EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 16, No. 4, 2023, 2368-2383
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Geodetic Roman Dominating Functions in a Graph

Rona Jane G. Fortosa[1,*], Sergio R. Canoy, Jr.[1,2]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics,*
[2] *Center for Graph Theory, Algebra, and Analysis- PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be a connected graph. A function $f : V(G) \to \{0, 1, 2\}$ is a *geodetic Roman dominating function* (or GRDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and $V_1 \cup V_2$ is a geodetic set in G. The weight of a geodetic Roman dominating function f, denoted by $\omega_G^{gR}(f)$, is given by $\omega_G^{gR}(f) = \sum_{v \in V(G)} f(v)$. The minimum weight of a GRDF on G, denoted by $\gamma_{gR}(G)$, is called the *geodetic Roman domination number* of G. In this paper, we give some properties of geodetic Roman domination and determine the geodetic Roman domination number of some graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Geodetic set,geodetic dominating set, Roman dominating function, Roman domination number, geodetic Roman dominating function, geodetic Roman domination number

---

1. Introduction

Roman domination was inspired by the strategies for defending the Roman Empire against invaders, as presented by Stewart [22] and ReVelle and Rosing [20]. Motivated by this strategy, Cockayne, Dreyer and Hedetniemi introduced the concept of Roman domination in 2004 [12]. Roman domination in a graph is a well studied concept under the topic of domination. As a protection strategy involving field armies, the Roman domination concept ensures that an unsecured location is made secured by sending an army to the location from an adjacent secured location subject to the constraint that one army must be left behind in the secured location. Other applications of the concept and some of its variations can be found in [1], [2], [3], [4], [5], [10], [12], [15], [16], [17], and [19].

Another variant of domination is the concept geodetic domination which was introduced by Buckley, Harary and Quintas [6]. Geodesics refers to the shortest paths between two vertices in a graph. The concept of geodesics is closely related to the

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v16i4.4962

Email addresses: ronajane.fortosa@g.msuiit.edu.ph (R.J. Fortosa),
sergio.canoy@g.msuiit.edu.ph (S. Canoy)

 © 2023 EJPAM All rights reserved.

notion of distance in a graph. As a matter of fact, the distance between two vertices is defined as the length of the shortest geodesic between them. In simple terms, the concept represents the minimum number of edges that must be traversed to travel from one vertex to another. Geodetic sets and geodetic domination have plenty of applications and researchers continue to investigate various concepts involving them. Some studies on geodetic sets and related concepts can be found in [7], [8], [11], [13], [14], [18], [21], [23] and [24].

In this study, we introduce the concept of geodetic Roman domination, a concept which combines the concepts of geodetic set and Roman domination. Geodetic Roman domination as a protection strategy (involving of field armies) guarantees, in addition to what the Roman domination requires, that every unsecured location lies along a shortest path between two secured locations.

2. Terminologies and Notations

Let G be a connected graph. For vertices u and v in G, a u-v geodesic is any shortest path in G joining u and v. The length of a u-v geodesic is called the *distance* $d_G(u,v)$ between u and v. For every two vertices u and v of G, the symbol $I_G[u,v]$ is used to denote the set consisting of u and v and the vertices lying on any of the u-v geodesics. The set $I_G(u,v)$ is the set $I_G[u,v] \setminus \{u,v\}$. The *geodetic closure* of a subset S of G is the set $I_G[S] = \cup_{u,v\in S} I_G[u,v]$. Also, $I_G(S) = \cup_{u,v\in S} I_G(u,v)$.

The *open neighborhood* of $u \in V(G)$ is given by $N_G(u) = \{v \in V(G) : vu \in E(G)$. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X is the set $N_G(X) = \cup_{u\in X} N_G(u)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$. The *degree* of a vertex v in G is given by $deg_G(v) = |N_G(u)|$. A vertex of a connected graph G is an *extreme or simplicial vertex* if its open neighborhood induces a complete subgraph of G. The set of extreme vertices of G is denoted by $Ext(G)$.

A set $S \subseteq V(G)$ is said to be a *dominating set* of a graph G if for every vertex $v \in V(G) \setminus S$ there exists an element of $w \in S$ such that $vw \in E(G)$, i.e., $N[S] = V(G)$. The smallest cardinality of a dominating set in G is called the *domination number* of G and is denoted by $\gamma(G)$. Any dominating set in G with cardinality $\gamma(G)$ is called a γ-*set* in G.

A set S of vertices in a graph G is a *geodetic set* if $I_G[S] = V(G)$. The minimum cardinality of a geodetic set in G, denoted by $g(G)$, is the *geodetic number* of G. A set $S \subseteq V(G)$ is called a *geodetic dominating set* if S is both a geodetic and a dominating set. The minimum cardinality of a geodetic dominating set in G, denoted by $\gamma_g(G)$, is the *geodetic domination number* of G. Any geodetic dominating set in G with cardinality $\gamma_g(G)$ is called a γ_g-*set* in G.

A set $S \subseteq V(G)$ of a graph G is 2-*path closure absorbing* if for each $x \in V(G) \setminus S$ there exist $u, v \in S$ such that $d_G(u,v) = 2$ and $x \in I_G(u,v)$. The minimum cardinality of a 2-path closure absorbing set in G is denoted by $\rho_2(G)$. Any 2-path closure absorbing set in G with cardinality $\rho_2(G)$ is called a ρ_2-set.

A function $f : V(G) \to \{0,1,2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF f is given by $\omega_G(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. Any RDF f on G with $\omega_G(f) = \gamma_R(G)$ is called a γ_R-function. If f is an RDF on G and $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$, then we denote f by $f = (V_0, V_1, V_2)$. In this case, $\omega_G(f) = |V_1| + 2|V_2|$.

A Roman dominating function $f = (V_0, V_1, V_2)$ on G is a *geodetic Roman dominating function* (or GRDF) if $V_1 \cup V_2$ is a geodetic set in G. The weight of a geodetic Roman dominating function $f = (V_0, V_1, V_2)$ on G is given by $\omega_G^{gR}(f) = |V_1| + 2|V_2|$. The minimum weight of a GRDF on G, denoted by $\gamma_{gR}(G)$, is called the *geodetic Roman domination number* of G. Any GRDF f on G with $\omega_G^{gR}(f) = \gamma_{gR}(G)$ is called a *γ_{gR}-function.*

The *join* of two graphs G and H, denoted by $G + H$, is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(G)\}$, where "$\cup$" refers to a disjoint union of sets.

3. Known Results

We state some results that will be needed in this study.

Remark 1. [9] *Every geodetic set in a graph contains the extreme vertices.*

Remark 2. [11] *Let n be a positive integer. Then*

(*i*) $\gamma_g(C_n) = \lceil \frac{n}{3} \rceil$

(*ii*) $\gamma_g(P_n) = \lceil \frac{n+2}{3} \rceil$.

Theorem 1. [13] *Let G be a connected graph of order $n \geq 2$. Then the following hold:*

(*i*) *$\gamma_g(G) = 2$ if and only if there exists a geodetic set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$.*

(*ii*) *$\gamma_g(G) = n$ if and only if G is the complete graph on n vertices.*

(*iii*) *$\gamma_g(G) = n-1$ if and only if there exists a vertex v in G such that $V(G) \setminus \{v\} \subseteq N_G(v)$ and $G \setminus v$ is the union of at least two complete graphs.*

Remark 3. [24] *Every 2-path closure absorbing set in a connected graph G is a dominating set in G.*

4. Results

Proposition 1. *Let G be a graph of order n and let $f = (V_0, V_1, V_2)$ be a γ_{gR}-function. Then each of the following statements holds:*

(*i*) *$V_1 \cup V_2$ contains all the extreme vertices of G.*

(*ii*) $|V_0| = 0$ *if and only if* $|V_2| = 0$.

