EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2323-2347
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A general family of fifth-order iterative methods for solving nonlinear equations

Ali Zein

Department of Applied Mathematics, Palestine Polytechnic University, Hebron, Palestine

Abstract. A family of fifth-order iterative methods is proposed for solving nonlinear equations using the weight function technique. This family offers flexibility through its structure and the choice of weight functions, resulting in a wide range of new specific schemes. It is demonstrated that this proposed family includes several well-known and recent methods as special cases. In addition, several new particular methods are designed to achieve better results than existing methods of the same type. Convergence analysis is conducted, and numerical examples in both real and complex domains are provided for several specific schemes within the proposed family. Comparisons between the existing methods within this family and the newly introduced methods generally indicate improved performance among the new members. Notably, the study of complex dynamics and basins of attraction reveals that our new specific schemes have broader sets of initial points that lead to convergence.

2020 Mathematics Subject Classifications: 65H05, 65B99

Key Words and Phrases: Fifth-order method, Weight function, Basins of attraction, Efficiency index, Nonlinear equations, Iterative methods

1. Introduction

Solving nonlinear equations is a very important problem in various fields of science and technology [2]. It is mostly impossible to find exact or analytical solutions for these equations. Therefore, iterative methods are used to find approximate solutions for them. In the literature, numerous iterative methods have been developed for approximating a root of a nonlinear equation $f(x) = 0$. Newton's method is the most popular and widely used iterative scheme defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots,$$

where x_0 is the initial approximation to a root α. It is well known that this method is quadratically convergent to simple roots. A root α is simple if $f(\alpha) = 0$ but $f'(\alpha) \neq 0$.

DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4949

Email address: `alizein@ppu.edu` (A. Zein)

In the last two decades, many higher order iterative methods have been derived and analyzed for solving nonlinear equations, see [1–4, 6–12, 14–23] and the references therein. In particular, many fifth-order iterative methods are found in the literature, see [6, 7, 9–12, 14, 18, 20].

Commonly, the *order of convergence* and the *efficiency index* are important in rating iterative methods. The efficiency index $E.I.$ is defined as $E.I. = p^{\frac{1}{d}}$, where p is the order of convergence and d is the total number of new function evaluations per iteration. According to the *Kung and Traub conjecture* [13] a multi-point method which uses d evaluations could achieve, at most, a convergence order $p = 2^{d-1}$. Methods that reach this bound are known as *optimal*. According to the Kung -Traub conjecture, a fifth-order scheme requires a minimum of four function evaluations per iteration.

To achieve fifth-order convergence, several authors have modified classical methods such as Newton's method, Chebyshev's method, and Chebyshev-Halley's method [7, 11, 12, 18]. But these methods require the evaluation of the second derivative, which can be expensive. Recently, several fifth-order methods have been developed by using the weight function technique [10, 14, 20].

In this paper, a two-step family of fifth-order iterative methods is presented. Convergence analysis is established, and a general error equation is derived for the family. The methods in this family require four new function evaluations per iteration, which is the minimum number of new function evaluations per iteration for fifth-order iterative methods, in the sense of Kung -Traub conjecture. Thus, the efficiency index is $E.I. = 5^{\frac{1}{4}} = 1.495$.

The presented family has the flexibility of choice in its formation and in selecting suitable weight functions, offering a large number of particular schemes. It is shown that several well-known and recent schemes are considered as special cases of the family. Namely, these methods include the Ham and Chun method [9], the Fang et al. method [6], the Sivakumar and Jayaraman method [20], the Liu et al. method [14], and the recent method of Khirallah and Alkhomsan [10].

Furthermore, the flexibility of choice within the presented family enables us to establish several specific schemes that exhibit better stability compared to existing methods of the same type. Indeed, four new particular schemes are deduced from the general family. Numerical results and the *basins of attraction* show better performance for the new schemes compared with the existing schemes mentioned above.

This paper is organized as follows: Section 2 provides preliminaries. Section 3 presents the construction and convergence analysis of the new family of fifth-order methods. In Section 4, some special cases of the proposed family are established. Also, some well-known schemes are listed as particular cases of the proposed family. Section 5 is devoted to study the stability of particular methods by using the basins of attraction technique. In section 6, numerical results for particular methods in real domain are given and comparisons are made.

2. Preliminaries

Definition 1. *[5] Suppose $\{x_n\}$ is a sequence that converges to a limit α. If there exist* *an integer constant $p \geq 1$ and a non-zero constant C such that*

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = C,$$

then p is called the order of convergence and C is called the asymptotic error constant.

Let $e_n = x_n - \alpha$ is the error in the n^{th} iteration. The equation

$$e_{n+1} = Ce_n^p + \mathcal{O}(e_n^{p+1})$$

is called *the error equation* for the method, where p is the order of convergence, see [23].

Definition 2. *[23] Let α be a root of $f(x) = 0$ and suppose that $x_{n-1}, x_n,$ and x_{n+1} are three consecutive iterations closer to the root α. Then the* computational order of convergence *can be computed by using the formula*

$$p \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}.$$

3. Construction of the new family and convergence analysis

Assume that α is a simple root of $f(x) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a sufficiently differentiable function. The new family of fifth-order iterative methods consists of two steps as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

$$x_{n+1} = y_n - G(\eta_n)\frac{f(y_n)}{Af'(x_n)+Bf'(y_n)}, \tag{2}$$

where A and B are parameters, and G is a weight function in terms of

$$\eta_n = 1 - \frac{f'(y_n)}{f'(x_n)}. \tag{3}$$

The first step of this family involves the classical Newton's method. In the second step, the parameters A and B are chosen arbitrarily. Then, the weight function G is designed to achieve fifth-order convergence, as demonstrated in Theorem 1. Note that, the symbolic computations in all proofs in this paper are performed by using Maple software.

Remark 1. *In this paper, we remark that:*

(i) The notation y_n given in (1) *represents the Newton step.*

(ii) The conventional notation $c_j = \dfrac{f^{(j)}(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, ...$ is employed.

Theorem 1. *Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, and the weight function $G(\eta)$ satisfies*

$$G(0) = A + B, \quad G'(0) = A, \quad G''(0) = \frac{5A+B}{2}, \tag{4}$$

then the iterative family (2) *converges to α with order of convergence five and satisfies the error equation*

$$e_{n+1} = \left(\left(14A + 4B - \frac{4}{3}G'''(0)\right)\frac{c_2^4}{A+B} - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6). \tag{5}$$

Provided $A + B \neq 0$ and $|G'''(0)| < \infty$.

Proof. Expanding $f(x_n)$ and $f'(x_n)$ about α by using Taylor's series, we get

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5] + \mathcal{O}(e_n^6) \tag{6}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5] + \mathcal{O}(e_n^6). \tag{7}$$

From (6) and (7), we get

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ + (8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + \mathcal{O}(e_n^6).\end{aligned} \tag{8}$$

Subtracting α from both sides of (1), then using $e_n = x_n - \alpha$ and inserting (8), we have

$$\begin{aligned} y_n - \alpha &= e_n - \frac{f(x_n)}{f'(x_n)} \\ &= c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 - (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &\quad - (8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + \mathcal{O}(e_n^6). \end{aligned} \tag{9}$$

Expanding $f(y_n)$ and $f'(y_n)$ about α and using (9) we obtain

$$\begin{aligned} f(y_n) = f'(\alpha)[c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ - 2(6c_2^4 - 12c_2^2c_3 + 5c_2c_4 + 3c_3^2 - 2c_5)e_n^5] + \mathcal{O}(e_n^6) \end{aligned} \tag{10}$$

and

$$\begin{aligned} f'(y_n) = f'(\alpha)[1 + 2c_2^2e_n^2 - (4c_2^3 - 4c_2c_3)e_n^3 + (8c_2^4 - 11c_2^2c_3 + 6c_2c_4)e_n^4 \\ - 2c_2(8c_2^4 - 14c_2^2c_3 + 10c_2c_4 - 4c_5)e_n^5] + \mathcal{O}(e_n^6). \end{aligned} \tag{11}$$

From (7), (10), and (11), we have

$$\begin{aligned}\frac{f(y_n)}{Af'(x_n)+Bf'(y_n)} &= \frac{c_2e_n^2}{A+B} + \Big[(-4c_2^2+2c_3)A - 2(c_2^2-c_3)B\Big]\frac{e_n^3}{(A+B)^2}\\ &+ \Big[(13c_2^3-14c_2c_3+3c_4)A^2 + (12c_2^3-21c_3c_2+6c_4)AB\\ &+ (3c_2^3-7c_3c_2+3c_4)B^2\Big]\frac{e_n^4}{(A+B)^3}\\ &+ \Big[(-38c_2^4+64c_2^2c_3-20c_2c_4-12c_3^2+4c_5)A^3\\ &- (48c_2^4-124c_2^2c_3+50c_2c_4+30c_3^2-12c_5)BA^2\\ &- (22c_2^4-76c_2^2c_3+40c_2c_4+24c_3^2-12c_5)B^2A\\ &- (4c_2^4-16c_2^2c_3+10c_2c_4+6c_3^2-4c_5)B^3\Big]\frac{e_n^5}{(A+B)^4} + \mathcal{O}(e_n^6). \quad (12)\end{aligned}$$

