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Metrical Fixed Point Results on b-multiplicative metric spaces employing binary relaion

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Abstract. In this manuscrit, we establish some results on the existence and uniqueness of fixed points by using b-multiplicative metric spaces(MMS) endowed with a binary relation. We also find result on the coincidence of points involving a pair of mappings. Finally some examples are presented to illustrate the suitability of our results.

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1. Introduction and Prilimaries

In 1922, Banach [1] laid the important result of fixed point theory in metric spaces. Later on, several authors generalized the Banach contraction principle, see[2–4]. Inspired by Turinici [5] work, Ran and Reurings [6] in 2004 worked on Banach contraction principle in ordered metric space and assumed the contractive condition only to hold on the comparable elements instead of the whole space. Fixed point in ordered metric space has been extensively studied in the literature [7–9].
The idea of MMS, which is a generalization of metric space, was first introduced by Bashirov et al. [10] in 2008. The main idea behind introducing MMS was to replace usual triangular inequality by the multiplicative triangle inequality. Later on, many research papers were written on fixed points in MMS [11–16, 18–20]. Czerwik [17] introduced the notion of b-metric space which is a generalization of metric space. There are some fixed point results in b-metric space. Later on, Muhammad Usman et al. [21] introduce the

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new notion of b-multiplicative metric space and proved fixed point theorems for single and multivalued mapping on b-multiplicative metric spaces, endowed with a graph.

In this paper we prove fixed point theorems for mapping on b-multiplicative metric space endowed with a binary relation and also prove a coincidence of points involving a pair of mapping and provide some examples to demonstrate our results.

Definition 1. *[21]. Let $\ddot{\mathbb{H}}$ be a non-empty set and let $k \geq 1$ be a given real number. A mapping $p : \ddot{\mathbb{H}} \times \ddot{\mathbb{H}} \to \mathbb{R}$ is called a b-multiplicative metric with coefficient k, if the following conditions hold:*

(M1) $p(\varpi, \rho) \geq 1$ *for all* $\varpi, \rho \in \ddot{\mathbb{H}}$ *and* $p(\varpi, \rho) = 1$ *if and only if* $\varpi = \rho$;

(M2) $p(\varpi, \rho) = p(\varpi, \rho)$ *for all* $\varpi, \rho \in \ddot{\mathbb{H}}$;

(M3) $p(\varpi, \rho) \leq p(\varpi, z)^k.p(z, \rho)^k$ *for all* $\varpi, \rho, z \in \ddot{\mathbb{H}}$.

The triplet ($\ddot{\mathbb{H}}$,p,k) is called a b-multiplicative metric space.

Definition 2. *[4]. Let $(\ddot{\mathbb{H}}, p, k)$ be any b-MMS, $\{\varpi_n\}$ be a sequence in $\ddot{\mathbb{H}}$ and $\varpi \in \ddot{\mathbb{H}}$. If for every multiplicative open ball $B_\epsilon(z) = \{\rho : p(\varpi, \rho) < \epsilon\}, \epsilon > 1$, there exists a natural number $N \in \mathbb{N}$ such that $n \geq N$ and $\varpi_n \in B_\epsilon(\varpi)$. Then the sequence $\{\varpi_n\}$ is said to be multiplicative converging to ϖ. We denote as $\varpi_n \to \varpi$ $(n \to +\infty)$.*

Lemma 1. *[21] let $(\ddot{\mathbb{H}}, p, k)$ is a b-multiplicative metric space. If a sequence $\{\varpi_n\}$ is a multiplicative convergent, then the multiplicative limit point is unique. Let $(\ddot{\mathbb{H}}, p)$ be a MMS, $\{\varpi_n\}$ be a sequence in $\ddot{\mathbb{H}}$ and $\varpi \in \ddot{\mathbb{H}}$. Then*

$$\varpi_n \to \varpi(n \to +\infty) \Leftrightarrow p(\varpi_n, \varpi) \to 1(n \to +\infty).$$

Definition 3. *[4]. Let $(\ddot{\mathbb{H}}, p)$ be a MMS and $\{\varpi_n\}$ be a sequence in $\ddot{\mathbb{H}}$.*

- *Then $\{\varpi_n\}$ is said to be multiplicative Cauchy sequence if for $\epsilon > 1$, there exists a positive integer $N \in \mathbb{N}$ such that $d(\varpi_m, \varpi_n) < \epsilon$ for all $n, m \geq N$.*
- *Then $\{\varpi_n\}$ is said to be multiplicative Cauchy if and only if $p(\varpi_n, \varpi_m) \to 1(n, m \to +\infty)$.*

Definition 4. *[4]. If every multiplicative Cauchy sequence in $(\ddot{\mathbb{H}}, p)$ is multiplicative convergent in $\ddot{\mathbb{H}}$, then MMS $(\ddot{\mathbb{H}}, p)$ is said to be multiplicative complete*

Definition 5. *[22]. Let $\ddot{\mathbb{H}}$ be a nonempty set. A subset $\ddot{\mathbb{R}}$ of $\ddot{\mathbb{H}}^2$ is called a binary relation on $\ddot{\mathbb{H}}$. The subsets, $\ddot{\mathbb{H}}^2$ and ϕ of $\ddot{\mathbb{H}}^2$ are called the universal relation and empty relation respectively.*

Definition 6. *[22]. Let $\ddot{\mathbb{R}}$ be a binary relation on a nonempty set $\ddot{\mathbb{H}}$. For $\varpi, \rho \in \ddot{\mathbb{H}}$, we say that ϖ and ρ are $\ddot{\mathbb{R}}$-comparative if either $(\varpi, \rho) \in \ddot{\mathbb{R}}$ or $(\rho, \varpi) \in \ddot{\mathbb{R}}$. We denote it by $[\varpi, \rho] \in \ddot{\mathbb{R}}$*