(*iii*) *If* $|V_0| = 0$, *then* $\gamma_{gR}(G) = n$.

(*iv*) *If* $|V_1| = 0$, *then* V_2 *is* γ_g*-set of* G *and* $\gamma_{gR}(G) = 2\gamma_g(G)$.

Proof.

(*i*) By Remark 1, $V_1 \cup V_2$ contains all the extreme vertices of G.

(*ii*) Suppose $|V_0| = 0$. Suppose further that $|V_2| \neq 0$. Define $g = (\varnothing, V(G), \varnothing)$. Then $\omega_G^{gR}(g) = n = |V_1| + |V_2| < |V_1| + 2|V_2| = \gamma_{gR}(G)$, a contradiction. Thus, $|V_2| = 0$. The converse is clear.

(*iii*) Suppose $|V_0| = 0$. Then $|V_2| = 0$ by (*ii*). Hence, $\gamma_{gR}(G) = |V_1| = n$.

(*iv*) Suppose $|V_1| = 0$. Then V_2 is a geodetic dominating set of G. Suppose V_2 is not a γ_g-set. Let D be a γ_g-set of G. Then $|D| < |V_2|$. Define $h = (V(G) \setminus D, \varnothing, D)$. Then h is a GRDF on G. Hence, $\omega_G^{gR}(h) = 2|D| < 2|V_2| = \gamma_{gR}(G)$, a contradiction. Thus V_2 is a γ_g-set in G and $\gamma_{gR}(G) = 2|V_2| = 2\gamma_g(G)$. □

Proposition 2. *For any graph* G, $1 \leq \gamma_g(G) \leq \gamma_{gR}(G) \leq \min\{n, 2\gamma_g(G)\}$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{gR}-function. Then $V_1 \cup V_2$ is a geodetic dominating set of G. Hence, $1 \leq \gamma_g(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{gR}(G)$. Now, let $V_0' = V_2' = \varnothing$ and $V_1' = V(G)$. Then $g = (V_0', V_1', V_2')$ is a GRDF on G and $\gamma_{gR}(G) \leq |V_1'| = n$. Finally, let S be a γ_g-set of G. Define $h = (V_0'', V_1'', V_2'')$ by setting $V_2'' = S$, $V_0'' = V(G) \setminus S$, and $V_1'' = \varnothing$. Then h is a GRDF on G and $\gamma_{gR}(G) \leq \omega_G^{gR}(G) = 2|V_2''| = 2\gamma_g(G)$. Therefore, $1 \leq \gamma_g(G) \leq \gamma_{gR}(G) \leq \min\{n, 2\gamma_g(G)\}$. □

Theorem 2. *Let* G *be any graph of order* n. *Then each of the following statements holds.*

(*i*) $\gamma_{gR}(G) = 1$ *if and only if* $G = K_1$.

(*ii*) $\gamma_{gR}(G) = 2$ *if and only if* $G = K_2$ *or* $G = \overline{K}_2$.

(*iii*) $\gamma_{gR}(G) = 3$ *if and only if* $G \in \{K_3, \overline{K}_3, K_1 \cup K_2\}$ *or* $G = \overline{K}_2 + H$ *for some graph* H *of order* $n - 2$.

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{gR}-function on G.

(*i*) Assume that $\gamma_{gR}(G) = 1$. Then $|V_1| = 1$ and $|V_0| = 0$. Hence $G = K_1$.
The converse is clear.

(*ii*) Suppose $\gamma_{gR}(G) = 2$. Then $\omega_G^{gR}(f) = |V_1| + 2|V_2| = 2$. Suppose $|V_2| = 1$. Then $|V_1| = 0$ and $\varnothing \neq V_0 = V(G) \setminus V_2 \subseteq N_G(V_2)$. This implies that V_2 is not a geodetic set in G, a contradiction. Hence, $|V_2| = 0$ and $|V_1| = n$. It follows that $G = K_2$ or $G = \overline{K}_2$.

Conversely, if $G = K_2$ or $G = \overline{K}_2$, $\gamma_{gR}(G)$=2.

(iii) Suppose $\gamma_{gR}(G) = 3$. Then $|V_1| + 2|V_2| = 3$. Hence $|V_2| \leq 1$. Consider the following cases:

Case 1: $|V_2| = 0$
Then $|V_0| = 0$ and $|V_1| = n = 3$. This implies that $G \in \{P_3, K_3, \overline{K}_3, K_1 \cup K_2\}$.

Case 2: $|V_2| = 1$
Then $|V_1| = 1$. Let $V_1 = \{x\}$ and $V_2 = \{y\}$. Then $V_0 = V(G) \setminus \{x, y\} \subseteq N_G(y)$. Since $V_1 \cup V_2$ is a geodetic set, $xy \notin E(G)$. Let $w \in N_G(x)$. Then $w \in V_0$. This implies that $d_G(x, y) = 2$. Since $V_0 \subseteq I_G(x, y)$, $V_0 = N_G(x) \cap N_G(y)$. Let $H = \langle V_0 \rangle$. Then $G = \langle \{x, y\} \rangle + H$ (isomorphic to $\overline{K}_2 + H$).

For the converse, suppose that $G \in \{K_3, \overline{K_3}, K_1 \cup K_2\}$. Then $\gamma_{gR}(G) = 3$. Next, suppose that $G = \overline{K}_2 + H$ for some graph H. Let $V(\overline{K}_2) = \{p, q\}$ and let $V_0 = V(H)$, $V_1 = \{p\}$, and $V_2 = \{q\}$. Then $g = (V_0, V_1, V_2)$ is a GRDF on G. It follows that $\gamma_{gR}(G) \leq \omega_G^{gR}(g) = 3$. By (i) and (ii), $\gamma_{gR}(G) = 3$. □

Lemma 1. *Let G be a graph of order n. Then $\gamma_g(G) = n$ if and only if every component of G is complete.*

Proof. Suppose $\gamma_g(G) = n$. If G is connected, then G is complete by Theorem 1(ii). Suppose G is disconnected with components $G_1, G_2, \ldots, G_k$. Suppose G has a component G_j that is not complete. Then $\gamma_g(G_j) < |V(G_j)|$ by Theorem 1(ii). Hence, $\gamma_g(G) = \sum_{i=1}^{k} \gamma_g(G_i) < n$, a contradiction. Thus, every component of G is complete.

The converse is clear. □

Theorem 3. *Let G be a graph of order n. Then $\gamma_g(G) = \gamma_{gR}(G)$ if and only if every component of G is complete.*

Proof. Suppose $\gamma_g(G) = \gamma_{gR}(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_{gR}-function on G. Then $\gamma_g(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{gR}(G)$. Since $\gamma_g(G) = \gamma_{gR}(G)$, it follows that $\gamma_g(G) = |V_1| + |V_2| = |V_1| + 2|V_2|$. Consequently, $|V_2| = 0$, $|V_0| = 0$, and $|V_1| = |V(G)|$. Thus, $\gamma_g(G) = n$. By Lemma 1, every component of G is complete.