Using (7) and (11) we can write the weight function variable η given in (3) as

$$\begin{aligned}\eta_n &= 1 - \frac{f'(y_n)}{f'(x_n)}\\ &= 2c_2e_n + (-6c_2^2+3c_3)e_n^2 + (16c_2^3-16c_2c_3+4c_4)e_n^3\\ &\quad +(-40c_2^4+61c_2^2c_3-22c_2c_4-9c_3^2+5c_5)e_n^4\\ &\quad +(96c_2^5-198c_2^3c_3+88c_2^2c_4+66c_2c_3^2-28c_2c_5-24c_3c_4+6c_6)e_n^5 + \mathcal{O}(e_n^6). \quad (13)\end{aligned}$$

Then, expanding the weight function $G(\eta_n)$ around zero, we get

$$\begin{aligned}G(\eta_n) &= G(0) + G'(0)\eta_n + \frac{G''(0)}{2}\eta_n^2 + \frac{G'''(0)}{6}\eta_n^3 + \dots\\ &= G(0) + 2G'(0)c_2e_n + \Big[(-6c_2^2+3c_3)G'(0) + 2G''(0)c_2^2\Big]e_n^2\\ &+ \left[(48G'(0)-36G''(0)+4G'''(0))\frac{c_2^3}{3} - (16G'(0)-6G''(0))c_2c_3 + 4G'(0)c_4\right]e_n^3 + \cdots . \quad (14)\end{aligned}$$

Because the least order in equation (12) is two, we expand $G(\eta_n)$ to the third order.

Finally, according to (9), (12) and (14), the error equation of the scheme (2) is

$$\begin{aligned}e_{n+1} &= y_n - \alpha - G(\eta_n)\frac{f(y_n)}{Af'(x_n)+Bf'(y_n)}\\ &= [A+B-G(0)]c_2\frac{e_n^2}{A+B} + \Big[(-2c_2^2+2c_3)A^2\\ &+ \Big((-4c_2^2+4c_3)B + (4G(0)-2G'(0))c_2^2 - 2G(0)c_3\Big)A\\ &- 2\Big((c_2^2-c_3)B + (-G(0)+G'(0))c_2^2 + G(0)c_3\Big)B\Big]\frac{e_n^3}{(A+B)^2}\end{aligned}$$

$$
\begin{aligned}
&+ \Big[\Big(4A^3 + (12B - 13G(0) + 14G'(0) - 2G''(0))A^2 + (12B - 12G(0) + 24G'(0) \\
&- 4G''(0))AB + (4B - 3G(0) + 10G'(0) - 2G''(0))B^2\Big)\, c_2^3 \\
&- 7(A+B)(A^2 + (2B - 2G(0) + G'(0))A + B(B - G(0) + G'(0)))c_2c_3 \\
&+ 3c_4(A+B)^2(A+B-G(0))\Big]\, \frac{e_n^4}{(A+B)^3} + \xi e_n^5 + \mathcal{O}(e_n^6), \qquad (15)
\end{aligned}
$$

where $\xi = \xi\,(A, B, c_2, ..., c_5, G(0), G'(0), G''(0), G'''(0))$ is the coefficient of e_n^5.

To obtain a fifth-order convergence, we need the coefficients of e_n^2, e_n^3 and e_n^4 in (15) to be zero. It is clear that the first condition in (4) makes the coefficient of e_n^2 zero. To simplify the calculations, we substitute $G(0) = A + B$ in (15), to get

$$
\begin{aligned}
e_{n+1} &= 2c_2^2[A - G'(0)]\frac{e_n^3}{A+B} - c_2\,\Big[\Big(9A^2 + (4B - 14G'(0) + 2G''(0))\,A \\
&- \,(B + 10G'(0) - 2G''(0))B\Big)\, c_2^2 - (7c_3(A - G'(0))(A+B))\Big]\, \frac{e_n^4}{(A+B)^2} \\
&+ \gamma e_n^5 + \mathcal{O}(e_n^6), \qquad (16)
\end{aligned}
$$

where $\gamma = \gamma(A, B, c_2, ..., c_5, G'(0), G''(0), G'''(0))$.

Obviously, the second condition in (4) eliminates the third-order error. Thus, applying $G'(0) = A$ in (16) leads to

$$
e_{n+1} = c_2^3(5A + B - 2G''(0))\frac{e_n^4}{A+B} + \mu e_n^5 + \mathcal{O}(e_n^6), \qquad (17)
$$

where $\mu = \mu(A, B, c_2, ..., c_5, G''(0), G'''(0))$.

By applying the third condition in (4), the fourth-order error vanishes in the error equation (17). That is, using $G''(0) = \dfrac{5A+B}{2}$ results in

$$
e_{n+1} = \left(\left(14A + 4B - \frac{4}{3}G'''(0)\right)\frac{c_2^4}{A+B} - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6).
$$

This ends the proof of the theorem.

The case $A+B = 0$ is excluded from Theorem 1. In this case, without loss of generality, the family of methods (2) can be written as

$$
x_{n+1} = y_n - G(\eta_n)\frac{f(y_n)}{f'(x_n) - f'(y_n)}. \qquad (18)
$$

For the scheme (18), we have the following convergence result.

Theorem 2. *Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, and the weight function $G(\eta)$ satisfies*

$$
G(0) = 0, \quad G'(0) = 1, \quad G''(0) = 2, \quad G'''(0) = \frac{15}{2}, \qquad (19)
$$

then the iterative scheme (18) *converges to α with order of convergence five and satisfies the error equation*

$$e_{n+1} = \left(\frac{1}{3}\left(42 - G^{(4)}(0)\right)c_2^4 - c_2^2 c_3\right)e_n^5 + \mathcal{O}(e_n^6). \tag{20}$$

Provided $|G^{(4)}(0)| < \infty$.

Proof. Following the proof of Theorem 1, according to (7), (10) and (11), we have

$$\frac{f(y_n)}{f'(x_n) - f'(y_n)} = \frac{e_n}{2} + \frac{-2c_2^2 + c_3}{4c_2}e_n^2 + \cdots. \tag{21}$$

Since the least order in (21) is one, we expand $G(\eta_n)$ to the fourth order, i.e.

$$G(\eta_n) = G(0) + G'(0)\eta_n + G''(0)\frac{\eta_n^2}{2} + G'''(0)\frac{\eta_n^3}{6} + G^{(4)}(0)\frac{\eta_n^4}{24} + \cdots. \tag{22}$$

Now, combining (9), (13), (21) and (22), we get the error equation for the scheme (18)

$$\begin{aligned} e_{n+1} &= y_n - \alpha - G(\eta_n)\frac{f(y_n)}{f'(x_n) - f'(y_n)} \\ &= -G(0)\frac{e_n}{2} + \left[(2G(0) - 4G'(0) + 4)c_2^2 - G(0)c_3\right]\frac{e_n^2}{4c_2} + \cdots. \end{aligned} \tag{23}$$

To make the coefficients of e_n and e_n^2 zero, we obtain the first two conditions in (19), i.e. $G(0) = 0$ and $G'(0) = 1$. Substituting these values in the error equation (23), we get

$$e_{n+1} = -c_2^2(G''(0) - 2)e_n^3 + 7c_2\left[\left(G''(0) - \frac{2G'''(0)}{21} - \frac{9}{7}\right)c_2^2 - \frac{c_3(G''(0) - 2)}{2}\right]e_n^4 + \cdots.$$

It is clear that setting $G''(0) = 2$ eliminates the third-order error. Subsequently, the fourth-order error is eliminated when $G'''(0) = \dfrac{15}{2}$. Therefore, we establish the third and fourth conditions in (19), leading to the following form of the error equation

$$e_{n+1} = \left(\frac{1}{3}\left(42 - G^{(4)}(0)\right)c_2^4 - c_2^2 c_3\right)e_n^5 + \mathcal{O}(e_n^6).$$

Thus, the proof is complete.