Proposition 1. *If $(\ddot{\mathbb{H}}, p, k \geq 1)$ is a b-metric space, $\ddot{\mathbb{R}}$ is a binary relation on $\ddot{\mathbb{H}}$, $\ddot{\mathfrak{F}}$ is a self-mapping on $\ddot{\mathbb{H}}$ and $\lambda \in [0, \frac{1}{k})$, then these conditions are equivalent.*

(I) $p(\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) \leq p(\varpi, \rho)^{\lambda}$ *for all* $\varpi, \rho \in \ddot{\mathbb{H}}$ *with* $(\varpi, \rho) \in \ddot{\mathbb{R}}$,

(II) $p(\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) \leq p(\varpi, \rho)^{\lambda}$ *for all* $\varpi, \rho \in \ddot{\mathbb{H}}$ *with* $[\varpi, \rho] \in \ddot{\mathbb{R}}$.

Proof. The implication (II) $\implies$ (I) is trivial. Coversely, we assume that (I) holds. Take $\varpi, \rho \in \ddot{\mathbb{H}}$ with $[\varpi, \rho] \in \ddot{\mathbb{R}}$. if $(\varpi, \rho) \in \ddot{\mathbb{R}}$, then (II) directly follows from (1). But, if $(\rho, \varpi) \in \ddot{\mathbb{R}}$, then using the symmetry of p and (I), we obtain

$$p(\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) = p(\ddot{\mathfrak{F}}\rho, \ddot{\mathfrak{F}}\varpi) \leq p(\rho, \varpi)^{\lambda} = p(\rho, \varpi)^{\lambda}.$$

which shows that (I) $\implies$ (II).

Proposition 2. *If $(\ddot{\mathbb{H}}, p, k \geq 1)$ is a b-metric space, $\ddot{\mathbb{R}}$ is a binary relation on $\ddot{\mathbb{H}}$, $\ddot{\mathfrak{F}}$ and S are self-mapping on $\ddot{\mathbb{H}}$ and $\lambda \in [0, \frac{1}{k})$, then these conditions are equivalent.*

(1) $p(\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) \leq p(S\varpi, S\rho)^{\lambda}$ *for all* $\varpi, \rho \in \ddot{\mathbb{H}}$ *with* $(\varpi, \rho) \in \ddot{\mathbb{R}}$,

(2) $p(\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) \leq p(S\varpi, S\rho)^{\lambda}$ *for all* $\varpi, \rho \in \ddot{\mathbb{H}}$ *with* $[\varpi, \rho] \in \ddot{\mathbb{R}}$.

Definition 7. *[23]. "Let $\ddot{\mathbb{H}}$ be a non-empty set and $\ddot{\mathbb{R}}$ be a binary relation on $\ddot{\mathbb{H}}$.*
(1) The inverse, transpose or dual relation of $\ddot{\mathbb{R}}$, denoted by $\ddot{\mathbb{R}}^{-1}$ is defined by

$$\ddot{\mathbb{R}}^{-1} = \{(\varpi, \rho) \in \ddot{\mathbb{H}}^2 : (\rho, \varpi) \in \ddot{\mathbb{R}}\}$$

(2) The reflexive closure of $\ddot{\mathbb{R}}$, denoted by $\ddot{\mathbb{R}}^{\#}$, is defined to be the set $\ddot{\mathbb{R}} \cup \triangle_{\varpi}$ (i.e., $\ddot{\mathbb{R}}^{\#} := \ddot{\mathbb{R}} \cup \triangle_{\varpi}$).
(3) The symmetric closure of $\ddot{\mathbb{R}}$, denoted by $\ddot{\mathbb{R}}^{s}$, is defined to be the set $\ddot{\mathbb{R}} \cup \ddot{\mathbb{R}}^{-1}$ (i.e., $\ddot{\mathbb{R}}^{\#} := \ddot{\mathbb{R}} \cup \ddot{\mathbb{R}}^{-1}$).

Proposition 3. *[24] For a binary relation $\ddot{\mathbb{R}}$ defined on a nonempty set $\ddot{\mathbb{H}}$,*

$$(\varpi, \rho) \in \ddot{\mathbb{R}}^{s} \iff [\varpi, \rho] \in \ddot{\mathbb{R}}.$$

Definition 8. *[24]. Let $\ddot{\mathbb{H}}$ be a non-empty set and $\ddot{\mathbb{R}}$ a binary relation on $\ddot{\mathbb{H}}$. A sequence $\varpi_n \subset \ddot{\mathbb{H}}$ is called $\ddot{\mathbb{R}}$- preserving if*

$$(\varpi_n, \varpi_{n+1}) \in \ddot{\mathbb{R}} \; for \; all \; n \in \mathbb{N}_0.$$

Definition 9. *[24] Let ($\ddot{\mathbb{H}}$,p) be a metric space. A binary relation $\ddot{\mathbb{R}}$ defined on $\ddot{\mathbb{H}}$ is called p-selfclosed if whenever $\{\varpi_n\}$ is an $\ddot{\mathbb{R}}$-preserving sequence and*

$$\varpi_n \xrightarrow{p} \varpi$$

then there exists a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ with $[\varpi_{n_k}, \varpi] \in \ddot{\mathbb{R}}$ for all $k \in \mathbb{N}_0$.