For the converse, suppose that every component of G is complete. Then $\gamma_g(G) = n$ by Lemma 1. Thus, $\gamma_{gR}(G) = n$ by Proposition 2. □

Proposition 3. *Let n be a positive integer. Then*

$$(i)\ \ \gamma_{gR}(C_n) = \begin{cases} 3, & \textit{if } n = 3 \\ \frac{2n}{3}, & \textit{if } n \equiv 0 (mod\ 3) \\ \frac{2n+1}{3}, & \textit{if } n \equiv 1 (mod\ 3) \\ \frac{2n+2}{3}, & \textit{if } n \equiv 2 (mod\ 3) \end{cases}$$

$$(ii)\ \gamma_{gR}(P_n) = \begin{cases} 1, & if\ n = 1 \\ \frac{2n+3}{3}, & if\ n \equiv 0(mod\ 3) \\ \frac{2n+4}{3}, & if\ n \equiv 1(mod\ 3) \\ \frac{2n+2}{3}, & if\ n \equiv 2(mod\ 3) \end{cases}$$

Proof.

(i) Clearly, $\gamma_{gR}(C_3) = 3$. Let $n \geq 4$ and let $C_n = [v_1, v_2, \ldots, v_n, v_1]$. Consider the following cases:

Case 1: $n \equiv 0(mod\ 3)$

Let $n = 3r$ for some positive integer r. Let $V_1 = \varnothing$, $V_2 = \{v_1, v_4, v_7, \ldots, v_{3r-2}\}$, and $V_0 = V(C_n) \setminus V_2$. Then $f = (V_0, V_1, V_2)$ is a GRDF on C_n. Hence, $\gamma_{gR}(C_n) \leq \omega_{C_n}(f) = 2|V_2| = \frac{2n}{3}$.

Let $g = (V_0', V_1', V_2')$ be a γ_{gR}-function on C_n. Since $\gamma_{gR}(G) \leq \frac{2n}{3}$, it follows that $V_2' \neq \varnothing$. Suppose $|V_1'| = k$. Then $k + 2|V_2'| \leq \frac{2n}{3}$. This implies that $|V_2'| \leq r - \frac{k}{2}$ and $|V_0'| = n - (|V_1'| + |V_2'|) \geq 2r - \frac{k}{2}$. Suppose that $k \geq 1$. Then $|V_2'| \leq r - \frac{k}{2}$ implies that $|V_0'| \leq 2|V_2'| \leq 2r - k$. This contradicts the fact that $|V_0'| \geq 2r - \frac{k}{2} > 2r - k$. Therefore, $k = 0$. By Proposition 1(iv) and Remark 2(i), $\gamma_{gR}(C_n) = \omega^{gR}_{C_n}(g) = \frac{2n}{3}$.

Case 2: $n \equiv 1(mod\ 3)$

Let $n = 3s+1$ for some positive integer s. Let $V_1 = \{v_{3s}\}$, $V_2 = \{v_1, v_4, v_7, \ldots, v_{3s-2}\}$, and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Thus $f = (V_0, V_1, V_2)$ is a GRDF in C_n. Hence, $\gamma_{gR}(C_n) \leq \omega^{gR}_{C_n}(f) = \frac{2n+1}{3}$.

Suppose $g = (V_0', V_1', V_2')$ is a γ_{gR}-function on C_n. Since $\gamma_{gR}(G) \leq \frac{2n+1}{3}$, it follows that $V_2' \neq \varnothing$. Suppose $|V_1'| = k$. Then $k + 2|V_2'| \leq \frac{2n+1}{3}$. Thus $|V_2'| \leq s - \frac{1}{2}(k-1)$ and $|V_0'| = n - (|V_1'| + |V_2'|) \geq 2s - \frac{1}{2}(k-1)$. If $k = 0$, then $|V_2'| \leq s + \frac{1}{2}$ and $|V_0'| \geq 2s + \frac{1}{2}$. Hence, $|V_2'| \leq s$ and $|V_0'| \geq 2s + 1$. This is not possible. Hence $k \geq 1$. Suppose $k \geq 2$. Then $|V_2'| \leq s - \frac{1}{2}(k-1)$ implies that $|V_0'| \leq 2s - (k-1)$. However, $|V_0'| \geq 2s - \frac{1}{2}(k-1) > 2s - (k-1)$, a contradiction. Therefore, $k = 1$ and $\gamma_{gR}(C_n) = \omega^{gR}_{C_n}(g) = \frac{2n+1}{3}$.

Case 3: $n \equiv 2(mod\ 3)$

Let $n = 3t+2$ for some positive integer t. Let $V_1 = \{v_{3t}, v_{3t+1}\}$, $V_2 = \{v_1, v_4, v_7, \ldots, v_{3t-2}\}$, and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Thus $f = (V_0, V_1, V_2)$ is a GRDF in C_n. Hence, $\gamma_{gR}(C_n) \leq \omega^{gR}_{C_n} = \frac{2n+2}{3}$.

Let $g = (V_0', V_1', V_2')$ be a γ_{gR}-function on C_n. Since $\gamma_{gR}(G) \leq \frac{2n+2}{3}$, it follows that $|V_2'| \neq \varnothing$. Suppose $|V_1'| = k$. Then $k + 2|V_2'| \leq \frac{2n+2}{3}$. Thus $|V_2'| \leq t + \frac{2-k}{2}$ and $|V_0'| = n - (|V_1'| + |V_2'|) \geq 2t + \frac{2-k}{2}$. If $k = 0$, then $|V_2'| \leq t + 1$ and $|V_0'| \geq 2t + 1$. Hence, $|V_2'| \leq t$ and $|V_0'| \geq 2t + 1$. If $k = 1$, then $|V_2'| \leq t + \frac{1}{2}$ and $|V_0'| \geq 2t + \frac{1}{2}$. Hence, $|V_2'| \leq t$ and $|V_0'| \geq 2t + 1$, which is not possible. Thus, $k \neq 1$. Suppose $k \geq 3$. Then $|V_2'| \leq t + \frac{2-k}{2}$ implies that $|V_0'| \leq 2|V_2'| \leq 2t + 2 - k$. However, $|V_0'| \geq 2t + \frac{2-k}{2} > 2t + 2 - k$, a contradiction. Therefore, $k = 0$ or $k = 2$. If $k = 0$, by Proposition 1(iv) and Remark

$2(i)$, $\gamma_{gR}(C_n) = \omega^{gR}_{C_n}(g) = \frac{2n+2}{3}$. If $k = 2$, then g is of the same type as the function f defined earlier. Hence, $\gamma_{gR}(C_n) = \omega^{gR}_{C_n}(g) = \frac{2n+2}{3}$.

(ii) Let $P_n = [v_1, v_2, \ldots, v_n]$. Clearly, $\gamma_{gR}(P_1) = 1$. Suppose $n \geq 2$. Consider the following cases:

Case 1: $n \equiv 0(mod\ 3)$
Let $n = 3r$ for some positve integer r. Let $V_2 = \{v_1, v_3, \ldots, v_{3r-2}\}$, $V_1 = \{v_{3r}\}$ and $V_0 = V(P_n) \setminus (V_1 \cup V_2)$. Thus $f = (V_0, V_1, V_2)$ is a GRDF on P_n. Hence, $\gamma_{gR}(P_n) \leq \omega^{gR}_{P_n}(f) = |V_1| + 2|V_2| = 1 + 2(\frac{n}{3}) = \frac{2n+3}{3}$.