4. Particular cases of the proposed family

Many fifth-order iterative methods can be derived from the family (2) because of the arbitrary nature of A and B. Furthermore, several weight functions can satisfy (4) for a specific choice of A and B. Here, some well-known methods are listed as particular cases of the family (2):

(i) Let $A=-1$ and $B=5$, then the weight function $G(\eta)$ satisfies the condition (4) if:

$$G(0)=4, \quad G'(0)=-1, \quad G''(0)=0.$$

Taking $G(\eta)=4-\eta$, the scheme (2) becomes

$$x_{n+1}=y_n-\left[\frac{f'(y_n)+3f'(x_n)}{5f'(y_n)-f'(x_n)}\right]\frac{f(y_n)}{f'(x_n)}, \tag{24}$$

which is the method proposed by Ham and Chun (HCM) [9].

(ii) If $A=1$ and $B=0$, then (4) holds if $G(\eta)$ satisfies

$$G(0)=1, \quad G'(0)=1, \quad G''(0)=\frac{5}{2}. \tag{25}$$

Taking $G(\eta)=\dfrac{5+3(1-\eta)^2}{1+7(1-\eta)^2}$, the scheme (2) takes the form

$$x_{n+1}=y_n-\left[\frac{5f'^2(x_n)+3f'^2(y_n)}{f'^2(x_n)+7f'^2(y_n)}\right]\frac{f(y_n)}{f'(x_n)}, \tag{26}$$

which is the method of Fang et al. (FLM) [6].

Another choice for the weight function satisfying (25) is $G(\eta)=1+\eta+\frac{5}{4}\eta^2-\frac{1}{6}\eta^3$, leading to the following scheme

$$x_{n+1}=y_n-\left[2-\frac{f'(y_n)}{f'(x_n)}+\frac{5}{4}\left(1-\frac{f'(y_n)}{f'(x_n)}\right)^2-\frac{1}{6}\left(1-\frac{f'(y_n)}{f'(x_n)}\right)^3\right]\frac{f(y_n)}{f'(x_n)}, \tag{27}$$

this is a special iterative scheme of the second family of fifth-order iterative methods introduced by Liu et al. (LM) [14].

(iii) Suppose $A=0$ and $B=1$, by (4)

$$G(0)=1, \quad G'(0)=0, \quad G''(0)=\frac{1}{2}.$$

We can choose $G(\eta)=1+\frac{1}{4}\eta^2$, then the corresponding method is given as

$$x_{n+1}=y_n-\left[\frac{5}{4}-\frac{f'(y_n)}{2f'(x_n)}+\frac{f'^2(y_n)}{4f'^2(x_n)}\right]\frac{f(y_n)}{f'(y_n)}, \tag{28}$$

which is the method of Sivakumar and Jayaraman (PJM) [20].

(iv) Let $A = 1$ and $B = 1$, then $G(\eta)$ satisfies (4) if

$$G(0) = 2, \quad G'(0) = 1, \quad G''(0) = 3.$$

Assuming the weight function as $G(\eta) = \dfrac{4-4\eta}{2-3\eta}$, we get the method of Khirallah and Alkhomsan (AMK1) [10]

$$x_{n+1} = y_n + \left[\frac{4f'(y_n)}{f'(x_n) - 3f'(y_n)}\right] \frac{f(y_n)}{f'(x_n) + f'(y_n)}. \tag{29}$$

Next, we present four new particular methods of the family (2) by considering alternative choices for A and B as follows:

1. If $A = -2$ and $B = 7$, the weight function $G(\eta)$ satisfies the condition (4) if

$$G(0) = 5, \quad G'(0) = -2, \quad G''(0) = -\frac{3}{2}.$$

Taking $G(\eta) = \dfrac{40-31\eta}{8-3\eta}$, the scheme (2) takes the form

$$x_{n+1} = y_n - \left[\frac{9f'(x_n) + 31f'(y_n)}{5f'(x_n) + 3f'(y_n)}\right] \frac{f(y_n)}{7f'(y_n) - 2f'(x_n)}. \tag{30}$$

We denote this new scheme as AZ1.

2. Let $A = -1$ and $B = 4$, the condition (4) holds if the weight function $G(\eta)$ satisfies

$$G(0) = 3, \quad G'(0) = -1, \quad G''(0) = -\frac{1}{2}.$$

Assuming $G(\eta) = \dfrac{12-7\eta}{4-\eta}$, then the resulting scheme (AZ2) is given as

$$x_{n+1} = y_n - \left[\frac{5f'(x_n) + 7f'(y_n)}{3f'(x_n) + f'(y_n)}\right] \frac{f(y_n)}{4f'(y_n) - f'(x_n)}. \tag{31}$$

3. Let $A = 10$ and $B = 3$, then the weight function $G(\eta)$ satisfies the condition (4) if

$$G(0) = 13, \quad G'(0) = 10, \quad G''(0) = \frac{53}{2}.$$

Using $G(\eta) = \dfrac{1040-578\eta}{80-106\eta}$ leads to the following scheme (AZ3)

$$x_{n+1} = y_n - \left[\frac{231f'(x_n) + 289f'(y_n)}{-13f'(x_n) + 53f'(y_n)}\right] \frac{f(y_n)}{10f'(x_n) + 3f'(y_n)}. \tag{32}$$

4. If $A = 1$ and $B = -1$, i.e. $A + B = 0$, then the weight function $G(\eta)$ is selected to satisfy the condition (19) in Theorem 2 to obtain fifth-order schemes of type (18). Several forms for $G(\eta)$ can be designed to satisfy (19). For example, assuming

$$G(\eta) = \frac{-4\eta}{\eta^3 + \eta^2 + 4\eta - 4},$$

we obtain the following scheme (AZ4)

$$x_{n+1} = y_n + \frac{4f'^2(x_n)f(y_n)}{\left(f'(x_n) - f'(y_n)\right)^2 \left(2f'(x_n) - f'(y_n)\right) - 4f'^2(x_n)f'(y_n)}. \tag{33}$$

The error equations for the methods mentioned above are derived using the general error equation (5) when $A+B \neq 0$ and the error equation (20) when $A+B = 0$. As shown, error equation (5) involves the evaluation of $G'''(0)$, while error equation (20) requires the evaluation of $G^{(4)}(0)$. Table 1 presents the mentioned methods along with their respective error equations

Table 1: Error equations for particular methods within the proposed family of iterative methods (2).

Method	**Equation**	**Error equation**
HCM	(24)	$e_{n+1} = \left(\frac{3}{2}c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
FLM	(26)	$e_{n+1} = \left(\frac{7}{2}c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
LM	(27)	$e_{n+1} = \left(\frac{46}{3}c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
PJM	(28)	$e_{n+1} = \left(4c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
AMK1	(29)	$e_{n+1} = -c_2^2c_3e_n^5 + \mathcal{O}(e_n^6)$
AZ1	(30)	$e_{n+1} = \left(\frac{9}{20}c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
AZ2	(31)	$e_{n+1} = \left(\frac{5}{6}c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
AZ3	(32)	$e_{n+1} = \left(\frac{231}{260}c_2^4 - c_2^2c_3\right)e_n^5 + \mathcal{O}(e_n^6)$
AZ4	(33)	$e_{n+1} = -c_2^2c_3e_n^5 + \mathcal{O}(e_n^6)$

5. Basins of attraction

Any iterative method can converge if the initial estimate is chosen appropriately, regardless of the order of convergence. Therefore, applying an iterative method to some test functions for one or more initial estimates does not fully illustrate the stability of the method. To address this, many authors have considered complex dynamics and basins of attraction for testing the stability and reliability of iterative methods, see [2, 3, 8, 10]. Indeed, basins of attraction shows how an iterative method converges based on different initial estimates in a specified region. This technique generates a picture in complex plane that provides a visual comparison for different methods.

To plot the basins of attraction, we specify a rectangular region, denoted as D, in the complex plane that contains all the roots of the considered complex polynomial $g(z)$. Different colors are assigned to different roots at a point $z_0 \in D$ based on the corresponding roots to which the method converges when starting from z_0. The brightness of the colors depends on the number of iterations required for convergence; a brighter color indicates fewer iterations, while a darker color indicates that the method requires more iterations. Points in the region D are assigned the black color if the method does not converge according to the convergence criteria. In this section, we assume a tolerance of 10^{-3} with a maximum of 20 iterations.

Table 2: Complex polynomials and their roots.