Definition 10. *[24] Let $\ddot{\mathbb{H}}$ be a nonempty set and $\ddot{\mathfrak{F}}$ a self-mapping on $\ddot{\mathbb{H}}$. A binary relation $\ddot{\mathbb{R}}$ defined on $\ddot{\mathbb{H}}$ is called $\ddot{\mathfrak{F}}$-closed if for any $\varpi, \rho \in \ddot{\mathbb{H}}$*

$$(\varpi, \rho) \in \ddot{\mathbb{R}} \implies (\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) \in \ddot{\mathbb{R}}.$$

Proposition 4. *[24] Let $\ddot{\mathbb{H}}$, $\ddot{\mathfrak{F}}$ and $\ddot{\mathbb{R}}$ be same as in definition 1.10. $\ddot{\mathbb{R}}^s$ must also be $\ddot{\mathfrak{F}}$-closed if $\ddot{\mathbb{R}}$ is $\ddot{\mathfrak{F}}$-closed.*

Definition 11. *[24] Let S and V are self mappings on a nonempty set $\ddot{\mathbb{H}}$. A binary relation $\ddot{\mathbb{R}}$ on $\ddot{\mathbb{H}}$ is called (S,V)-closed if for all $\varpi, \rho \in \ddot{\mathbb{H}}$, $(V_\varpi, V_\rho) \in \ddot{\mathbb{R}}$ yield that (S_ϖ, S_ρ) belong to $\ddot{\mathbb{R}}$.*

if we take V= identity mapping, then we conclude that $\ddot{\mathbb{R}}$ is S-closed.
if $\ddot{\mathbb{R}}$ is S-closed, then $\ddot{\mathbb{R}}^s$ is also S-closed.

Definition 12. *[25] Let $(\ddot{\mathbb{H}}, p, k \geq 1)$ be a b-metric space and let $\ddot{\mathbb{R}}$ a binary relation on $\ddot{\mathbb{H}}$.*

(i) we say that (ϖ, p) is $\ddot{\mathbb{R}}$-complete if every $\ddot{\mathbb{R}}$-preserving b-Cauchy sequence in $\ddot{\mathbb{H}}$ converges.

(ii) A subset G of $\ddot{\mathbb{H}}$ is called $\ddot{\mathbb{R}}$-closed if every $\ddot{\mathbb{R}}$-preserving b-convergent sequence in G converges to a point of G.

Definition 13. *[25] Let $(\ddot{\mathbb{H}}, p, k \geq 1)$ be a b-metric space and let $V : \ddot{\mathbb{H}} \to \ddot{\mathbb{H}}$. A binary relation $\ddot{\mathbb{R}}$ defined on $\ddot{\mathbb{H}}$ is called (V, b_p)-self closed if, whenever $\{\varpi_n\}$ is an $\ddot{\mathbb{R}}$-preserving sequence and $\varpi_n \to_p \varpi$, there exists a subsequence $\{\varpi_{n_i}\}$ of $\{\varpi_n\}$ with $[V\varpi_{n_i}, V\varpi] \in \ddot{\mathbb{R}}$ for all $i \in \mathbb{N}$.*

If V is the identity mapping, then we get the following definitions:

Definition 14. *[25] Let $(\ddot{\mathbb{H}}, p, k \geq 1)$ be a b-metric space. A binary relation $\ddot{\mathbb{R}}$ defined on $\ddot{\mathbb{H}}$ is called b_p-self closed if, whenever $\{\varpi_n\}$ is an $\ddot{\mathbb{R}}$-preserving sequence and $\varpi_n \to_p \varpi$, there exists a subsequence $\{\varpi_{n_j}\}$ of $\{\varpi_n\}$ with $(\varpi_{n_j}, \varpi) \in \ddot{\mathbb{R}}$ for all $j \in \mathbb{N}$.*

Definition 15. *[26] Let $\ddot{\mathbb{H}}$ be a nonempty set and $\ddot{\mathbb{R}}$ a binary relation on $\ddot{\mathbb{H}}$. A subset G of $\ddot{\mathbb{H}}$ is called $\ddot{\mathbb{R}}$-directed if for each $\varpi, \rho \in G$, there exists $z \in \ddot{\mathbb{H}}$ such that $(\varpi, z) \in \ddot{\mathbb{R}}$ and $(\rho, z) \in \ddot{\mathbb{R}}$.*

Definition 16. *[27] Let $\ddot{\mathbb{H}}$ be a nonempty set and $\ddot{\mathbb{R}}$ a binary relation on $\ddot{\mathbb{H}}$. For $\varpi, \rho \in \ddot{\mathbb{H}}$, a path of length k (where k is a natural number) in $\ddot{\mathbb{R}}$ from ϖ to ρ is a finite sequence $\{t_0, t_1, t_2, ...t_k\} \subset \ddot{\mathbb{H}}$ satisfying the following conditions:*
(i) $t_0 = \varpi$ and $t_k = \rho$
(ii) $(t_j, t_{j+1}) \in \ddot{\mathbb{R}}$ for each j $(0 \leq j \leq k-1)$.
Note that although they are not necessarily distinct, a path of length k involves k+1 elements of $\ddot{\mathbb{H}}$".

Definition 17. *[25] Let* $(\ddot{\mathbb{H}}, p, k \geq 1)$ *be a b-metric space, let* $\ddot{\mathbb{R}}$ *be a binary relation on* $\ddot{\mathbb{H}}$*, and let* S *and* V *be two self-mappings on* $\ddot{\mathbb{H}}$*. we say that* S *and* V *are* $\ddot{\mathbb{R}}$*-compatible if, for any sequence* $\{\varpi_n\} \in \ddot{\mathbb{H}}$ *such that* $\{S\varpi_n\}$ *and* $\{V\varpi_n\}$ *are* $\ddot{\mathbb{R}}$*-preserving and*

$$\lim_{\varpi \to +\infty} V(\varpi_n) = \lim_{\varpi \to +\infty} S(\varpi_n),$$

we have

$$\lim_{\varpi \to +\infty} d(VP(\varpi_n), PV\varpi_n) = 0.$$

Lemma 2. *[28] Let* $\ddot{\mathbb{H}}$ *be a non empty set and let* $\ddot{\mathfrak{F}}$ *be a self mapping on* $\ddot{\mathbb{H}}$*. Then there exists a subset* $G \subseteq \ddot{\mathbb{H}}$ *such that* $\ddot{\mathfrak{F}}(G) = \ddot{\mathfrak{F}}(\ddot{\mathbb{H}})$ *and* $\ddot{\mathfrak{F}} : G \to \ddot{\mathbb{H}}$ *is one-to one.*