Let $g = (V_0', V_1', V_2')$ be a γ_{gR}-function. Since $\gamma_{gR}(G) \leq \frac{2n+3}{3}$, it follows that $V_2' \neq \varnothing$. Suppose $|V_1'| = k$. Then $k + 2|V_2'| \leq \frac{2n+3}{3}$. Thus $|V_2'| \leq r - \frac{1}{2}(k-1)$ and $|V_0'| = n - (|V_1'| + |V_2'|) \geq 2r - \frac{1}{2}(k+1)$. Suppose $k = 0$. Then $|V_2'| \leq r + \frac{1}{2}$ and $|V_0'| \geq 2r - \frac{1}{2}$. This implies that $|V_2'| \leq r$ and $|V_0'| \geq 2r$. Since $|V_1'| = 0$, $|V_0'| < 2|V_2'|$ (as $v_1 \in V_2'$ or $v_n \in V_2'$; hence, at least one of them has only one neighbor in V_0'). Thus, $|V_2'| \leq r$ implies that $|V_0'| < 2r$. This contradicts the fact that $|V_0'| \geq 2r$. Suppose $k = 2$. Then $|V_2'| \leq r - \frac{1}{2}$ and $|V_0'| \geq 2r - \frac{3}{2}$. This implies that $|V_2'| \leq r-1$ and $|V_0'| \geq 2r-1$. This is not possible. Suppose $k \geq 4$. Then $|V_2'| \leq r - \frac{1}{2}(k-1)$ implies that $|V_0'| \leq 2r-(k-1)$. However, $|V_0'| \geq 2r - \frac{1}{2}(k+1) > 2r - (k-1)$, a contradiction. Thus, $k = 1$ or $k = 3$. If $k = 1$, then g is of the same type as the function f defined earlier. Hence $\gamma_{gR}(P_n) = \frac{2n+3}{3}$. If $k = 3$, then we may consider $h = (V_0'', V_1'', V_2'')$ where $V_1'' = \{v_1, v_2, v_{3r}\}$, $V_2'' = \{v_4, v_7, \ldots, v_{3r-2}\}$ and $V_0'' = V(P_n) \setminus (V_1'' \cup V_2'')$. Hence, h is a GRDF on P_n and $\omega^{gR}_{P_n}(h) = \frac{2n+3}{3}$.

Case 2: $n \equiv 1(mod\ 3)$
Let $n = 3s+1$ for some positive integer s. Let $V_1 = \varnothing$ $V_2 = \{v_1, v_4, v_7, \ldots, v_{3s+1}\}$, and $V_0 = V(P_n) \setminus (V_1 \cup V_2)$. Thus $f = (V_0, V_1, V_2)$ is a GRDF in P_n. Hence, $\gamma_{gR}(P_n) \leq \omega^{gR}_{P_n} = 2(\frac{n+2}{3}) = \frac{2n+4}{3}$.

Let $g = (V_0', V_2', V_2')$ be a γ_{gR}-function on P_n. Since $\gamma_{gR}(P_n) \leq \frac{2n+4}{3}$, it follows that $V_2' \neq \varnothing$. Suppose that $|V_1'| = k$. Then $k + 2|V_2'| \leq \frac{2n+4}{3}$. Thus $|V_2'| \leq s - \frac{1}{2}(k-2)$ and $|V_0'| = n - (|V_1'| + |V_2'|) \geq 2s - \frac{k}{2}$. Suppose that $k = 1$. Then $|V_2'| \leq s + \frac{1}{2}$ and $|V_0'| \geq 2s - \frac{1}{2}$. This implies that $|V_2'| \leq s$ and $|V_0'| \geq 2s$. Since $|V_1'| = 1$, $|V_0'| < 2|V_2'|$ (as $v_1 \in V_2'$ or $v_n \in V_2'$; hence, at least one of them has only one neighbor in V_0'). Thus, $|V_2'| \leq s$ implies that $|V_0'| < 2s$. This contradicts the fact that $|V_0'| \geq 2s$. Suppose that $k = 3$. Then $|V_2'| \leq s - \frac{1}{2}$ and $|V_0'| \geq 2s - \frac{3}{2}$. This implies that $|V_2'| \leq s-1$ and $|V_0'| \geq 2s-1$. This is not possible. Suppose $k \geq 5$. Then $|V_2'| \leq s - \frac{1}{2}(k-2)$ implies that $|V_0'| \leq 2s - (k-2)$. However, $|V_0'| \geq 2s - \frac{k}{2} > 2s - (k-2)$, a contradiction. Thus, $k = 0$ or $k = 2$ or $k = 4$. If $k = 0$, by Proposition 1(iv) and Remark 2(ii), $\gamma_{gR}(P_n) = \frac{2n+4}{3}$. If $k = 2$, then we may consider $j = (V_0'', V_1'', V_2'')$ where $V_1'' = \{v_1, v_{3s+1}\}$, $V_2'' = \{v_3, v_6, v_9, \ldots, v_{3s}\}$ and $V_0'' = V(P_n) \setminus (V_1'' \cup V_2'')$. Hence, j is a GRDF on P_n and $\omega^{gR}_{P_n}(j) = \frac{2n+4}{3}$. If $k = 4$, then we may consider $l = (V_0^*, V_1^*, V_2^*)$ where $V_1^* = \{v_1, v_2, v_{3s}, v_{3s+1}\}$, $V_2^* = \{v_4, v_7, \ldots, v_{3s-1}\}$ and $V_0^* = V(P_n) \setminus (V_1^* \cup V_2^*)$. Hence, l is a GRDF on P_n and $\omega^{gR}_{P_n}(l) = \frac{2n+4}{3}$. Therefore, $\gamma_{gR}(P_n) = \frac{2n+4}{3}$.

Case 3: $n \equiv 2(mod\ 3)$
Let $n = 3t+2$ for some positive integer t. Define $V_1 = \{v_1, v_{3t+2}\}$, $V_2 = \{v_3, v_6, v_9, \ldots, v_{3t}\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Thus, $f = (V_0, V_1, V_2)$ is a GRDF in P_n. Hence, $\gamma_{gR}(P_n) \leq \omega_G^{gR}(f) = |V_1| + 2|V_2| = 3 + 2(\frac{n-2}{3}) = \frac{2n+2}{3}$.

Let $g = (V_0, V_1, V_2)$ be a γ_{gR}-function on P_n. Since $\gamma_{gR}(G) \leq \frac{2n+2}{3}$, it follows that $V_2' \neq \varnothing$. Suppose that $|V_1'| = k$. Then $k + 2|V_2'| \leq \frac{2n+2}{3}$. Thus, $|V_2'| \leq t - \frac{1}{2}(k-2)$ and $|V_0'| = n - (|V_1'| + |V_2'|) \geq 2t - \frac{1}{2}(k-2)$. If $k = 0$, then $|V_2'| \leq t+1$ and $|V_0'| \geq 2t+1$. This is not possible. Suppose that $k = 1$. Then $|V_2'| \leq t + \frac{1}{2}$ and $|V_0'| \geq 2t + \frac{1}{2}$. This implies that $|V_2'| \leq t$ and $|V_0'| \geq 2t+1$. This is also not possible. Suppose that $k \geq 3$. Then $|V_2'| \leq t - \frac{1}{2}(k-2)$ implies that $|V_0'| \leq 2t-(k-2)$. However, $|V_0'| \geq 2t - \frac{1}{2}(k-2) > 2t-(k-2)$, a contradiction. Thus, $k = 2$ and $\gamma_{gR}(P_n) = \frac{2n+2}{3}$. □

Theorem 4. *Let $G_1, \ldots, G_k$ $(k \geq 2)$ be the components of G. Then*

$$\gamma_{gR}(G) = \sum_{j=1}^{k} \gamma_{gR}(G_j).$$

Proof. Let $G_1, \ldots, G_k$ be the components of G. For each $j \in \{1, 2, \ldots, k\}$, let $g_j = (V_0^j, V_1^j, V_2^j)$ be a γ_{gR}-functions of G_j. Let $V_0 = \cup_{j=1}^k V_0^j$, $V_1 = \cup_{j=1}^k V_1^j$, and $V_2 = \cup_{j=1}^k V_2^j$. Then $g = (V_0, V_1, V_2)$ is a *GRDF* on G. Hence,

$$\gamma_{gR}(G) \leq \omega_G^{gR}(g) = |V_1| + 2|V_2| = \sum_{j=1}^{k} \gamma_{gR}(G_j).$$