Function	**Roots**
$g_1(z) = z^2 - 1$	± 1
$g_2(z) = z^3 - 1$	$1, \ -0.5 \pm 0.866025i$
$g_3(z) = z^3 + z + i$	$-0.56228 - 0.662359i,\ 1.32472i,\ 0.56228 - 0.662359i$
$g_4(z) = z^4 - 1$	$\pm 1, \ \pm i$
$g_5(z) = z^4 - z + i$	$-0.759845 + 0.592595i, \ -0.532605 - 1.08829i,$ $0.181924 + 0.732098i,\ 1.11052 - 0.236405i$
$g_6(z) = z^5 - 1$	$1, \ -0.809017 \pm 0.587785i, \ 0.309017 \pm 0.951057i$
$g_7(z) = z^6 - 1$	$\pm 1, \ -0.5 \pm 0.866025i, \ 0.5 \pm 0.866025i$
$g_8(z) = z^{11} - 1$	$1, \ -0.959493 \pm 0.281733i, -0.654861 \pm 0.75575i,$ $-0.142315 \pm 0.989821i,\ 0.415415 \pm 0.909632i,$ $0.841254 \pm 0.540641i$
$g_9(z) = z^6 + 10z^3 - 8$	$-2.20663,\ 0.906359, -0.45318 \pm 0.78493i, 1.10332 \pm 1.911i$
$g_{10}(z) = z(z^2+1)(z^2-4)$	$0, \ \pm i, \pm 2$

In Table 2, ten complex polynomials are listed with their roots as test cases. We use the region $D = [-2, 2] \times [-2, 2]$ for polynomials $g_1, g_2, ..., g_8$, and $D = [-3, 3] \times [-3, 3]$ for g_9 and g_{10}. A grid of 320×320 points is used in all experiments.

We compare the results of our new particular methods from the proposed family, namely AZ1 (30), AZ2 (31), AZ3 (32), and AZ4 (33), with those of well-known methods: HCM (24), FLM (26), LM (27), PJM (28), and AMK1 (29). To provide a summary, these methods are listed in Table 3.

Table 3: Particular methods within the family (2). The first step is $y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}$ in all methods.

Method	**Second step of the method**
HCM	$x_{n+1} = y_n - \left[\dfrac{f'(y_n) + 3f'(x_n)}{5f'(y_n) - f'(x_n)}\right] \dfrac{f(y_n)}{f'(x_n)}$
FLM	$x_{n+1} = y_n - \left[\dfrac{5f'^2(x_n) + 3f'^2(y_n)}{f'^2(x_n) + 7f'^2(y_n)}\right] \dfrac{f(y_n)}{f'(x_n)}$
LM	$x_{n+1} = y_n - \left[2 - \dfrac{f'(y_n)}{f'(x_n)} + \dfrac{5}{4}\left(1 - \dfrac{f'(y_n)}{f'(x_n)}\right)^2 - \dfrac{1}{6}\left(1 - \dfrac{f'(y_n)}{f'(x_n)}\right)^3\right] \dfrac{f(y_n)}{f'(x_n)}$
PJM	$x_{n+1} = y_n - \left[\dfrac{5}{4} - \dfrac{f'(y_n)}{2f'(x_n)} + \dfrac{f'^2(y_n)}{4f'^2(x_n)}\right] \dfrac{f(y_n)}{f'(y_n)}$
AMK1	$x_{n+1} = y_n + \left[\dfrac{4f'(y_n)}{f'(x_n) - 3f'(y_n)}\right] \dfrac{f(y_n)}{f'(x_n) + f'(y_n)}$
AZ1	$x_{n+1} = y_n - \left[\dfrac{9f'(x_n) + 31f'(y_n)}{5f'(x_n) + 3f'(y_n)}\right] \dfrac{f(y_n)}{7f'(y_n) - 2f'(x_n)}$
AZ2	$x_{n+1} = y_n - \left[\dfrac{5f'(x_n) + 7f'(y_n)}{3f'(x_n) + f'(y_n)}\right] \dfrac{f(y_n)}{4f'(y_n) - f'(x_n)}$
AZ3	$x_{n+1} = y_n - \left[\dfrac{231f'(x_n) + 289f'(y_n)}{-13f'(x_n) + 53f'(y_n)}\right] \dfrac{f(y_n)}{10f'(x_n) + 3f'(y_n)}$
AZ4	$x_{n+1} = y_n + \dfrac{4f'^2(x_n)f(y_n)}{(f'(x_n) - f'(y_n))^2\,(2f'(x_n) - f'(y_n)) - 4f'^2(x_n)f'(y_n)}$

In Table 4, the number of black points is given for all test cases with the average value across all cases for each method. In Figures 1, 2, 3, and 4, we illustrate the basins of attraction for the functions g_2, g_4, g_5, and g_6, respectively.

It is clear that our four new methods and the AMK1 method exhibit good dynamical behavior, i.e they are more stable than other methods. Based on the number of black points for the given test functions, we can rank the best methods as follows: AZ1, followed by AMK1, then AZ2 and AZ3 being nearly equal, and then AZ4.

Table 4: Number of black points for iterative methods for each test case and average across all cases.

MD	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	**Average**
HCM	320	415	108	3688	54	6997	11002	18087	1714	104	4249
FLM	488	1035	450	9296	282	13627	18556	31282	5572	272	8086
LM	612	10929	4969	27936	10469	29856	32916	41961	22971	5742	18836
PJM	326	2928	1066	14844	1343	16903	20024	32524	8446	928	9933
AMK1	320	0	49	640	0	141	726	6541	8	0	842
AZ1	320	0	0	640	0	77	740	6017	0	0	779
AZ2	320	0	6	656	0	200	982	6543	10	0	872
AZ3	320	0	8	656	0	163	930	6659	0	0	874
AZ4	320	0	29	648	0	197	926	7161	11	0	929