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2. Main Result

In this manuscript, we utilize the following notations:

(i) $\mathrm{F}(\ddot{\mathfrak{F}})$ = the set of all fixed points of $\ddot{\mathfrak{F}}$,

(ii) $\ddot{\mathbb{H}}(\ddot{\mathfrak{F}}; \ddot{\mathbb{R}}) := \{\varpi \in \ddot{\mathbb{H}} : (\varpi, \ddot{\mathfrak{F}}\varpi) \in \ddot{\mathbb{R}}\}$,

(iii) $\gamma\ (\varpi, \rho, \ddot{\mathbb{R}})$:= the class of all paths in $\ddot{\mathbb{R}}$ from ϖ to ρ

Theorem 1. *Let* $(\ddot{\mathbb{H}}, p, k \geq 1)$ *be a b-complete b-multiplicative metric space and* $\ddot{\mathbb{R}}$ *a binary relation on* $\ddot{\mathbb{H}}$*.* $\ddot{\mathfrak{F}}$*:* $\ddot{\mathbb{H}} \times \ddot{\mathbb{H}}$ *be a self-mapping satisfying the following conditions given below.*

(i) $\ddot{\mathbb{H}}(\ddot{\mathfrak{F}}; \ddot{\mathbb{R}})$ *is non-empty.*

(ii) $\ddot{\mathbb{R}}$ *is* $\ddot{\mathfrak{F}}$*-closed.*

(iii) Either $\ddot{\mathfrak{F}}$ *is b-continuous or* $\ddot{\mathbb{R}}$ *is* b_p*-self closed.*

(iv) There exists $\lambda \in [0, \frac{1}{k})$ *such that.*

$$p(\ddot{\mathfrak{F}}\varpi, \ddot{\mathfrak{F}}\rho) \leq p(\varpi, \rho)^{\lambda}$$

Then $\ddot{\mathfrak{F}}$ *has a fixed point. i.e., there exists* $\varpi* \in \ddot{\mathbb{H}}$ *such that* $\ddot{\mathfrak{F}}\varpi* = \varpi*$*.*

(v) $\gamma(\varpi, \rho, \ddot{\mathbb{R}}^s)$ *is non- empty, for each* $\varpi, \rho \in \ddot{\mathbb{H}}$*, then* $\ddot{\mathfrak{F}}$ *has a unique fixed point.*

Proof. Let $\varpi_0 \in (\ddot{\mathfrak{F}};\ \ddot{\mathbb{R}})$ be an arbitrary element. Now we define the sequence ϖ_n of picard iterates i.e., $\varpi_n = \ddot{\mathfrak{F}}\varpi_{n-1} = \ddot{\mathfrak{F}}^n\varpi_0$ for all $n \in \mathbb{N}$. As $(\varpi_0, \ddot{\mathfrak{F}}\varpi_0) \in \ddot{\mathbb{R}}$ and $\ddot{\mathbb{R}}$ is $\ddot{\mathfrak{F}}$-closed, we get.

$$(\ddot{\mathfrak{F}}\varpi_0, \ddot{\mathfrak{F}}^2\varpi_0), (\ddot{\mathfrak{F}}^2\varpi_0, \ddot{\mathfrak{F}}^3\varpi_0), ..., (\ddot{\mathfrak{F}}^n\varpi_0, \ddot{\mathfrak{F}}^{n+1}\varpi_0), ..., \in \ddot{\mathbb{R}}$$

so that

$$(\varpi_n, \varpi_{n+1}) \in \ddot{\mathbb{R}}, \quad for\ all\ n\ \in \mathbb{N} \tag{1}$$

therefore the sequence ϖ_n is $\ddot{\mathbb{R}}$-preserving. Applying the contractivity condition (iv) to (1). We deduce, for all $n \in \mathbb{N}$. that

$$p(\varpi_n, \varpi_{n+1}) \leq p(\varpi_{n-1}, \varpi_n)^{\lambda},$$

which by induction yield that

$$p(\varpi_n, \varpi_{n+1}) \leq p(\varpi_0, \ddot{\mathfrak{F}}\varpi_0)^{\lambda^n} \qquad for\ all\ n \in \mathbb{N} \tag{2}$$

By using (2) and multiplicative triangular inequality, for all $n \in \mathbb{N}$, $r \in \mathbb{N}$, we have

$$\begin{aligned} p(\varpi_n, \varpi_{n+r}) &\leq p(\varpi_n, \varpi_{n+1})^{k^n} \cdot p(\varpi_{n+1}, \varpi_{n+2})^{k^{n+1}} \cdots \cdot p(\varpi_{n+r-1}, \varpi_{n+r})^{k^{n+r-1}} \\ &\leq p(\varpi_n, \varpi_{n+1})^{\lambda^n k^n} \cdot d(\varpi_{n+1}, \varpi_{n+2})^{\lambda^{n+1} k^{n+1}} \cdots \cdot p(\varpi_{n+r-1}, \varpi_{n+r})^{\lambda^{n+r-1} k^{n+r-1}} \\ &\leq p(\varpi_0, \ddot{\mathfrak{F}}\varpi_0)^{(\lambda k)^n + (\lambda k)^{n+1} + \cdots + (\lambda k)^{n+r-1}} \\ &\leq p(\varpi_0, \ddot{\mathfrak{F}}\varpi_0)^{\frac{(\lambda k)^n}{1-(\lambda k)}}. \end{aligned}$$

This implies that $p(\varpi_n, \varpi_{n+r}) \rightarrow_b 1$, (as $n \rightarrow +\infty$) Hence, the sequence ϖ_n is multiplicative Cauchy sequence in $\ddot{\mathbb{H}}$. As ($\ddot{\mathbb{H}}$, p, $k \geq 1$) is b-complete, there exists $\varpi* \in \ddot{\mathbb{H}}$ such that

$$\varpi_n \ \rightarrow \varpi^*.$$

Now, in lieu of (iii) assume that $\ddot{\mathfrak{F}}$ is b-continuous, we have

$$\varpi_{n+1} \ = \ \ddot{\mathfrak{F}}\varpi_n \ \xrightarrow{p} \ \ddot{\mathfrak{F}}\varpi^*.$$

owing to the uniqueness of limit, we obtain $\ddot{\mathfrak{F}}\varpi* = \varpi^*$ i.e., ϖ^* is a fixed point of $\ddot{\mathfrak{F}}$.