Next, suppose that $f = (V_0, V_1, V_2)$ is a γ_{gR}-function on G. Then $f_j = (V_0^j, V_1^j, V_2^j)$, where $V_0^j = V_0 \cap V(G_j)$, $V_1^j = V_1 \cap V(G_j)$, and $V_2^j = V_2 \cap V(G_j)$, is a GRDF on G_j for each $j \in \{1, 2, \ldots k\}$. Thus, $\gamma_{gR}(G_j) \leq \omega_{G_j}^{gR}(f_j)$ for all $j \in \{1, 2, \ldots, k\}$. Hence, $\sum_{j=1}^k \gamma_{gR}(G_j) \leq \gamma_{gR}(G)$. This establishes the desired equality. □

Proposition 4. *Let G be a graph of order n. If $\gamma_g(G) = n-1$, then $\gamma_{gR}(G) = n$.*

Proof. Suppose $\gamma_g(G) = n-1$. Let $f = (V_0, V_1, V_2)$ be a γ_{gR}-function on G. Since $V_1 \cup V_2$ is a geodetic set, $V_1 \cup V_2 = V(G)$ or $V_1 \cup V_2 = V(G) \setminus \{x\}$ for some $x \in V(G)$. If $V_1 \cup V_2 = V(G)$, then $|V_0| = 0$. Thus, $|V_2| = 0$ and $|V_1| = n$. Hence, $\gamma_{gR}(G) = n$. Suppose $V_1 \cup V_2 = V(G) \setminus \{x\}$ for some $x \in V(G)$. Then $V_0 = \{x\}$. Since f is a γ_{gR}-function, $|V_2| = 1$. Therefore, $\gamma_{gR}(G) = |V_1| + 2|V_2| = (n-2) + 2 = n$. □

Corollary 1. *For any positive integer n, $\gamma_{gR}(K_{1,n}) = n$.*

The next result follows from Theorem 3, Corollary 1, and Theorem 4.

Corollary 2. *Let G be a graph of order n. If every component of G is either complete or a star, then $\gamma_{gR}(G) = n$.*

Proposition 5. *If G is a graph of order $n \geq 5$ and $\gamma_{gR}(G) = n$, then G has no induced subgraph P_5.*

Proof. Suppose G has an induced subgraph $P_5 = [v_1, v_2, v_3, v_4, v_5]$. Define $V_0 = \{v_2, v_4\}$, $V_2 = \{v_3\}$ and $V_1 = V(G) \setminus \{v_2, v_3, v_4\}$. Then $V_1 \cup V_2$ is a geodetic dominating set in G and $V_0 \subseteq N_G(v_3)$. This implies that $f = (V_0, V_1, V_2)$ is a GRDF on G. Thus, $\omega_G^{gR}(f) = |V_1| + 2|V_2| = (n-3) + 2(1) = n - 1$, a contradiction. Therefore, G is P_5-free. □

The converse of Proposition 5 is not true. The cycle C_5 has no induced subgraph P_5 but $\gamma_{gR}(C_5) = 4 \neq 5$ by Proposition 3.

Proposition 6. *Let G be a connected graph such that $\gamma_g(G) \neq \gamma_{gR}(G)$. Then $\gamma_{gR}(G) = \gamma_g(G) + 1$ if and only if one of the following holds:*

- (i) *There exists a vertex v in G such that $V(G) \setminus \{v\} \subseteq N_G(v)$ and $G \setminus v$ is the union of at least two complete graphs.*
- (ii) *There exists a vertex v in G and $S \subseteq V(G)$ such that $S \subseteq N_G(v)$ and $V(G) \setminus S$ is a γ_g-set in G.*

Proof. Suppose $\gamma_g(G) + 1 = \gamma_{gR}(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_{gR}-function. Consider the following cases:
Case 1: $\gamma_g(G) < |V_1| + |V_2|$
Then $\gamma_g(G) + 1 \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| \leq \gamma_{gR}(G)$. The assumption that $\gamma_{gR}(G) = \gamma_g(G) + 1$ implies that $|V_2| = 0$. By Proposition 1(ii), $|V_0| = 0$ and $\gamma_{gR}(G) = n$. It follows that $\gamma_g(G) = n - 1$. By Theorem 1(iii), (i) follows.

Case 2: $\gamma_g(G) = |V_1| + |V_2|$
Then $\gamma_g(G) + 1 = |V_1| + |V_2| + 1 = |V_1| + 2|V_2| = \gamma_{gR}(G)$. Hence, $|V_2| = 1$ and $|V_1| = \gamma_g(G) - 1$. This implies that $|V_0| = |V(G) \setminus (V_1 \cup V_2)| = n - \gamma_g(G)$. Let $V_2 = \{v\}$ and $S = V_0$. Then $S \subseteq N_G(v)$. Moreover, $V(G) \setminus S = V_1 \cup V_2$ is a γ_g-set because it is a geodetic set and $|V_1 \cup V_2| = \gamma_g(G)$. Therefore (ii) holds.

For the converse, suppose first that (i) holds. Let $S = V(G) \setminus \{v\}$. Let $w \in S$. Since $G \setminus v = \langle S \rangle$ is the union of at least two complete graphs, the component C of $G \setminus v$ containing w as a vertex is complete. This implies that $S = Ext(G)$. Now, let C_1 and C_2 be distinct components of $G \setminus v$ and let $x \in V(C_1)$ and $y \in V(C_2)$. Then $v \in I_G(x, y)$. Hence, $S = Ext(G)$ is the unique γ_g-set of G and $\gamma_g(G) = n - 1$. By Proposition 4, we have $\gamma_{gR}(G) = n = \gamma_g(G) + 1$. Next, suppose that (ii) holds. Let $V_0 = S$, $V_2 = \{v\}$ and $V_1 = V(G) \setminus (S \cup \{v\})$. Then $V_1 \cup V_2 = V(G) \setminus S$ is a γ_g-set of G and $V_0 \subseteq N_G(v)$. It follows that $g = (V_0, V_1, V_2)$ is a GRDF on G and

$$\gamma_{gR}(G) \leq \omega_G^{gR}(g) = |V_1| + 2|V_2| = \gamma_g(G) - 1 + 2 = \gamma_g(G) + 1.$$

Since $\gamma_g(G) < \gamma_{gR}(G)$, $\gamma_g(G) + 1 \leq \gamma_{gR}(G)$. Thus, $\gamma_{gR}(G) = \gamma_g(G) + 1$. □

Theorem 5. *Let $G = K_{n_1,\dots,n_k}$ be the complete k-partite graph with $1 \le n_1 \le n_2 \dots \le n_k$ and $|\{n_j : n_j \neq 1\}| \ge 2$. Then*

$$\gamma_{gR}(G) = \min\{n(G)+1, 6\},$$

where $n(G) = \min\{n_j : n_j \ge 2\}$.

