \frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + b_1(x_i)\frac{3}{4\sin(\frac{3h}{2})} + b_2(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}\\
\epsilon_4(x_i) &= b_3(x_i)\frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + b_4(x_i)\frac{-3}{4\sin(\frac{3h}{2})} + b_5(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}\\
\rho_4(x_i) &= b_3(x_i)\frac{-3\cot^2(\frac{h}{2})}{2+4\cos(h)} + b_4(x_i)(0) + b_5(x_i)\frac{2}{1+2\cos(h)}\\
\tau_4(x_i) &= b_3(x_i)\frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} + b_4(x_i)\frac{3}{4\sin(\frac{3h}{2})} + b_5(x_i)\frac{\sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}
\end{aligned}$$

Error analysis of the boundary value problem (1) and implementation on method is discussed in the following section through three numerical examples.

4. Numerical Results and Discussions

Some numerical examples are considered in this section to demonstrate the competency of the proposed trigonometric spline method. Numerical results found by the method are compared with existing methods in the literature such as [6] and [19] and with the analytical solution at knots x_i using different N. It was establish that proposed method in contrast with these methods is more accurate. Absolute errors of the proposed method is calculated numerically by the following formula [25–27]

$$L_\infty = \max_i \mid u(x_i) - U(x_i) \mid \qquad \text{or} \qquad L_\infty = \max_i \mid v(x_i) - V(x_i) \mid$$

$$L_2 = \sqrt{\sum_{i=1}^{N}(u(x_i) - U(x_i))^2} \quad \text{or} \quad L_2 = \sqrt{\sum_{i=1}^{N}(v(x_i) - V(x_i))^2}$$

Example 1.Consider the following system [6]

$$\begin{cases} u^{''}(x) + (2x-1)u^{'}(x) + \cos(\pi x)v^{'}(x) = f_1(x) \\ v^{''}(x) + xu(x) = f_2(x) \\ u(0) = u(1) = 0, v(0) = v(1) = 0, \end{cases} \tag{12}$$

where $0<x<1$, $f_1(x) = -\pi^2\sin(\pi x) + (2x-1)\pi\cos(\pi x) + (2x-1)\cos(\pi x)$, and $f_2(x) = 2 + x\sin(\pi x)$. The exact solutions are $u(x) = \sin(\pi x)$ and $v(x) = x^2 - x$. The errors are different knots found using CTBSM at $h = 1/40$ are tabulated in Table 2 and compared with existing methods. It is concluded that the present CTBSM is acceptable and accurate than both VIM and CBSM. The numerical results obtained by CTBSM are illustrated in Fig. 1 and Fig. 2.

Table 2: Comparison of present method with VIM [6] and CBSM [19] for Example 1 for $u(x)$ and $v(x)$

x	VIM $U(x)$	CBSM $U(x)$	CBSM $V(x)$	CTBSM $U(x)$	CTBSM $V(x)$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	$3.30E-04$	$1.40E-04$	$5.74E-06$	$4.27E-05$	$6.53E-06$
0.2	$2.51E-03$	$2.80E-04$	$1.13E-05$	$5.45E-05$	$2.12E-06$
0.3	$7.84E-03$	$3.90E-04$	$1.64E-05$	$6.56E-05$	$2.45E-06$
0.4	$1.66E-02$	$4.60E-04$	$2.03E-05$	$7.29E-05$	$2.88E-06$
0.5	$2.77E-02$	$4.80E-04$	$2.26E-05$	$7.48E-05$	$3.21E-06$
0.6	$3.87E-02$	$4.60E-04$	$2.26E-05$	$7.29E-05$	$3.22E-06$
0.7	$4.59E-02$	$3.90E-04$	$2.01E-05$	$6.56E-05$	$2.87E-06$
0.8	$4.49E-02$	$2.80E-04$	$1.51E-05$	$5.45E-05$	$2.44E-06$
0.9	$3.09E-02$	$1.50E-04$	$8.14E-06$	$4.27E-05$	$8.87E-06$
1.0	0.00000	0.00000	0.00000	$2.07E-13$	$1.83E-13$