Alternately, suppose that $\ddot{\mathbb{R}}$ is $b_p - selfclosed.$ since ϖ_n is an $\ddot{\mathbb{R}}$-preserving sequence and

$$\varpi_n \ \xrightarrow{p} \ \varpi.$$

by the $b_p - selfcloseness$ of $\ddot{\mathbb{R}}$, there exists a subsequence $\{\varpi_{n_j}\}$ of $\{\varpi_n\}$ with

$$[\varpi_{n_j, \varpi}] \in \ddot{\mathbb{R}} \quad for\ all\ j\ \in\ \mathbb{N}$$

using (iv), Proposition (1.1), we obtain

$$p(\varpi^*, \ddot{\mathfrak{F}}\varpi^*) \leq [p(\varpi^*, \varpi_{n+1}) \cdot p(\varpi_{n+1}, \ddot{\mathfrak{F}}\varpi^*)]^k$$

$$= [p(\varpi^*, \varpi_{n+1}) \cdot p(\varpi_{n+1}, \ddot{\mathfrak{F}}\varpi^*)]^k$$
$$\leq\ [p(\varpi^*, \varpi_{n+1}) \cdot p(\varpi_{n+1}, \ddot{\mathfrak{F}}\varpi^*)^{\lambda}]^k \to\ 1\ \ as\ \ n\ \to\ \ +\infty.$$

Hence, $\ddot{\mathfrak{F}}\varpi^* = \varpi^*$ and ϖ^* is a fixed point of $\ddot{\mathfrak{F}}$.
suppose that ρ^* is another fixed point of $\ddot{\mathfrak{F}}$.

By assumption (v), there exists a path (say $\{t_0, t_1, t_2, ...t_k,\}$) of some finite length k in $\ddot{\mathbb{R}}^s$ from ϖ to ρ so that

$$t_0 = \varpi,\ t_k = \rho,\ [t_j, t_{j+1}] \in \ddot{\mathbb{R}}\ for\ each\ j\ (0 \leq j \leq k-1). \tag{3}$$

As $\ddot{\mathbb{R}}$ is $\ddot{\mathfrak{F}}$-closed, by using proposition (1.3) , we have

$$[\ddot{\mathfrak{F}}^n t_j, \ddot{\mathfrak{F}}^n t_{j+1}] \in \ddot{\mathbb{R}}\ for\ each\ j\ (0 \leq j \leq k-1)\ and\ for\ each\ n \in \mathbb{N} \tag{4}$$

Making use of (3), (4), (5), triangular inequality, assumption (iv) and proposition (1.1), we obtain

$$\begin{aligned}
p(\varpi, \rho) &= p(\ddot{\mathfrak{F}}^n t_0, \ddot{\mathfrak{F}}^n t_k) \leq \prod_{j=0}^{k-1} (\ddot{\mathfrak{F}}^n t_j, \ddot{\mathfrak{F}}^n t_{j+1}) \\
&\leq \prod_{j=0}^{k-1} p(\ddot{\mathfrak{F}}^{n-1} t_j, \ddot{\mathfrak{F}}^{n-1} t_{j+1})^{\lambda} \\
&\leq \prod_{j=0}^{k-1} p(\ddot{\mathfrak{F}}^{n-2} t_j, \ddot{\mathfrak{F}}^{n-2} t_{j+1})^{\lambda^2} \\
&\leq \cdots \leq \prod_{j=0}^{k-1} p(t_j, t_{j+1})^{\lambda^n} \\
&\to 1\ as\ n \to +\infty
\end{aligned} \tag{5}$$

so, that ϖ=ρ. Hence $\ddot{\mathfrak{F}}$ has a unique fixed point.

Theorem 2. *Let ($\ddot{\mathbb{H}}$,d,$k \geq 1$) be a b-complete b-multiplicative metric space and $\ddot{\mathbb{R}}$ a binary relation on $\ddot{\mathbb{H}}$. S, V: $\ddot{\mathbb{H}} \to \ddot{\mathbb{H}}$ be a self-mapping satisfying the following conditions given below.*

(i) $\ddot{\mathbb{H}}(S, V; \ddot{\mathbb{R}})$ *are non-empty and* $S(\ddot{\mathbb{H}}) \subseteq V(\ddot{\mathbb{H}})$;

(ii) $\ddot{\mathbb{R}}$ *is* (S, V)*-closed.*

(iii) There exists $\lambda \in [0, \frac{1}{k})$ *such that*

$$d(S\varpi, S\rho) \leq d(V\varpi, V\rho)^{\lambda}$$

(iv) Either S is (V,$\ddot{\mathbb{R}}$)-continuous or S and V are continuous.
or

(iv') S and V are $\ddot{\mathbb{R}}$- compatible, V is $\ddot{\mathbb{R}}$- continuous, and either S is $\ddot{\mathbb{R}}$-continuous or $\ddot{\mathbb{R}}$ is $(V, b_d) - self - closed$,
Then S and V have a point of coincidence.