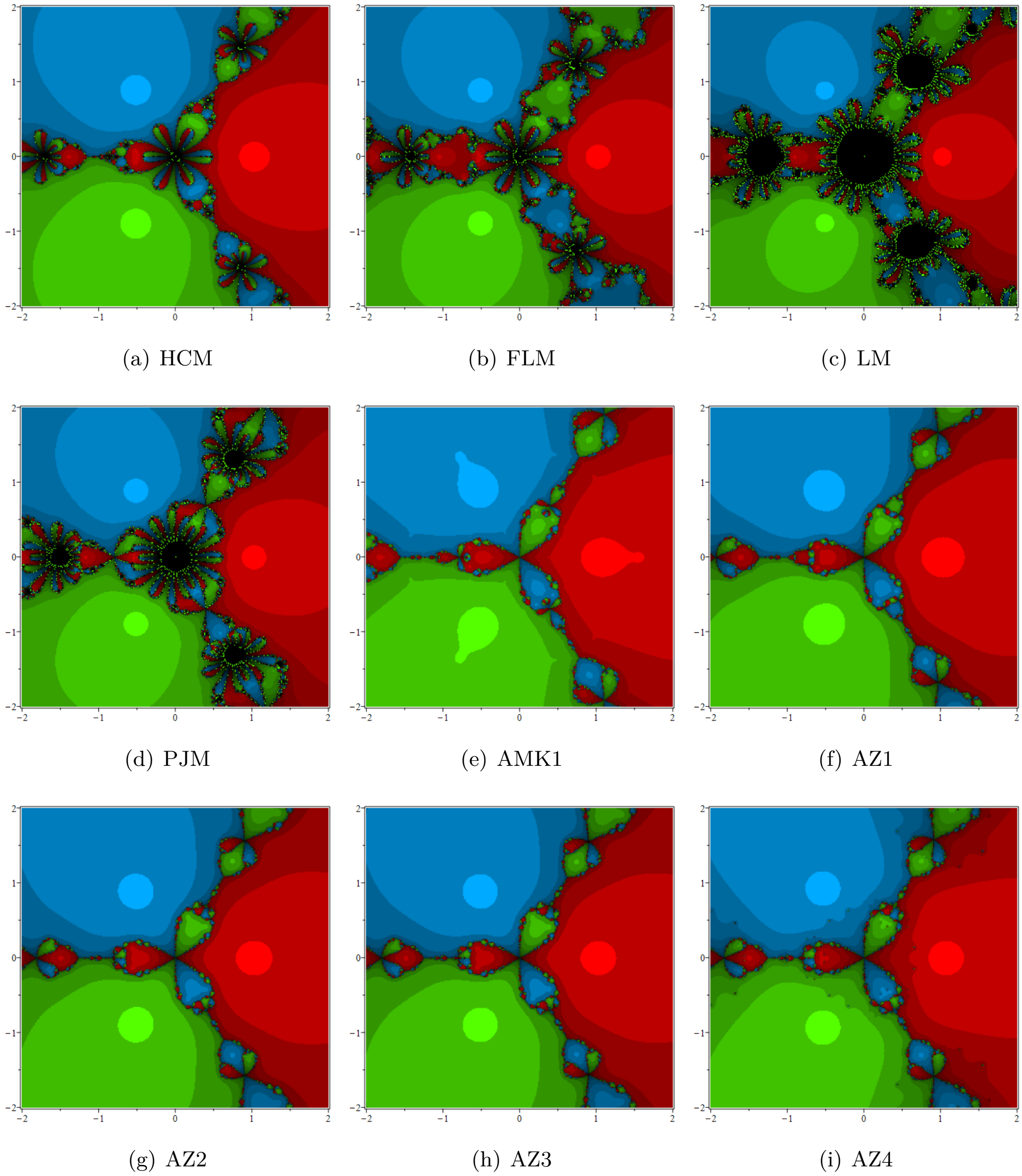
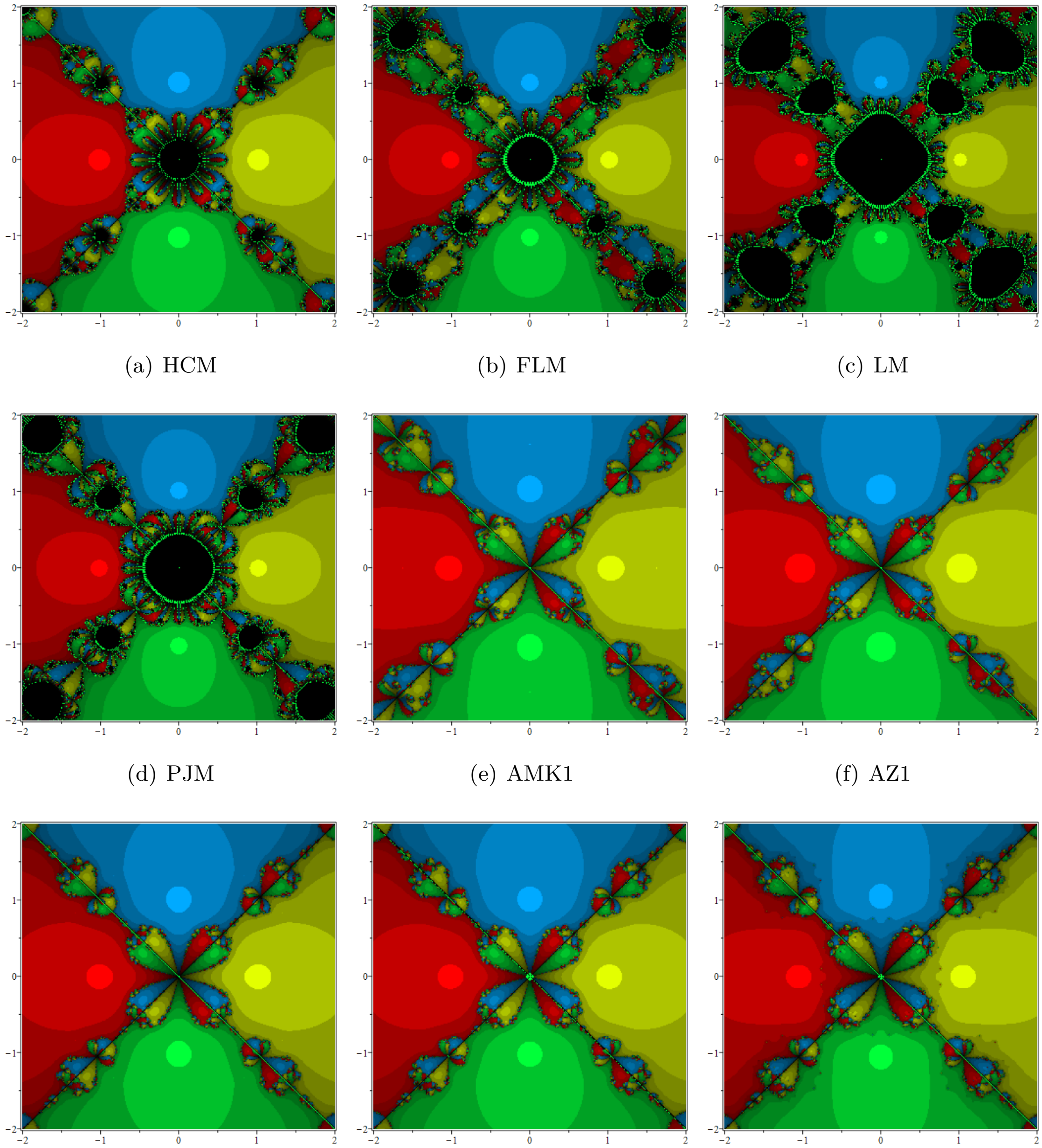