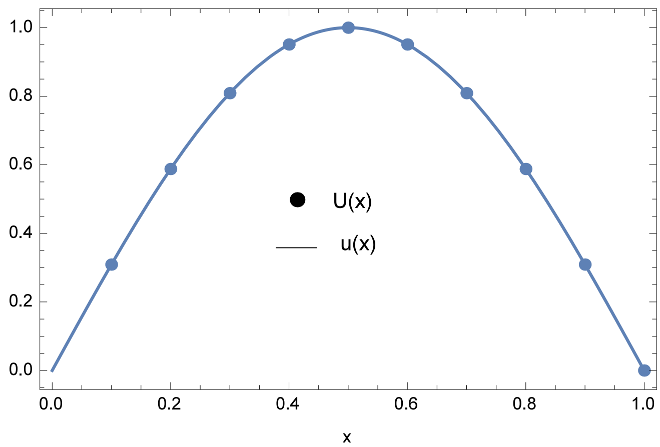

Figure 1: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 1 with $n = 5$

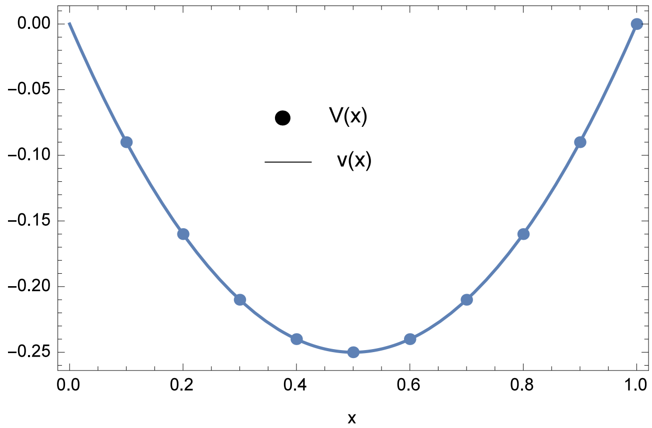

Figure 2: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 1 with $n = 5$

Example 2. Consider the following equations [17, 28]

$$\begin{cases} u^{''}(x) + u^{'}(x) + xu(x) + v^{'}(x) + 2xv(x) = f_1(x) \\ v^{''}(x) + v(x) + 2u^{'}(x) + x^2u(x) = f_2(x) \\ u(0) = u(1) = 0, v(0) = v(1) = 0 \end{cases} \tag{13}$$

where $0 \leq x \leq 1$, $f_1(x) = -2(x+1)\cos(x) + \pi\cos(\pi x) + 2x\sin(\pi x) + (4x - 2x^2 - 4)\sin(x)$, and $f_2(x) = -4(x-1)\cos(x) - 2(2 - x^2 + x^3)\sin(x) - (\pi^2 - 1)\sin(\pi x)$. The exact solutions are $u(x) = 2(x-1)sin(x)$, and $v(x) = \sin(\pi x)$. The Table 3 reports the maximum value of the errors of present method compared with the results obtained in [8, 9] and exact solutions. It is concluded that the present CTBSM is more accurate than the methods developed in [8, 9]. Fig. 3 and Fig. 4 depict the numerical results obtained by CTBSM at $n = 25$.

Table 3: Maximum errors for Example 2 with $n = 25$

x	reproducing kernel [8]		Sinc method [9]		CTBSM	
	$u(x)$	$v(x)$	$u(x)$	$v(x)$	$u(x)$	$v(x)$
0.08	$3.3E-03$	$7.7E-03$	$3.2E-03$	$1.5E-03$	$1.1E-04$	$3.2E-04$
0.24	$7.7E-03$	$2.2E-02$	$9.4E-04$	$7.0E-03$	$1.5E-04$	$7.8E-04$
0.40	$9.7E-03$	$2.7E-02$	$2.0E-03$	$7.4E-03$	$1.3E-04$	$1.1E-03$
0.56	$9.5E-03$	$2.7E-02$	$2.2E-04$	$1.0E-02$	$1.2E-04$	$1.1E-03$
0.72	$7.3E-03$	$2.0E-02$	$4.1E-03$	$4.4E-03$	$8.2E-05$	$0.8E-03$
0.88	$3.4E-03$	$9.4E-03$	$1.0E-02$	$2.1E-02$	$1.3E-05$	$3.6E-04$
0.96	$1.1E-03$	$3.1E-03$	$2.1E-03$	$6.9E-03$	$2.7E-07$	$1.3E-04$

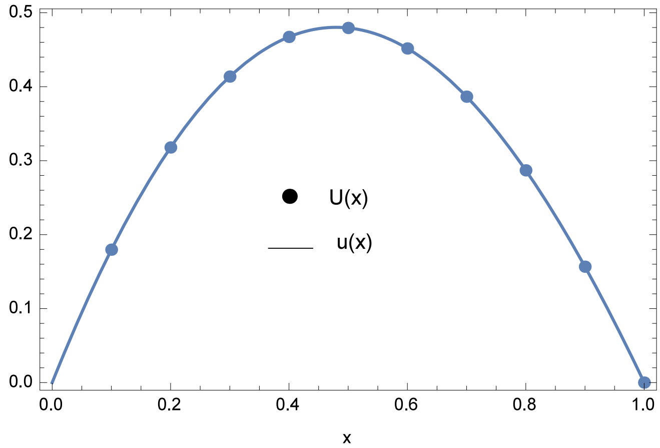

Figure 3: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 2 with $n = 5$

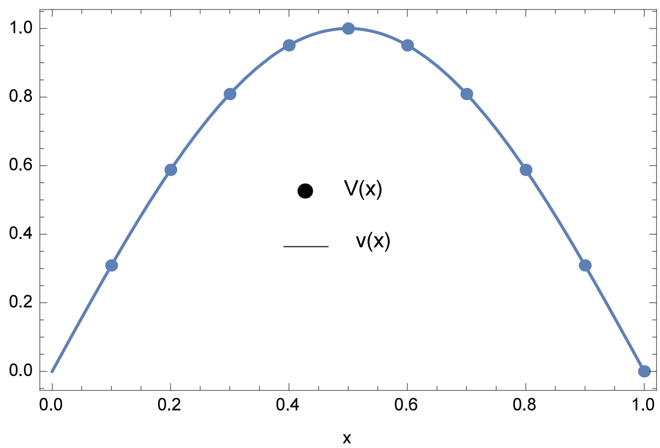

Figure 4: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 2 with $n = 5$

Example 3.Finally, we consider the system [19]