Proof. let $\varpi_0 \in \ddot{\mathbb{H}}$(S,V,$\ddot{\mathbb{R}}$) be an arbitrary element. Then $(V\varpi_0, S\varpi_0) \in \ddot{\mathbb{R}}$. If $V(\varpi_0) = S(\varpi_0)$, then ϖ_0 is a coincidence point of S and V and, hence, we are through.
otherwise, if $V(\varpi_0) \neq S(\varpi_0)$, then, in view of $S(\ddot{\mathbb{H}}) \subseteq V(\ddot{\mathbb{H}})$, we can choose $\varpi_1 \in \ddot{\mathbb{H}}$ such that $V(\varpi_1) = S(\varpi_0)$. Again from $S(\ddot{\mathbb{H}}) \subseteq V(\ddot{\mathbb{H}})$, we can choose $\varpi_2 \in \ddot{\mathbb{H}}$ such that $V(\varpi_2) = S(\varpi_1)$. construct the sequence $\{\varpi_n\} \subset \ddot{\mathbb{H}}$ such that

$$V(\varpi_{n+1}) = S(\varpi_n) \quad for\ all\ n \in \mathbb{N} \tag{6}$$

Now,we claim that $\{V\varpi_n\}$ is $\ddot{\mathbb{R}}$- preserving sequence, i.e.,

$$(V\varpi_n, V\varpi_{n+1}) \in \ddot{\mathbb{R}} \qquad for\ all \qquad n \in \mathbb{N} \tag{7}$$

we can show this fact by induction. By equation (6) (with n= 0) and fact that $\varpi_0 \in \ddot{\mathbb{H}}(S, V, \ddot{\mathbb{R}})$, We conclude that $(V\varpi_0, V\varpi_1) \in \ddot{\mathbb{R}}$. which means that (7) holds for n=0.

suppose (7) is true for $n = r \geq 0$ i.e., $(V\varpi_r, V\varpi_{r+1}) \in \ddot{\mathbb{R}}$. As $\ddot{\mathbb{R}}$ is (S,V)- closed, we get $(S\varpi_r, S\varpi_{r+1}) \in \ddot{\mathbb{R}}$. by using , this yield that $(V\varpi_{r+1}, V\varpi_{r+2}) \in \ddot{\mathbb{R}}$, i.e., inclusion (7) holds for n=r+1. Hence by induction, inclusion (7) is valid for all $n \in \mathbb{N}$.
in view of (6) and (7), the sequence $\{S\varpi_n\}$ is also an $\ddot{\mathbb{R}}$-preserving, i.e.,

$$(S\varpi_n, S\varpi_{n+1}) \in \ddot{\mathbb{R}} \qquad for\ all \qquad n \in \mathbb{N}$$

By using (6), (7)and assumption (iii), we find

$$p(V\varpi_n, V\varpi_{n+1}) = p(S\varpi_{n-1}, S\varpi_n) \leq p(V\varpi_{n-1}, V\varpi_n)^{\lambda} \qquad for\ all \qquad n \in \mathbb{N} \tag{8}$$

which by induction yield that

$$p(V\varpi_n, V\varpi_{n+1}) = p(S\varpi_{n-1}, S\varpi_n) \leq p(V\varpi_{n-1}, V\varpi_n)^{\lambda^n} \qquad for\ all \qquad n \in \mathbb{N} \tag{9}$$

By using (9) and multiplicative triangular inequality, for all $n \in \mathbb{N}$, $r \in \mathbb{N}$, we have

$$\begin{aligned}
p(V\varpi_n, V\varpi_{n+r}) &\leq p(V\varpi_n, V\varpi_{n+1})^{k^n}\cdot p(V\varpi_{n+1}, V\varpi_{n+2})^{k^{n+1}}\cdots p(V\varpi_{n+r-1}, V\varpi_{n+r})^{k^{n+r-1}}\\
&\leq p(V\varpi_n, V\varpi_{n+1})^{\lambda^n k^n}\cdot p(V\varpi_{n+1}, V\varpi_{n+2})^{\lambda^{n+1}k^{n+1}}\cdots .\\
&\quad p(V\varpi_{n+r-1}, V\varpi_{n+r})^{\lambda^{n+r-1}k^{n+r-1}}\\
&\leq p(V\varpi_0, V\varpi_1)^{(\lambda k)^n+(\lambda k)^{n+1}+\cdots+(\lambda k)^{n+r-1}}\\
&\leq p(V\varpi_0, V\varpi_1)^{\frac{(\lambda k)^n}{1-(\lambda k)}}.
\end{aligned}$$

This implies that $p(V\varpi_n, V\varpi_{n+r}) \to_b 1$, (as $n \to +\infty$) Hence, the sequence $V\varpi_n$ is multiplicative Cauchy sequence in $\ddot{\mathbb{H}}$. By using (3), we have $V\varpi_n \subseteq S(\ddot{\mathbb{H}})$ and hence $V\varpi_n$ is an $\ddot{\mathbb{R}}$-preserving b-multiplicative Cauchy sequence in $\ddot{\mathbb{H}}$. As ($\ddot{\mathbb{H}}$, p, $k \geq 1$) is b-complete, there exists $u \in V(\ddot{\mathbb{H}})$ such that

$$\lim_{\varpi\to+\infty} V(\varpi_n) = V(u) \tag{10}$$

By using (6) and (10), we get

$$\lim_{\varpi\to+\infty} S(\varpi_n) = V(u) \tag{11}$$

Now we show that u is a coincidence point of S and V.