Figure 1: Basins of attraction associated with $g_2(z) = z^3 - 1$

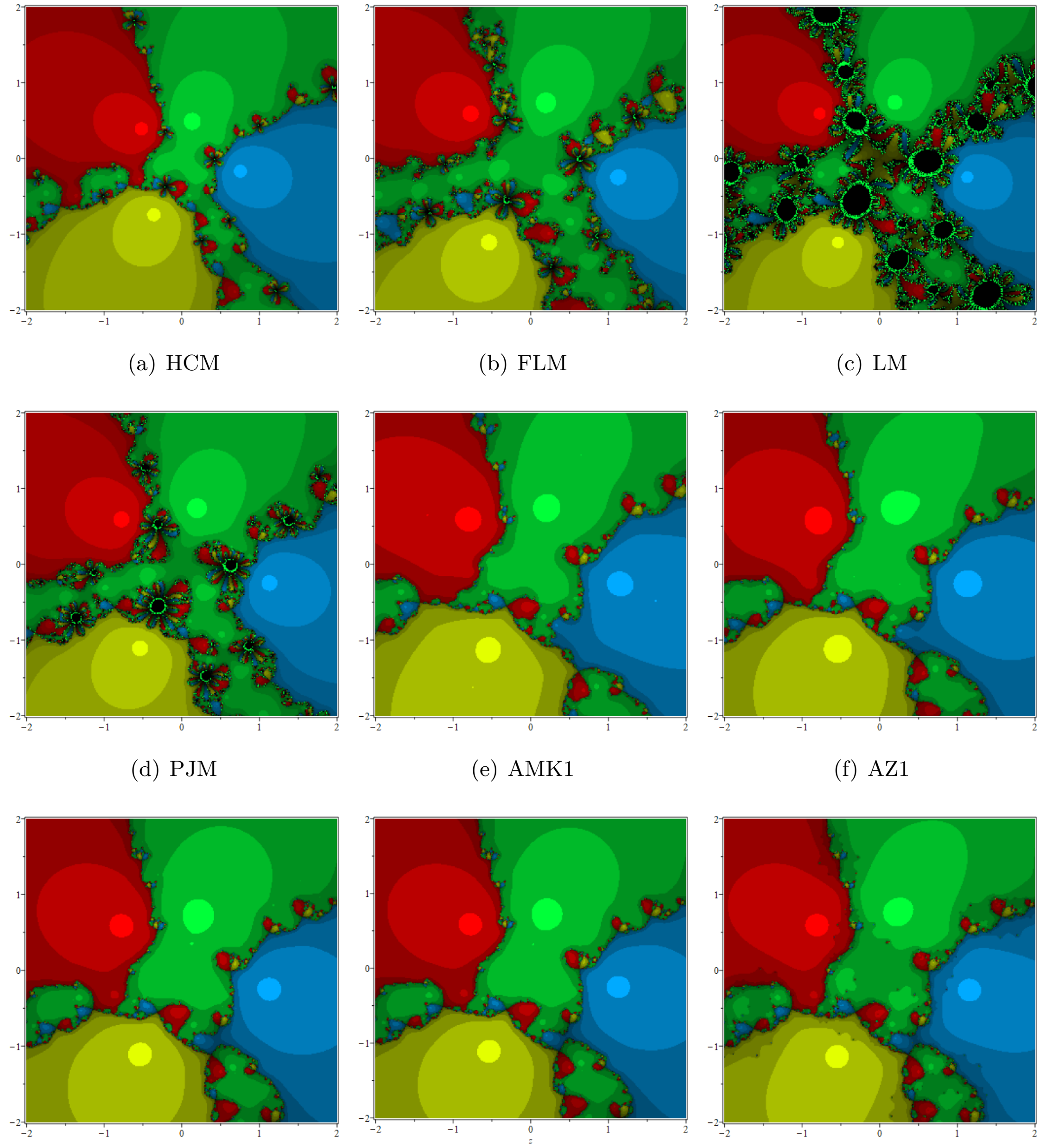


(a) HCM (b) FLM (c) LM

(d) PJM (e) AMK1 (f) AZ1

(g) AZ2 (h) AZ3 (i) AZ4

Figure 2: Basins of attraction associated with $g_4(z) = z^4 - 1$

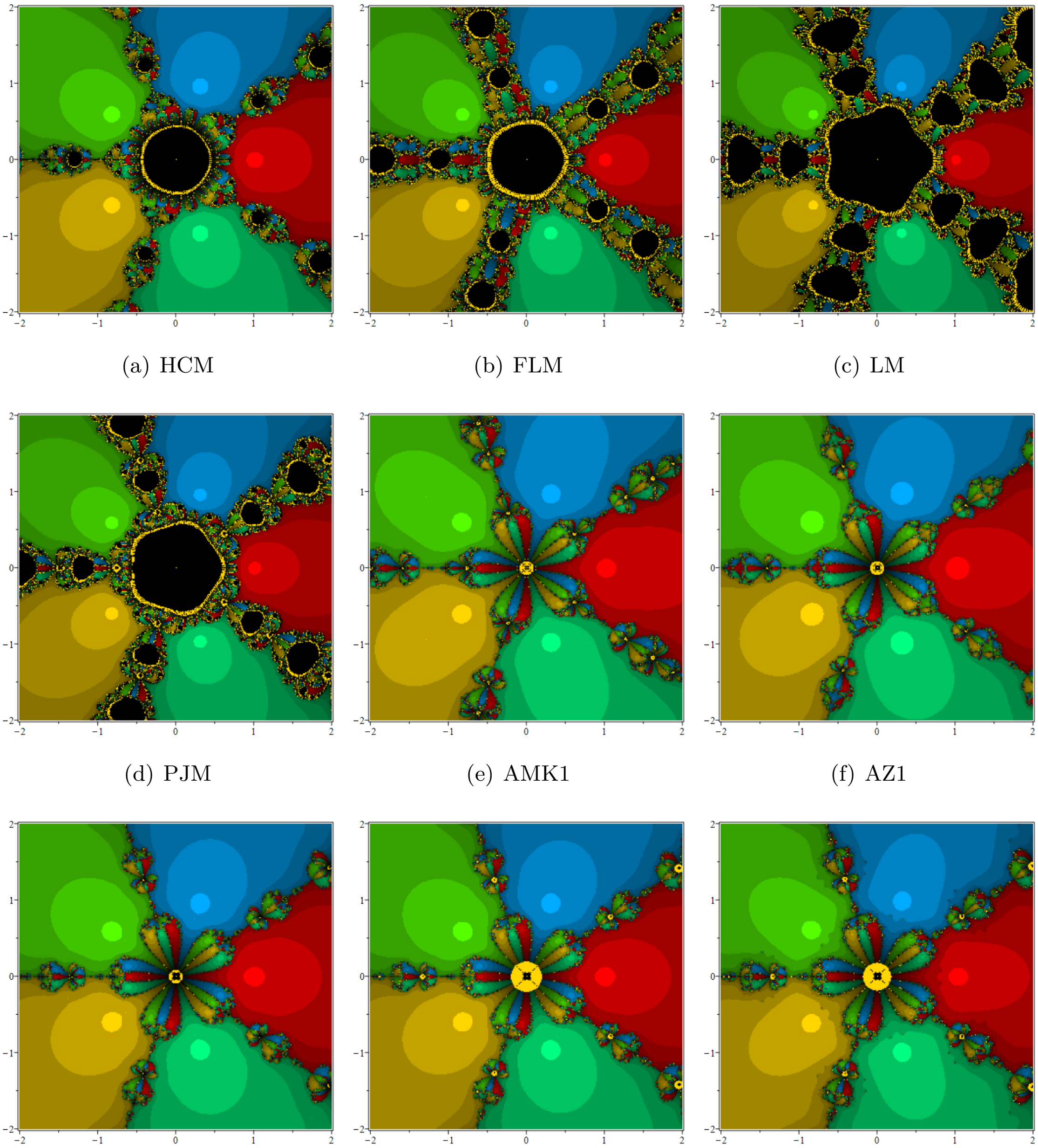


(a) HCM (b) FLM (c) LM

(d) PJM (e) AMK1 (f) AZ1

(g) AZ2 (h) AZ3 (i) AZ4

Figure 3: Basins of attraction associated with $g_5(z) = z^4 - z + i$



(a) HCM (b) FLM (c) LM

(d) PJM (e) AMK1 (f) AZ1

(g) AZ2 (h) AZ3 (i) AZ4

Figure 4: Basins of attraction associated with $g_6(z) = z^5 - 1$

6. Numerical results

In this section, we consider numerical examples to confirm the order of convergence and demonstrate the efficiency of the newly constructed methods: AZ1 (30), AZ2 (31), AZ3 (32), and AZ4 (33). We compare these new methods with the well-known methods: HCM (24), FLM (26), LM (27), PJM (28), and AMK1 (29). All these methods, as shown, belong to the proposed family (2).

All computations are performed using Maple 2021 with a precision of 2000 significant digits. The applied stopping criterion is defined as $|x_n - x_{n-1}| < 10^{-50}$. The computational order of convergence ρ is approximated by using the formula [4]

$$\rho = \frac{\ln |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}.$$

Table 5 lists eight of the most frequently used test functions in research. Their simple roots are provided up to 25 decimal places. Tables 6, 7, 8, and 9 display the following metrics: the number of iterations (**N**) at which the stopping criterion is satisfied, the computational order of convergence (ρ), the error $|x_N - x_{N-1}|$, $|f(x_N)|$, and the processing time in seconds. The processing time is determined as the mean of 10,000 executions to obtain reasonably accurate values.

Table 5: The test functions and their simple roots.

Function	**Root**
$f_1(x) = x^3 + 4x^2 - 10$	1.3652300134140968457608068
$f_2(x) = \sin^2 x - x^2 + 1$	1.4044916482153412260350868
$f_3(x) = x^2 - e^x - 3x + 2$	0.2575302854398607604553673
$f_4(x) = \ln(x^2 + x + 2) - x + 1$	4.1525907367571582749969894
$f_5(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$	$-1.2076478271309189270094167$
$f_6(x) = e^{\sin x} - x + 1$	2.6306641479279036339753265
$f_7(x) = x^5 + x - 10000$	6.3087771299726890947675717
$f_8(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3$	2.3319676558839640103080436

Clearly, the computational order of convergence for all considered methods confirms the theoretical analysis. The numerical results categorize the methods into two groups:

- The methods AZ1, AZ2, AZ3, AZ4 and AMK1.
- The methods HCM, FLM, LM, PJM.

In general, for the given test functions, the first category yields better results than the second category, with fewer iterations in some test cases and higher accuracy in most cases. Indeed, the results for the methods in the first category are comparable. It appears that for some cases, AMK1 is the best method, while AZ1 is the best choice for other cases.

Based on the processing time, it seems that the methods HCM, AMK1, AZ1, AZ2, and AZ3 require less execution time compared to the other methods.

Table 6: Numerical results for test functions $f_1(x)$ and $f_2(x)$.

Function	**Method**	x_0	**N**	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	**Time**
f_1	HCM	0.8	4	9.15×10^{-51}	7.63×10^{-251}	5.00	0.0019
	FLM		5	1.80×10^{-183}	5.92×10^{-914}	5.00	0.0031
	LM		28	1.89×10^{-101}	3.46×10^{-503}	4.99	0.0203
	PJM		5	2.49×10^{-137}	3.44×10^{-683}	5.00	0.0030
	AMK1		4	1.34×10^{-68}	1.02×10^{-340}	4.99	0.0021
	AZ1		4	3.34×10^{-86}	7.82×10^{-429}	5.00	0.0025
	AZ2		4	7.14×10^{-70}	1.03×10^{-346}	5.00	0.0022
	AZ3		4	1.93×10^{-67}	1.62×10^{-334}	5.00	0.0021
	AZ4		5	1.02×10^{-222}	2.63×10^{-1111}	5.00	0.0032
f_1	HCM	2.0	4	6.00×10^{-74}	9.22×10^{-367}	4.99	0.0019
	FLM		4	3.12×10^{-63}	9.16×10^{-313}	4.99	0.0024
	LM		5	1.41×10^{-248}	7.94×10^{-1239}	5.00	0.0040
	PJM		4	4.60×10^{-63}	7.39×10^{-312}	4.99	0.0024
	AMK1		4	1.97×10^{-111}	7.12×10^{-555}	5.00	0.0021
	AZ1		4	2.72×10^{-90}	2.80×10^{-449}	5.00	0.0025
	AZ2		4	3.76×10^{-81}	4.18×10^{-403}	4.99	0.0022
	AZ3		4	2.63×10^{-80}	7.62×10^{-399}	4.99	0.0022
	AZ4		4	1.31×10^{-88}	9.11×10^{-441}	4.99	0.0029
f_2	HCM	1.0	5	2.64×10^{-195}	1.63×10^{-973}	5.00	0.0036
	FLM		5	1.65×10^{-128}	3.83×10^{-639}	5.00	0.0045
	LM		6	1.17×10^{-135}	3.10×10^{-674}	5.00	0.0063
	PJM		5	1.26×10^{-66}	1.16×10^{-329}	5.00	0.0051
	AMK1		4	7.61×10^{-63}	3.42×10^{-312}	4.99	0.0032
	AZ1		4	1.11×10^{-83}	4.82×10^{-416}	4.99	0.0033
	AZ2		4	1.03×10^{-58}	7.55×10^{-291}	5.00	0.0030
	AZ3		4	1.71×10^{-56}	1.01×10^{-279}	5.00	0.0032
	AZ4		5	4.28×10^{-168}	1.92×10^{-838}	5.00	0.0054
f_2	HCM	3.0	5	9.57×10^{-195}	1.02×10^{-970}	5.00	0.0039
	FLM		5	3.00×10^{-167}	7.69×10^{-833}	5.00	0.0039
	LM		5	2.01×10^{-128}	4.63×10^{-638}	4.99	0.0050
	PJM		5	7.36×10^{-167}	7.77×10^{-831}	5.00	0.0042
	AMK1		5	7.32×10^{-250}	2.81×10^{-1247}	5.00	0.0044
	AZ1		5	1.41×10^{-226}	1.61×10^{-1130}	5.00	0.0046
	AZ2		5	2.87×10^{-210}	1.25×10^{-1048}	5.00	0.0039
	AZ3		5	1.41×10^{-208}	3.95×10^{-1040}	5.00	0.0046
	AZ4		5	1.24×10^{-249}	3.88×10^{-1246}	5.00	0.0044