Proof. Let $S_{n_1}, S_{n_2}, \dots, S_{n_k}$ be the partite sets in G and let $n(G) = \min\{n_j : n_j \ge 2\}$. Suppose $n(G) = 2$. Then $\gamma_{gR}(G) = 3 = n(G)+1$, by Theorem 2$(iii)$. Next, suppose that $n(G) \ge 3$. Pick $u \in S_n$. Let $V_2 = \{u\}$, $V_0 = V(G) \setminus S_{n(G)}$, and $V_1 = S_{n(G)} \setminus \{u\}$. Then $f = (V_0, V_1, V_2)$ is a GRDF on G. This implies that

$$\gamma_{gR}(G) \le \omega_G^{gR}(f) = (n(G)-1)+2 = n(G)+1.$$

Next, let $V_2^* = \{x, y\}$, $V_1^* = \{w, z\}$, and $V_0^* = V(G) \setminus (V_1^* \cup V_2^*)$ where $x, w \in S_{n_r}$ and $y, z \in S_{n_t}$ where $n_r \neq 1$ and $n_t \neq 1$. Then $f' = (V_0^*, V_1^*, V_2^*)$ is a GRDF on G and $\gamma_{gR}(G) \le \omega_G^{gR}(f') = |V_1^*| + 2|V_2^*| = 2 + 2(2) = 6$. Therefore, $\gamma_{gR}(G) \le \min\{n(G)+1, 6\}$. Now, let $g = (V_0'', V_1'', V_2'')$ be a γ_{gR}-function on G. Suppose that $\gamma_{gR}(G) < n(G)+1 \le 6$. Then $\gamma_{gR}(G) = \omega_G^{gR}(g) = |V_1''| + 2|V_2''| < n(G)+1$. This implies that $|V_2''| \le 2$. If $|V_2''| = 0$, then $|V_0''| = 0$ and $|V_1''| = \sum_{i=1}^{k} n_i \ge 6$, a contradiction. Suppose that $|V_2''| = 1$, say $V_2'' = \{v''\}$. We may assume that $v'' \in S_{n(G)}$. Then $S_{n(G)} \setminus \{v''\} \subseteq V_1''$. This implies that

$$n(G)+1 = |S_{n(G)} \setminus \{v''\}| + 2|V_2''| \le |V_1''| + 2|V_2''| < n(G)+1,$$

a contradiction. Suppose now that $|V_2''| = 2$. Suppose $|V_1''| = 1$. Then $n(G) = 5$. Let $V_2'' = \{p, q\}$ and $V_1'' = \{s\}$. Since $V_1'' \cup V_2''$ is a geodetic set, at least two of the vertices p, q, and s belong to the same partite set, say S_{n_i} where $i \in \{1, 2, \dots, k\}$. Choose any $z \in S_{n_i} \setminus \{p, q, s\}$(such z exists because $n_j \ge n(G) = 5$). Then $z \notin I_G(\{p, q, s\})$, a contradiction. Suppose $|V_1''| = 0$. Then $V_2'' \subseteq S_{n_j}$ for some $j \in \{1, 2, \dots, k\}$. Let $w \in S_{n_j} \setminus V_2''$. Then $w \in V_0'' \setminus N_G(V_2'')$, a contradiction. Hence, $\gamma_{gR}(G) \ge n(G)+1$. The same argument can be used to show that $\gamma_{gR}(G) \ge 6$ if $6 \le n(G)+1$. Accordingly, $\gamma_{gR}(G) = \min\{n(G)+1, 6\}$. □

Example 1. *For any two integers $m, n \ge 2$, $\gamma_{gR}(K_{m,n}) = \min\{m+1, n+1, 6\}$.*

The next result shows that every pair of positive integers (both at least 4) are realizable as the geodetic domination number and geodetic Roman domination number of a connected graph.

Theorem 6. *Let a and b be positive integers such that $4 \le a \le b \le 2a$. Then there exists a connected graph G such that $\gamma_g(G) = a$ and $\gamma_{gR}(G) = b$.*

Proof. Consider the following cases:

Case 1. $a = b$.
Let $G = K_a$. Then $\gamma_g(G) = \gamma_{gR}(G) = a$.

Case 2. $a < b$.

Subcase 1. $b = a + 1$.
Let $G = K_{1,a}$. Then $\gamma_g(G) = a$ and $\gamma_{gR}(G) = a + 1 = b$.

Subcase 2. $b = 2a - 1$.
Let $m = b - a = a - 1$ and let $G = P_{3m}$. Then $\gamma_g(P_{3m+1}) = m + 1 = a$ by Remark 2(ii), and by Proposition 3(ii), $\gamma_{gR}(P_{3m}) = 2m + 1 = 2a - 1 = b$.
Subcase 3. $b = 2a$.
Let $G = C_{3a}$. Then $\gamma_g(G) = \gamma_g(C_{3a}) = \lceil \frac{3a}{3} \rceil = a$ by Remark 2(i) and by Proposition 3(i), $\gamma_{gR}(G) = 2a$.

Subcase 4. $a + 2 \leq b < 2a - 1$
Then $2a - b - 1 \geq 1$, i.e., $2a - b \geq 2$. Let $m = b - a$. Consider the graph G in Figure 1 obtained from $P_{3m-2} = [v_1, v_2, v_3, v_4, \ldots, v_{3(m+1)-2}]$ by adding the edges $v_{3(m+1)-2}w_j$ for each $j \in \{1, 2, \ldots, 2a - b - 1\}$. Let $S_1 = \{v_1, v_4, \ldots, v_{3(m+1)-2}\}$. Then S_1 is a γ_g-set in $P_{3(m+1)-2}$. Hence, $S = \{v_1, v_4, \ldots, v_{3(m+1)-2}, w_1, w_2, \ldots, w_{2a-b-1}\}$ is a γ_g-set in G and $\gamma_g(G) = |S| = (m+1)+2a-b-1 = (b-a+1)+2a-b-1 = a$. Suppose $2a-b-1 = 1$. Then $b = 2a-2$, $m = a-2$. Then $G = P_{3(m+1)-1}$. By Remark 2(ii), $\gamma_g(P_{3(m+1)-1}) = m+2 = a$ and by Proposition 3(ii), $\gamma_{gR}(G) = \gamma_{gR}(P_{3(m+1)-1}) = 2(m + 1) = b$. Next, suppose that $2a - b - 1 \geq 2$. If $S_1 \cap V_2 = \varnothing$, then

$$\gamma_{gR}(G) = \gamma_{gR}(P_{3m+1}) + 2a - b - 1 = 2(m + 1) + a - (m + 1) = m + a = b.$$

Suppose $S_1 \cap V_2 \neq \varnothing$. Since f is a γ_{gR}-function on G, $|S_1 \cap V_2| = 1$ and $v_{3m+1} \in V_0$. Let $v_{3m+2} \in S_1 \cap V_2$. It is routine to show that $g = (V_0', V_1', V_2')$ where $V_1' = V_1 \setminus (S_1 \setminus \{v_{3m+2}\})$, $V_2' = V_2$ and $V_0' = V_0$ is a γ_{gR}-function on P_{3m+2}. Then

$$\gamma_{gR}(G) = \gamma_{gR}(P_{3m+2}) + 2a - b - 2 = 2(m + 1) + a - (m + 2) = b.$$

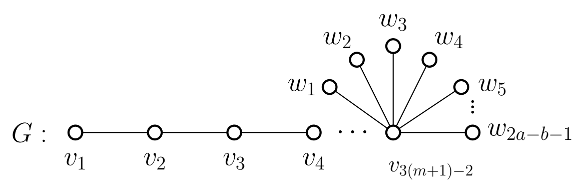

Figure 1: A graph G with $\gamma_g(G) = a$ and $\gamma_{gR}(G) = b$

This proves the assertion. □

Corollary 3. *Let n be a positive integer with $n \geq 2$. Then there exists a connected graph G such that $\gamma_{gR}(G) - \gamma_g(G) = n$. In other words, the difference $\gamma_{gR}(G) - \gamma_g(G)$ can be made arbitrarily large.*

Proposition 7. *Let G and H be non-complete graphs. Then $3 \leq \gamma_{gR}(G + H) \leq 6$.*

Proof. Since $G + H \notin \{K_1, K_2\}$, $\gamma_{gR}(G + H) \geq 3$, by (i) and (ii) of Theorem 2. Pick $u, v \in V(G)$ and $x, y \in V(H)$ such that $uv \notin E(G)$ and $xy \notin E(H)$. Let $V_1 = \{v, y\}$, $V_2 = \{u, x\}$ and $V_0 = V(G + H) \setminus (V_1 \cup V_2)$. Then $f = (V_0, V_1, V_2)$ is a GRDF on $G + H$. Hence, $\gamma_{gR}(G + H) \leq \omega^{gR}_{G+H}(f) = 6$. □

Lemma 2. *Let G and H be non-complete graphs and let $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$, be a geodetic set in $G+H$. Then each of the following statements holds.*

(i) If $|S_G| \geq 2$ and $|S_H| \leq 1$, then S_G is a 2-path closure absorbing set in G.