Now, in lieu of (iv) consider that p is $(V, \ddot{\mathbb{R}})$-continuous, Thus utilizing (7) and (10) we obtain

$$\lim_{\varpi\to+\infty} S(\varpi_n) = S(u) \tag{12}$$

In view of (11) and (12), we obtain V(u)= S(u). Hence, we are completed. second , we assume that S and V are continuous and owing to the Lemma 1.1, there exists a subset $G \subseteq \ddot{\mathbb{H}}$ such that $V(G) = V(\ddot{\mathbb{H}})$ and $V : G \to \ddot{\mathbb{H}}$ is one to one. Now we define $\ddot{\mathfrak{F}} : V(G) \to V(\ddot{\mathbb{H}})$ by $\ddot{\mathfrak{F}}(Va) = S(a)$ for all $V(a) \in V(G)$ where $a \in G$

As $V : G \to \ddot{\mathbb{H}}$ is injective and $S(\ddot{\mathbb{H}}) \subseteq V(\ddot{\mathbb{H}})$, we get to the conclusion that $\ddot{\mathfrak{F}}$ is well defined. Additionally, $\ddot{\mathfrak{F}}$ is continuous because S and V are continuous. As $V(\ddot{\mathbb{H}}) = V(G)$ and $S(\ddot{\mathbb{H}}) \in V(\ddot{\mathbb{H}})$, we get $S(\ddot{\mathbb{H}}) \in V(G)$. This means that, it is possible to construct $\{\varpi_n\} \in G$ satisfying relation (6) and we choose $u \in G$. Utilizing equation (10) and (11) and the continuity of $\ddot{\mathfrak{F}}$, we find

$$S(u) = \ddot{\mathfrak{F}}(Vu) = \ddot{\mathfrak{F}}(\lim_{n\to+\infty} V\varpi_n) = \lim_{n\to+\infty} \ddot{\mathfrak{F}}(V\varpi_n) = \lim_{n\to+\infty} S(\varpi_n) = V(u)$$

Hence, $u \in \ddot{\mathbb{H}}$ is a point of coincidence of a pair of maps. This end the proof.

Owing to (6), we have $\{V\varpi_n\} \subseteq S(\ddot{\mathbb{H}})$ and hence, $\{V\varpi_n\}$ is b-multiplicative Cauchy sequence in $\ddot{\mathbb{H}}$. As $\ddot{\mathbb{H}}$ is b-complete, there exists $u \in V(\ddot{\mathbb{H}})$ such that

$$\lim_{\varpi\to+\infty} V(\varpi_n) = V(u). \tag{13}$$

By using (6) and (13), we get

$$\lim_{\varpi\to+\infty} S(\varpi_n) = V(u). \tag{14}$$

As V is $\ddot{\mathbb{R}}$-continuous, we find

$$\lim_{\varpi\to+\infty} V(V\varpi_n) = V(\lim_{n\to+\infty} V(\varpi_n)) = V(V(u)) \tag{15}$$

moreover, we get

$$\lim_{\varpi\to+\infty} V(S\varpi_n) = V(\lim_{n\to+\infty} S(\varpi_n)) = V(V(u)) \tag{16}$$

since $\{S\varpi_n\}$ and $\{V\varpi_n\}$ are $\ddot{\mathbb{R}}$-preserving

$$\lim_{\varpi\to+\infty} S(\varpi_n) = V(u) = \lim_{n\to+\infty} V(\varpi_n) \tag{17}$$

and S and V are $\ddot{\mathbb{R}}$-compatible, we obtain

$$\lim_{\varpi\to+\infty} p(VS(\varpi_n), SV(\varpi_n)) = 0. \tag{18}$$

Now, we demonstrate that V(u) is a coincidence point of S and V.

We assume that S is $\ddot{\mathbb{R}}$-continuous. By using (7), we get

$$\lim_{\varpi\to+\infty} S(V\varpi_n) = S \lim_{n\to+\infty} V(\varpi_n) == S(V(u)) \tag{19}$$

suppose that V(u) = z, utilizing triangle inequality, we get

$$\begin{aligned} p(Vz, Sz) &\leq [p(Vz, V(S\varpi_n)) \cdot p(V(S\varpi_n), Sz)]^k \\ &\leq p(Vz, V(S\varpi_n))^k \cdot [p(V(S\varpi_n), S(V\varpi_n) \cdot p(S(V\varpi_n), Sz)^{k^2} \end{aligned}$$

Making $n \to +\infty$, we get p(Vz,Sz) = 1, which implies Vz = Sz, i.e., z = V(u) is coincidence point of S and V.

Alternatively, assume that $\ddot{\mathbb{R}}$ is (V, b_p)-self closed. Since $\{V\varpi_n\}$ is $\ddot{\mathbb{R}}$-preserving and $V\varpi_n \to Vu$, in view of the (V, b_p)-self closeness of $\ddot{\mathbb{R}}$, there exists a subsequence $\{V\varpi_{n_i}\}$ of $\{V\varpi_n\}$ such that $[VV\varpi_{n_i}, VVu]$ belongs to $\ddot{\mathbb{R}}$ for all $i \in \mathbb{N} \cup \{0\}$. Since $V\varpi_{n_i} \to Vu$, in the view of proposition 1.3, we get

$$p(SV\varpi_{n_i}, SVu) \leq p(VV\varpi_{n_i}, VVu)^{\lambda} \qquad \textit{for all } i \in \mathbb{N} \cup \{0\}$$

we choose Vu = z. By the triangle inequality , we get

$$\begin{aligned} p(Vz,Sz) &\leq [p(Vz,V(S\varpi_{n_i}))\cdot p(V(S\varpi_{n_i}),Sz)]^k \\ &\leq p(Vz,V(S\varpi_{n_i}))^k\cdot[p(V(S\varpi_{n_i}),S(V\varpi_{n_i})\cdot p(S(V\varpi_{n_i}),Sz)]^{k^2} \\ &\leq p(Vz,V(S\varpi_{n_i}))^k\cdot p(V(S\varpi_{n_i}),S(V\varpi_{n_i})^{k^2}\cdot p(S(V\varpi_{n_i}),Sz).^{\lambda k^2} \end{aligned}$$

Making $i\to+\infty$, we get p(Vz,Sz) = 1, which implies Vz = Sz, that is, z = V(u) is a coincidence point of S and V.