Table 7: Numerical results for test functions $f_3(x)$ and $f_4(x)$.

Function	**Method**	x_0	**N**	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	**Time**
f_3	HCM	-1.0	4	5.87×10^{-104}	1.01×10^{-519}	4.99	0.0021
	FLM		4	2.26×10^{-95}	5.12×10^{-477}	5.00	0.0024
	LM		4	1.35×10^{-74}	1.12×10^{-372}	5.00	0.0026
	PJM		4	1.40×10^{-95}	3.91×10^{-478}	4.99	0.0027
	AMK1		4	2.91×10^{-88}	3.93×10^{-441}	4.99	0.0022
	AZ1		4	8.63×10^{-92}	8.38×10^{-459}	4.99	0.0020
	AZ2		4	2.55×10^{-95}	1.77×10^{-476}	4.99	0.0022
	AZ3		4	6.94×10^{-96}	2.62×10^{-479}	4.99	0.0024
	AZ4		4	2.41×10^{-91}	1.52×10^{-456}	4.99	0.0024
f_3	HCM	1.5	4	8.53×10^{-80}	6.53×10^{-399}	5.00	0.0021
	FLM		4	3.57×10^{-82}	5.09×10^{-411}	5.00	0.0023
	LM		4	1.11×10^{-84}	4.32×10^{-423}	5.00	0.0028
	PJM		4	9.64×10^{-83}	6.08×10^{-414}	4.99	0.0023
	AMK1		4	2.46×10^{-78}	1.70×10^{-391}	5.00	0.0022
	AZ1		4	8.94×10^{-79}	1.00×10^{-393}	5.00	0.0023
	AZ2		4	3.80×10^{-79}	1.29×10^{-395}	5.00	0.0022
	AZ3		4	3.36×10^{-79}	6.93×10^{-396}	5.00	0.0024
	AZ4		4	1.65×10^{-78}	2.28×10^{-392}	5.00	0.0028
f_4	HCM	2.0	4	1.80×10^{-67}	5.05×10^{-339}	5.00	0.0023
	FLM		4	6.50×10^{-59}	4.92×10^{-296}	5.00	0.0027
	LM		5	4.57×10^{-145}	2.73×10^{-726}	5.00	0.0036
	PJM		4	2.31×10^{-54}	3.05×10^{-273}	5.00	0.0031
	AMK1		4	1.15×10^{-81}	2.90×10^{-410}	5.00	0.0024
	AZ1		4	9.02×10^{-77}	1.08×10^{-385}	5.00	0.0022
	AZ2		4	3.68×10^{-73}	1.43×10^{-367}	5.00	0.0023
	AZ3		4	1.14×10^{-72}	4.21×10^{-365}	5.00	0.0026
	AZ4		4	5.79×10^{-78}	9.44×10^{-392}	4.99	0.0026
f_4	HCM	6.0	4	2.42×10^{-118}	2.20×10^{-593}	4.99	0.0024
	FLM		4	3.09×10^{-113}	1.19×10^{-567}	4.99	0.0027
	LM		4	3.81×10^{-101}	1.10×10^{-506}	4.99	0.0029
	PJM		4	1.37×10^{-112}	2.20×10^{-564}	4.99	0.0031
	AMK1		4	2.25×10^{-124}	8.39×10^{-624}	4.99	0.0025
	AZ1		4	3.82×10^{-122}	1.48×10^{-612}	4.99	0.0021
	AZ2		4	1.42×10^{-120}	1.22×10^{-604}	4.99	0.0024
	AZ3		4	2.28×10^{-120}	1.34×10^{-603}	4.99	0.0027
	AZ4		4	1.81×10^{-123}	2.84×10^{-619}	4.99	0.0026

Table 8: Numerical results for test functions $f_5(x)$ and $f_6(x)$.

Function	**Method**	x_0	**N**	$\|x_N - x_{N-1}\|$	$\|f(x_N)\|$	ρ	**Time**
f_5	HCM	-2.5	6	3.11×10^{-61}	1.84×10^{-301}	4.99	0.0082
	FLM		7	2.88×10^{-102}	5.38×10^{-506}	4.99	0.0105
	LM		8	3.12×10^{-207}	4.42×10^{-1030}	5.00	0.0139
	PJM		7	1.22×10^{-120}	8.61×10^{-598}	4.99	0.0107
	AMK1		5	1.63×10^{-140}	1.07×10^{-697}	5.00	0.0067
	AZ1		6	2.25×10^{-200}	2.58×10^{-997}	5.00	0.0094
	AZ2		6	4.97×10^{-127}	1.70×10^{-631}	4.99	0.0093
	AZ3		6	7.25×10^{-123}	2.25×10^{-612}	5.00	0.0098
	AZ4		6	8.47×10^{-103}	3.99×10^{-509}	5.00	0.0099
f_5	HCM	-1.0	4	4.08×10^{-67}	7.16×10^{-331}	5.00	0.0051
	FLM		5	2.75×10^{-237}	4.27×10^{-1181}	5.00	0.0069
	LM		6	4.81×10^{-111}	3.86×10^{-549}	5.00	0.0091
	PJM		5	9.30×10^{-195}	2.24×10^{-968}	5.00	0.0069
	AMK1		4	2.71×10^{-64}	1.34×10^{-316}	4.99	0.0048
	AZ1		4	5.71×10^{-71}	2.74×10^{-350}	4.99	0.0054
	AZ2		4	3.33×10^{-87}	2.28×10^{-432}	4.99	0.0056
	AZ3		4	1.26×10^{-100}	3.61×10^{-501}	5.00	0.0058
	AZ4		5	3.78×10^{-241}	7.04×10^{-1201}	5.00	0.0086
f_6	HCM	1.5	5	3.22×10^{-54}	1.27×10^{-270}	4.98	0.0025
	FLM		6	3.34×10^{-249}	1.65×10^{-1245}	5.00	0.0042
	LM		9	1.44×10^{-92}	3.66×10^{-462}	4.99	0.0085
	PJM		6	1.04×10^{-235}	5.06×10^{-1178}	5.00	0.0045
	AMK1		5	1.40×10^{-57}	1.81×10^{-287}	4.98	0.0025
	AZ1		5	9.92×10^{-57}	3.33×10^{-283}	4.98	0.0030
	AZ2		5	6.55×10^{-56}	4.25×10^{-279}	4.98	0.0028
	AZ3		5	8.76×10^{-56}	1.82×10^{-278}	4.98	0.0030
	AZ4		5	9.62×10^{-60}	2.80×10^{-298}	4.99	0.0035
f_6	HCM	3.5	4	5.22×10^{-71}	1.41×10^{-354}	5.00	0.0021
	FLM		4	2.32×10^{-67}	2.70×10^{-336}	5.00	0.0023
	LM		5	2.05×10^{-243}	2.15×10^{-1216}	5.00	0.0039
	PJM		4	2.34×10^{-65}	2.83×10^{-326}	5.00	0.0030
	AMK1		4	1.63×10^{-74}	3.87×10^{-372}	5.00	0.0022
	AZ1		4	1.49×10^{-73}	2.54×10^{-367}	5.00	0.0022
	AZ2		4	1.12×10^{-72}	6.12×10^{-363}	5.00	0.0020
	AZ3		4	1.51×10^{-72}	2.76×10^{-362}	5.00	0.0025
	AZ4		4	1.24×10^{-77}	9.83×10^{-388}	5.00	0.0028

Table 9: Numerical results for test functions $f_7(x)$ and $f_8(x)$.