(ii) If $|S_H| \geq 2$ and $|S_G| \leq 1$, then S_H is a 2-path closure absorbing set in H.

Proof. Suppose $|S_G| \geq 2$ and $|S_H| \leq 1$. If $S_G = V(G)$, then we are done. Suppose that $S_G \neq V(G)$ and let $v \in V(G) \setminus S_G$. Since S is a geodetic set in $G+H$ and $|S_H| \leq 1$, there exist $p, q \in S_G$ such that $v \in I_{G+H}(p, q)$. This implies that $d_G(p, q) = 2$ and $v \in I_G(p, q)$. Hence, S_G is a 2-path closure absorbing set in G, showing that (i) holds. Similarly, (ii) holds. □

Theorem 7. *Let G and H be non-complete graphs. Then $\gamma_{gR}(G + H) = 3$ if and only if $G \in \{\overline{K}_2, \overline{K}_2 + G_1\}$ or $H \in \{\overline{K}_2, \overline{K}_2 + H_1\}$ for some graphs G_1 and H_1.*

Proof. Suppose $\gamma_{gR}(G + H) = 3$. Since G and H are non-complete graphs and $G + H$ is a connected graph, $G + H = \overline{K}_2 + F$ for some non-complete graph F by Theorem 2(iii). Let $\overline{K}_2 = \{a, b\}$. Then $a, b \in V(G)$ or $a, b \in V(H)$. We may assume that $a, b \in V(G)$. Then $G = \overline{K}_2$ or $G = \overline{K}_2 + G_1$ where $G_1 = \langle V(G) \setminus \{a, b\}\rangle$.

Conversely, if $G = \overline{K}_2$, then $\gamma_{gR}(G + H) = 3$. If $G = \overline{K}_2 + G_1$ for some graph G_1, then $G + H = \overline{K}_2 + (G_1 + H)$. By Theorem 2$(iii)$, $\gamma_{gR}(G + H) = 3$. The same conclusion holds when $H \in \{\overline{K}_2, \overline{K}_2 + H_1\}$ for some graph H_1. □

Theorem 8. *Let G and H be non-complete graphs. Then $\gamma_{gR}(G + H) = 3$ if and only if $\rho_2(G) = 2$ or $\rho_2(H) = 2$.*

Proof. Suppose $\gamma_{gR}(G + H) = 3$. By Theorem 7, $\rho_2(G) = 2$ or $\rho_2(G) = 2$.

Conversely, suppose that $\rho_2(G) = 2$ say $S = \{x, y\}$ is a 2-path closure absorbing set in G. If $|V(G)| = 2$, then $G = \overline{K}_2$. Suppose $G \neq \overline{K}_2$. Then for all $u \in V(G) \setminus \{x, y\}$, $d_G(x, y) = 2$ and $u \in I_G(x, y)$. This implies that $G = \{x, y\} + G_1$ for some graph G_1. By Theorem 2(iii), $\gamma_{gR}(G + H) = 3$. Similarly, if $\rho_2(H) = 2$, then $\gamma_{gR}(G + H) = 3$. □

Theorem 9. *Let G and H be non-complete graphs. Then $\gamma_{gR}(G + H) = 4$ if and only if one of the following conditions holds:*

(i) $\rho_2(H) \neq 2$ and there exists a ρ_2-set $\{x, y, z\}$ in G such that $V(G) \setminus \{x, y, z\} \subseteq N_G(x)$.

(ii) $\rho_2(G) \neq 2$ and there exists a ρ_2-set $\{x, y, z\}$ in H such that $V(H) \setminus \{x, y, z\} \subseteq N_H(x)$.

Proof. Suppose $\gamma_{gR}(G+H)=4$. Let $f=(V_0,V_1,V_2)$ be a γ_{gR}-function on $G+H$. Then $|V_1|+2|V_2|=4$. Suppose $|V_2|=0$. Then $|V_1|=|V(G+H)|=4$. Since G and H are non-complete graphs and $\gamma_{gR}(G+H)\neq 3$, this is not possible. Suppose $|V_2|=2$, say $V_2=\{v,w\}$. Then $V_2\subseteq V(G)$ or $V_2\subseteq V(H)$, since V_2 is a geodetic set of $G+H$. Suppose that $V_2\subseteq V(G)$. By Lemma 2, V_2 is a 2-path closure absorbing set. Hence, $\rho_2(G)=2$. By Theorem 8, this implies that $\gamma_{gR}(G+H)=3$, a contradiction. Hence, $|V_2|=1$ and $|V_1|=2$. Assume first that $V_2=\{x\}\subseteq V(G)$. Let $V_1=\{y,z\}$. Since $\gamma_{gR}(G+H)\neq 3$, $\rho_2(G)\neq 2$ and $\rho_2(H)\neq 2$. Hence, $V_1\subseteq V(G)$. Since f is a GRDF on $G+H$, $V(G)\setminus\{x,y,z\}\subseteq N_G(x)$ and $\{x,y,z\}$ is a ρ_2-set in G. This shows that (i) holds. Similarly, (ii) holds if $V_2=\{x\}\subseteq V(H)$.

Conversely, suppose (i) holds. By the preceding result, $\gamma_{gR}(G+H)\neq 3$. Thus, $\gamma_{gR}(G+H)\geq 4$. Let $V_2=\{x\}$, $V_1=\{y,z\}$ and $V_0=V(G+H)\setminus(V_1\cup V_2)$. Then $f=(V_0,V_1,V_2)$ is a GRDF on $G+H$. Hence, $\gamma_{gR}(G+H)\leq \omega_{G+H}^{gR}(f)=4$. Therefore, $\gamma_{gR}(G+H)=4$. The same conclusion holds if (ii) holds.

□

Theorem 10. *Let G and H be non-complete graphs such that $\gamma_{gR}(G+H)\notin\{3,4\}$. Then $\gamma_{gR}(G+H)=5$ if and only if one of the following holds:*

(*i*) $\gamma(G)=1$ *and* $\rho_2(H)=3$

(*ii*) $\gamma(H)=1$ *and* $\rho_2(G)=3$

(*iii*) *There exists nonadjacent vertices $v,w\in V(G)$ and $x,y\in V(H)$ such that $V(G)\setminus\{v,w\}\subseteq N_G(v)$.*

(*iv*) *There exists nonadjacent vertices $v,w\in V(G)$ and $x,y\in V(H)$ such that $V(H)\setminus\{x,y\}\subseteq N_H(x)$.*

(*v*) *There exist $v,w,x,y\in V(G)$ such that $V(G)\setminus\{v,w,x,y\}\subseteq N_G(v)$.*

(*vi*) *There exist $v,w,x,y\in V(H)$ such that $V(H)\setminus\{v,w,x,y\}\subseteq N_H(v)$.*

(*vii*) *There exist $v,w,x\in V(G)$ such that $V(G)\setminus\{v,w,x\}\subseteq N_G(\{v,w\})$.*