$$\begin{cases} u''(x) + xu(x) + xv(x) = 2 \\ v''(x) + 2xv(x) + 2xu(x) = -2 \\ u(0) = u(1) = 0, v(0) = v(1) = 0 \end{cases} \tag{14}$$

where $0<x<1$. The exact solutions are $u(x) = x^2 - x$ and $v(x) = x - x^2$. The approximate and exact solutions at the nodal point are displayed in Table 4. Also, from the table, the approximate solutions agree with the exact solutions. The errors at difficult knots and absolute errors obtained by using proposed CTBSM at $n = 21$ are tabulated in Table 5 and compared with existing methods [19]. It is concluded that the present

CTBSM is acceptable and accurate than CBSM [19]. The numerical results obtained by CTBSM are illustrated in Fig. 5 and Fig. 6.

Table 4: Comparison of present method with exact solution for Example 3 with $n = 5$

x	Exact Solution $u(x)$	Approx. Solution $U(x)$	Absolute error	Exact Solution $v(x)$	Approx. Solution $V(x)$	Absolute error
0.2	−0.160000	−0.160000	$3.763546E-16$	0.160000	0.160000	$4.216428E-16$
0.4	−0.240000	−0.240000	$2.352348E-17$	0.240000	0.240000	$5.807326E-16$
0.6	−0.240000	−0.240000	$9.682004E-16$	0.240000	0.240000	$2.472546E-16$
0.8	−0.160000	−0.160000	$3.258334E-15$	0.160000	0.160000	$3.371215E-15$

Table 5: Comparison of errors norms for CBSM and CTBSM for Example 3 with $n = 21$ for $u(x)$ and $v(x)$

	CBSM		CTBSM	
Errors	$u(x)$	$v(x)$	$u(x)$	$v(x)$
Max-Norm	$3.72E-13$	$2.53E-13$	$1.25E-13$	$1.22E-13$
$L2$-Norm	$4.37E-13$	$4.37E-13$	$2.44E-13$	$2.18E-13$

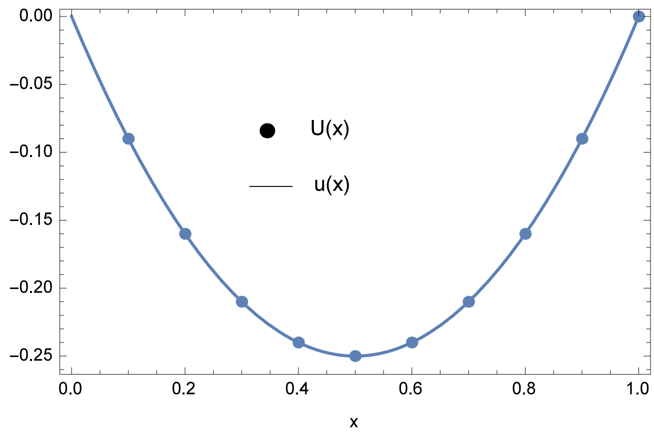

Figure 5: Numerical solution $U(x)$ and exact solution $u(x)$ for Example 3 with $n = 5$

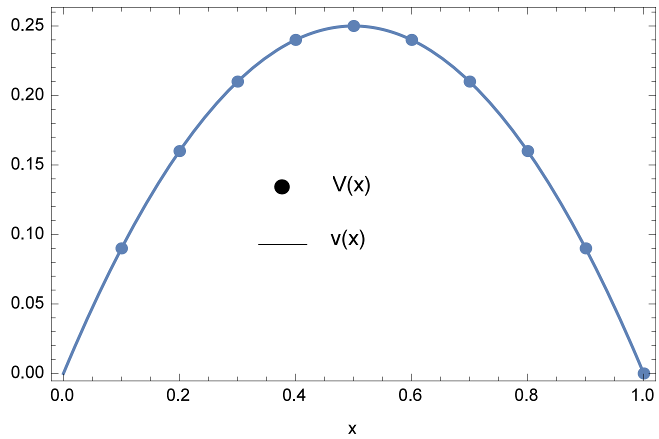

Figure 6: Numerical solution $V(x)$ and exact solution $v(x)$ for Example 3 with $n = 5$

5. Conclusions

In this paper, a numerical approach grounded on CTBSM has been utilized to solve linear system of second order boundary value problems. The CTBSM used in this paper is simple and straight forward to apply. The numerical results reported in the tables and depicted in the graphs illustrated the validity of the method and provided highly accurate solutions that are superior when compared with other available methods or compare favorably with them to say the least.

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