Function	**Method**	x_0	**N**	$\|x_N - x_{N-1}\|$	$\|f(x_N)\|$	ρ	**Time**
f_7	HCM	4.0	Div	-	-	-	-
	FLM		10	6.66×10^{-63}	3.15×10^{-309}	4.99	0.0065
	LM		Div	-	-	-	-
	PJM		34	4.60×10^{-84}	5.75×10^{-415}	4.99	0.0256
	AMK1		5	7.16×10^{-61}	7.54×10^{-300}	5.00	0.0033
	AZ1		5	8.08×10^{-72}	1.38×10^{-355}	4.99	0.0033
	AZ2		5	1.03×10^{-73}	3.14×10^{-364}	4.99	0.0030
	AZ3		5	2.35×10^{-77}	2.22×10^{-382}	4.99	0.0037
	AZ4		6	3.57×10^{-147}	2.33×10^{-731}	5.00	0.0046
f_7	HCM	8.0	5	1.26×10^{-248}	2.57×10^{-1238}	5.00	0.0033
	FLM		5	6.57×10^{-194}	2.94×10^{-964}	5.00	0.0034
	LM		5	2.73×10^{-142}	1.80×10^{-705}	5.00	0.0037
	PJM		5	2.18×10^{-195}	1.39×10^{-971}	5.00	0.0038
	AMK1		4	2.79×10^{-65}	6.80×10^{-322}	5.00	0.0024
	AZ1		4	3.41×10^{-72}	1.85×10^{-357}	4.99	0.0024
	AZ2		4	2.40×10^{-58}	2.13×10^{-287}	4.99	0.0027
	AZ3		4	3.07×10^{-57}	8.51×10^{-282}	4.99	0.0028
	AZ4		4	4.47×10^{-61}	7.16×10^{-301}	4.99	0.0028
f_8	HCM	1.0	4	2.20×10^{-151}	1.03×10^{-756}	5.00	0.0092
	FLM		4	4.58×10^{-152}	3.07×10^{-760}	5.00	0.0085
	LM		4	3.00×10^{-152}	3.17×10^{-761}	5.00	0.0095
	PJM		4	2.86×10^{-152}	2.67×10^{-761}	5.00	0.0090
	AMK1		4	5.61×10^{-151}	1.31×10^{-754}	5.00	0.0084
	AZ1		4	4.31×10^{-151}	3.34×10^{-755}	5.00	0.0099
	AZ2		4	3.40×10^{-151}	9.86×10^{-756}	5.00	0.0092
	AZ3		4	3.29×10^{-151}	8.24×10^{-756}	5.00	0.0092
	AZ4		4	5.60×10^{-151}	1.30×10^{-754}	5.00	0.0092
f_8	HCM	4.0	4	1.30×10^{-84}	7.37×10^{-423}	4.99	0.0083
	FLM		4	1.28×10^{-52}	5.28×10^{-263}	5.00	0.0079
	LM		5	3.12×10^{-209}	3.87×10^{-1046}	5.00	0.0114
	PJM		4	1.90×10^{-54}	3.43×10^{-272}	5.00	0.0082
	AMK1		5	3.60×10^{-245}	1.42×10^{-1225}	5.00	0.0111
	AZ1		4	6.97×10^{-55}	3.69×10^{-274}	4.99	0.0093
	AZ2		4	6.29×10^{-61}	2.12×10^{-304}	4.99	0.0092
	AZ3		4	5.93×10^{-62}	1.58×10^{-309}	4.99	0.0095
	AZ4		4	1.43×10^{-57}	1.42×10^{-287}	4.99	0.0095

7. Conclusion

We have used the weight function procedure to develop a two-step family of fifth-order iterative methods for solving nonlinear equations. This family has a wide range of choices due to its formation and the ability to select various weight functions. This flexibility allows us to design specific methods with improved stability and accuracy. Furthermore, we show that the proposed family includes several well-known and recent fifth-order iterative methods as special cases. We provide a detailed convergence analysis and assess the methods' stability using basins of attraction. The basins of attraction in the complex plane, along with numerical results in the real domain, reveal that the newly constructed methods AZ1, AZ2, AZ3, and AZ4, as well as the recent method AMK1 [10], exhibit superior stability and accuracy when compared to other known fifth-order methods within the same family.

Acknowledgements

The author would like to thank the anonymous reviewers for their comments and suggestions.

References

[1] H. A. Abro and M. M. Shaikh. A new time-efficient and convergent nonlinear solver, applied mathematics and computation. *Applied Mathematics and Computation*, 335:516–536, 2019.

[2] F. I. Chicharro, A. Cordero, N. Garrido, and J. R. Torregrosa. Wide stability in a new family of optimal fourth-order iterative methods. *Computational Mathematics and Mathematical Physics*, 1:e1023, 2019.

[3] A. Cordero, M. M.-Martinez, and J. R. Torregrosa. Chaos and stability in a new iterative family for solving nonlinear equations. *Algorithms*, 14:1–24, 2021.

[4] A. Cordero and J. R. Torregrosa. Variants of newton's method using fifth-order quadrature formulas. *Applied Mathematics and Computation*, 190:686–698, 2007.

[5] J. F. Epperson. *An Introduction to Numerical Methods and Analysis.* John Wiley & Sons, New Jersey, 2013.

[6] L. Fang, L. Sun, and G. P. He. An efficient newton-type method with fifth-order convergence for solving nonlinear equations. *Computational and Applied Mathematics*, 27:269–274, 2008.

[7] M. Grau and J. L. Diaz-Barrero. An improvement of the euler-chebyshev iterative method. *Journal of Mathematical Analysis and Applications*, 315:1–7, 2006.

[8] M. A. Hafiz and M.Q. Khirallah. An optimal fourth order method for solving nonlinear equations. *Journal of mathematics and Computer Science*, 23:86–97, 2021.

[9] Y. M. Ham and C. B. Chun. A fifth-order iterative method for solving nonlinear equations. *Applied Mathematics and Computation*, 194:287–290, 2007.

[10] M. Q. Khirallah and A. M. Alkhomsan. A new fifth-order iterative method for solving non-linear equations using weight function technique and the basins of attraction. *Journal of Mathematics and Computer Science*, 28:281–293, 2023.

[11] J. Kou and Y. Li. The improvements of chebyshev-halley methods with fifth-order convergence. *Applied Mathematics and Computation*, 188:143–147, 2007.

[12] J. Kou, Y. Li, and X. Wang. A family of fifth-order iterations composed of newton and third-order methods. *Applied Mathematics and Computation*, 186:1258–1262, 2007.

[13] H. T. Kung and J. F. Traub. Optimal order of one-point and multipoint iteration. *Journal of Association of Computer Mathematics*, 21:634–651, 1974.

[14] C. S. Liu, E. R. El-Zahar, and C.W. Chang. Three novel fifth-order iterative schemes for solving nonlinear equations. *Mathematics and Computers in Simulation*, 187:282–293, 2021.

[15] K.I. Noor and M.A. Noor. Predictor-corrector halley method for nonlinear equations. *Applied Mathematics and Computation*, 188:1587–1591, 2007.

[16] K.I. Noor, M.A. Noor, and S. Momani. Modified householder iterative method for nonlinear equations. *Applied Mathematics and Computation*, 190:1534–1539, 2007.

[17] M. A. Noor, K. I. Noor, E. Al-Said, and M. Waseem. Some new iterative methods for nonlinear equations. *Mathematical Problems in Engineering*, 2010:1–12, 2010.

[18] M.A. Noor and K.I. Noor. Fifth-order iterative methods for solving nonlinear equations. *Applied Mathematics and Computation*, 188:406–410, 2007.

[19] M. S. Petković, B. Neta, L. D. Petković, and J. Džunić. *Multipoint Methods for Solving Nonlinear Equations.* Elsevier, Amsterdam-Boston-Heidelberg-London-New York, 2013.

[20] P. Sivakumar and J. Jayaraman. Some new higher order weighted newton methods for solving nonlinear equation with applications. *Mathematical and Computational Applications*, 24:21 pages, 2019.

[21] O. S. Solaiman and I. Hashim. Two new efficient sixth order iterative methods for solving nonlinear equations. *Journal of King Saud University-Science*, 31:701–705, 2019.

[22] O. S. Solaiman and I. Hashim. An iterative scheme of arbitrary odd order and its basins of attraction for nonlinear systems. *Computers, Materials & Continua*, 66:1427–1444, 2021.

[23] S. Weerakoon and T.G.I. Fernado. A variant of newton's method with accerated third-order convergence. *Applied Mathematics Letters*, 13:87–93, 2000.