(*viii*) *There exist $v,w,x\in V(H)$ such that $V(H)\setminus\{v,w,x\}\subseteq N_H(\{v,w\})$.*

Proof. Let G and H be non-complete graphs such that $\gamma_{gR}(G+H)\neq\{3,4\}$. Suppose that $\gamma_{gR}(G+H)=5$. Let $f=(V_0,V_1,V_2)$ be a γ_{gR}-function on $G+H$. Then $|V_1|+2|V_2|=5$. Suppose that $|V_2|=0$. Then $|V_1|=|V(G+H)|=5$. Since G and H are non-complete graphs and $\gamma_{gR}(G+H)\neq\{3,4\}$, this is not possible. Suppose $|V_2|=1$, say $V_2=\{v\}$. Then $|V_1|=3$. Assume that $V_2=\{v\}\subseteq V(G)$. If $V_1\subseteq V(H)$. Then $V(G)\setminus\{v\}\subseteq N_G(v)$. This implies that $\gamma(G)=1$. Since V_1 is a 2-path closure absoring set in H and $\gamma_{gR}(G+H)\neq 3$, V_1 is a ρ_2-set in H. Hence, $\rho_2(H)=3$ and (i) holds. Suppose $|V_1\cap V(G)|=1$, say $w\in v\cap V(G)$. Then $V(G)\setminus\{v,w\}\subseteq N_G(v)$. Since H is non-complete and $\rho_2(H)\neq 2$, $vw\notin E(G)$. Since $\rho_2(G)\neq 2$, $xy\notin E(H)$. This shows that (iii) holds. Next, suppose that

$|V_1 \cap V(G)| \geq 2$. Since $\gamma_{gR}(G+H) \neq 4$, it follows that $V_1 \subseteq V(G)$, i.e. $|V_1 \cap V(G)| = 3$. Clearly, $V(G) \setminus \{v,w,x,y\} \subseteq N_G(v)$, showing that (v) holds. Suppose now that $|V_2| = 2$, say $V_2 = \{v,w\}$. Then $|V_1| = 1$. Let $V_1 = \{x\}$. Assume that $V_2 \cap V(G) \neq \varnothing$, say $v \in V(G)$. Since $\rho_2(G) \neq 2$ and $\rho_2(H) \neq 2$, $\{v,w,x\} \subseteq V(G)$. Hence $\{v,w,x\}$ is a ρ_2-set in G and $V(G) \setminus \{v,w,x\} \subseteq N_G(\{v,w\})$. This shows that (vii) holds. Similarly, (ii) or (iv) or (vi) or $(viii)$ holds.

The converse is clear. □

Conclusion

This study introduced the notion of geodetic Roman domination. Some properties of geodetic Roman dominating functions were explored and the geodetic Roman domination numbers of certain graphs were determined. It was also shown that any pair of positive integers (subject to a constraint) are realizable as the geodetic domination number and geodetic Roman domination number of a connected graph. This newly defined variant of Roman domination may be investigated further for other graphs including those ones resulting from some binary operations of graphs. One may also try exploring the relationship between this variant and the other variations of Roman domination.

Acknowledgements

The authors would like to thank the referees for the invaluable assistance they gave us through their comments and suggestions which led to the improvement of the paper. The authors are also grateful to the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines and MSU-Iligan Institute of Technology, Philippines for funding this research.

References

[1] M. Adabi, E. Ebrahimi Targhi, N. Jafari Rad, and M. Saied Moradi. Properties of independent Roman domination in graphs. *Australasian Journal of Combinatorics*, 52:11–18, 2012.

[2] H.A. Ahangar, M.A. Henning, V. Samodivkin, and I.G. Yero. Total Roman domination in graphs. *Applicable Analysis and Discrete Mathematics*, 10(2):501–517, 2016.

[3] M.P. Alvarez-Ruiz, T. Mediavilla-Gradolph, S.M. Sheikholeslami, J.C. Valenzuela-Tripodoro, and I.G. Yero. On the strong Roman domination number of graphs. *Discrete Applied Mathematics*, 231:44–59, 2017.

[4] S. Banerjee, J.M. Keil, and D. Pradhan. Perfect Roman domination in graphs. *Theoretical Computer Science*, 796:1–21, 2019.

[5] R.A. Beeler, T.W. Haynes, and S.T. Hedetnieme. Double Roman domination. *Discrete Applied Mathematics*, 211:23–29, 2016.

[6] F. Buckley, F. Harary, and L.V. Quintas. Extremal results on the geodetic number of graphs. *Discrete Applied Mathematics*, 211:23–29, 2016.

[7] G. Cagaanan and S.R. Canoy Jr. On the geodetic covers and geodetic bases of the composition G[Km]. *Ars Combinatoria*, 79:33–45, 2006.

[8] G. Cagaanan and S.R. Canoy Jr. Bounds for the geodetic number of the Cartesian product of graphs. *Utilitas Matematica*, 79:91–98, 2009.

[9] G. Chartrand, F. Harary, and P. Zhang. On the geodetic number of a graph. *Networks: An International Journal*, 39(1):1–6, 2002.

[10] M. Chellali, T.W. Haynes, and S.T. Hedetnieme. Roman {2}-domination. *Discrete Applied Math*, 204:22–28, 2016.

[11] S.R. Chellathurai and S.P. Vijaya. The geodetic domination number for the product of graphs. 3(4):19–30, 2014.

[12] E.J. Cockayne, P.A. Deryer, S.M. Hedetnieme, and S.T. Hedetnieme. Roman Domination in Graphs. *Discrete Mathematics*, 278(13):11–22, 2004.

[13] H. Escuardo, R. Hansberg, A. Jafari Rad, and L. Volkman. Geodetic domination in graphs. *JCMCC-Journal of Combinatorial Mathematics and Combinatorial Computing*, pages 77–89, 2011.

[14] A. Hansberg and L. Volkman. On the geodetic and geodetic domination numbers of a graph. *Discrete Mathematics*, 310(15-16):2140–2146, 2010.

[15] M.A. Henning, W.F. Klostermeyer, and G. MacGillivray. Perfect Roman domination in trees. *Discrete Applied Mathematics*, 236:235–245, 2018.

[16] K. Kammerling and L. Volkman. Roman k-domination in graphs. *Journal of the Korean Mathematical Society*, 46(6):1309–1318, 2009.

[17] M.H. Muddebiha and Sumangaladevi. Connected Roman domination in graphs. *International Journal of Research and Engineering Technology*, 2(10):333–340, 2013.

[18] J.J. Mulloor and V. Sangeetha. Restrained geodetic domination in graphs. *Discrete Mathematics, Algorithms and Applications*, 12(6):2050084, 2020.

[19] P.R.L. Pushpam and T.N.M. Malini Mai. Edge Roman domination in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 69:175–182, 2009.

[20] C.S. Revelle and K.E. Rosing. Defendens imperium romanum: a classical problem in military strategy. *American Mathematical Monthly*, 107(7):585–594, 2000.

[21] C.J. Saromines and S.R. Canoy Jr. Geodetic hop dominating sets in a graph. *European Journal of Pure and Applied Mathematics*, 16(1):5–17, 2023.

[22] I. Stewart. Defend the Roman empire! *Scientific American*, 281(6):136–138, 1999.

[23] P.A.P. Sudhahar, A. Ajitha, and A. Subramanian. Edge geodetic domination number of a graph. *International Journal of Mathematics and its Applications*, 4(3-B):45–50, 2016.

[24] T.L. Tacbobo, F.P. Jamil, and S.R. Canoy Jr. Monophonic and geodetic domination in the join, corona and composition of graphs. *Ars Combinatoria*, (112):13–31, 2013.