Now we can give examples in support of theorem 1.

Example 1. *Let $\ddot{\mathbb{H}}=\mathbb{R}^+$ and $p=|\frac{\varpi}{\rho}|$, then ($\ddot{\mathbb{H}}$,p) is a complete multiplicative metric space. Define binary relation $\ddot{\mathbb{R}}=\{(\varpi,\rho)\in\mathbb{R}^2_+$: $\frac{\varpi}{\rho}\geq 1$, $\varpi,\rho\in\mathbb{R}^+\}$ on $\ddot{\mathbb{H}}$. consider mapping $\ddot{\mathfrak{F}}$:$\ddot{\mathbb{H}}\to\ddot{\mathbb{H}}$ defined by*

$$\ddot{\mathfrak{F}}(\varpi)=\varpi^{\frac{2}{3}}$$

obviously, $\ddot{\mathbb{R}}$ is $\ddot{\mathfrak{F}}$ closed and $\ddot{\mathfrak{F}}$ is continuous. Now, for $\varpi,\rho\in\ddot{\mathbb{H}}$ with $(\varpi,\rho)\in\mathbb{R}^+$. We have

$$p(\ddot{\mathfrak{F}}\varpi,\ddot{\mathfrak{F}}\rho)=\left|\frac{\varpi^{\frac{2}{3}}}{\rho^{\frac{2}{3}}}\right|=\left|\frac{\varpi}{\rho}\right|^{\frac{2}{3}}=p(\varpi,\rho)^{\frac{2}{3}}<p(\varpi,\rho)^{\frac{3}{4}}$$

i.e., $\ddot{\mathfrak{F}}$ satisfies assumption (iv) of Theorem (2.1) for $\lambda=\frac{3}{4}$. Consequently, every conditions (i)-(iv) of Theorem (2.1) also holds and therefore, $\ddot{\mathfrak{F}}$ has a unique fixed point (for $\varpi=1$).

Example 2. *Let $\ddot{\mathbb{H}}=[0.1,1]$ and $p=|\frac{\varpi}{\rho}|$, then ($\ddot{\mathbb{H}}$,p) is complete B-MMS. Define binary relation $\ddot{\mathbb{R}}=\{(\varpi,\rho)\in[0.1,1]^2$: $\frac{\varpi}{\rho}\geq 1$, $\varpi,\rho\in\mathbb{R}^+\}$ on $\ddot{\mathbb{H}}$. consider mapping $\ddot{\mathfrak{F}}:\ddot{\mathbb{H}}\to\ddot{\mathbb{H}}$ defined by*

$$\ddot{\mathfrak{F}}(\varpi)=e^{\varpi-1-\frac{\varpi^3}{10}}$$

obviously, $\ddot{\mathbb{R}}$ is $\ddot{\mathfrak{F}}$-closed and $\ddot{\mathfrak{F}}$ is continuous. Now, for $\varpi,\rho\in[0.1,1]$. We have

$$p(\ddot{\mathfrak{F}}\varpi,\ddot{\mathfrak{F}}\rho)=\left|\frac{\ddot{\mathfrak{F}}\varpi}{\ddot{\mathfrak{F}}\rho}\right|\leq\left|\frac{\varpi}{\rho}\right|^{\lambda}=p(\varpi,\rho)^{\lambda}\quad \textit{for all } \varpi,\rho\ \in X$$

where, $\lambda=0.997$, finally , we can say that $\ddot{\mathfrak{F}}$ has a unique fixed point $0.7411317711\in X$.

Example 3. *Let* $\ddot{\mathbb{H}} = [1,3]$ *and* $p = |\frac{\varpi}{\rho}|$, *then* $(\ddot{\mathbb{H}}, p)$ *is complete b-MMS. Define binary relation* $\ddot{\mathbb{R}}$=*{(1,1),(2,1),(2,2),(3,1),(3,2)} on* $\ddot{\mathbb{H}}$ *and a mapping* $\ddot{\mathfrak{F}}$:$\ddot{\mathbb{H}} \to \ddot{\mathbb{H}}$ *defined by*

$$\ddot{\mathfrak{F}}(\varpi) = \begin{cases} 1, & if\ 1 \leq \varpi \leq 2 \\ 2 & if\ 2 < \varpi \leq 3, \end{cases}$$

Obviously, $\ddot{\mathbb{R}}$ *is* $\ddot{\mathfrak{F}}$ *- closed but* $\ddot{\mathfrak{F}}$ *is not continuous. Take an* $\ddot{\mathbb{R}}$*-preserving sequence* $\{\varpi_n\}$ *such that*

$$\varpi_n \xrightarrow{p} \varpi$$

so that $(\varpi_n, \varpi_{n+1}) \in \ddot{\mathbb{R}}$ *for all* $n \in \mathbb{N}$. *Here, one can observe that*

$$(\varpi_n, \varpi_{n+1}) \notin \{(3,1),(3,2)\}$$

So that

$$(\varpi_n, \varpi_{n+1}) \in \{(1,1),(2,1),(2,2)\}$$

which gives rise to $\{\varpi_n\} \subset \{1,2\}$.$\{1,2\}$ *is closed, we have* $[\varpi_n, \varpi] \in \ddot{\mathbb{R}}$. *Therefore,* $\ddot{\mathbb{R}}$ *is p-closed. Assumption can be verified (iv) of Theorem 2.1 with* $\lambda = \frac{1}{2}$. *Thus, all the condition (i)- (iv) of Theorem 2.1 are satisfies and* $\ddot{\mathfrak{F}}$ *has a fixed point in* $\ddot{\mathbb{H}}$ *(for* ϖ*=1).*

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