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Perfect Equitable Isolate Dominations in Graphs

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Abstract. A subset $S \subseteq V(G)$ is said to be a perfect equitable isolate dominating set of a graph G if it is both perfect equitable dominating set of G and isolate dominating set of G. The minimum cardinality of a perfect equitable isolate dominating set is called perfect equitable isolate domination number of G and is denoted by $\gamma_{pe0}(G)$. A perfect equitable isolate dominating set S of G is called γ_{pe0}-set of G. In this paper, the authors give characterizations of a perfect equitable isolate dominating set of some graphs and graphs obtained from the join and corona of two graphs. Furthermore, the perfect equitable isolate domination numbers of these graphs is determined, and the graphs with no perfect equitable isolate dominating sets are investigated.

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Key Words and Phrases: perfect domination, equitable domination, perfect equitable domination, isolate domination, perfect equitable isolate domination

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1. Introduction

Recently, there has been a growing interest in the applications of one of the widest research topics in graph theory, the study of domination in graphs, which was developed by Claude Berge in 1958 when he introduced the coefficient of external stability known today as domination [2]. Due to the richness of the research and applications to graphs, many variants of domination started to prosper and some of these variants are the isolate domination and perfect equitable domination in graph.

The concept of perfect domination was first introduced by Livingston and Stout [16] as an answer to the problem of the supplement study conducted by the same authors in [15]. This notion has been celebrated for years, and many studies of this kind have been introduced. Caay and Palahang [6] introduced the notion of perfect independent domination of graphs where they joined the notion of perfect domination and independent domination and investigate the existence of such variant and the corresponding number to graph. There are also many variants of perfect dominations of graphs which are found

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in the paper of [16], [10] and [11]. Another variant of domination is the equitable domination graph. The concept of equitable domination was believed to have been introduced by A. Anitha, et.al. in [8] and it was also discussed in the paper of G. Deepak, et.al. in [9]. This concept has extended further and so Caay and Durog in [5] introduced the notion of independent equitable domination in graphs. Furthermore, this concept has also been developed by Caay and Arugay when they introduced the notion of perfect equitable domination in [4] which studied about the domination that is perfect and equitable at the same time. Furthermore, in 2024, Caay in [3] introduced the notion of equitable rings domination in graphs.

In 2013, the concept of Isolate domination in graphs was studied by Hamid and Balamurugan [13]. Because this study gives a lot of opportunity to see many research topics, a lot of mathematician studied many variants of this. Armada and Hamja in [1] studied the perfect isolate domination in graphs where they defined a domination to be perfect and isolate at the same time. Many authors also have made a lot of studies on this different variants and can be found in [17].

In this paper, we study the perfect equitable isolate domination in graphs. A subset $S \subseteq V(G)$ is said to be a perfect equitable isolate dominating set if it is a isolate dominating set and if it is perfect equitable dominating set. To give clarity, the flow of our paper is as follows: in Section 2, we introduce the necessary notations and basic concepts that are used in this study. We also introduce the isolate domination and perfect equitable dominations, and some of their results from the references that are used in the discussion of the study. We also established the case when these two dominations imply each other and so we come up with our formal working definition. In Section 3, we show our results of our study.

2. Preliminaries and the working definitions

Throughout this paper, the graph we consider here is a connected simple graph. That means, there are no loops and multiple edges. A pair $G = (V(G), E(G))$ is called a graph (on V). The elements of $V(G)$ are called the vertices of G and the elements of $E(G)$ are called the edges of G. If no confusion arises, we can use V and E to denote the set of vertices and set of edges of G, respectively. Suppose $v \in V$, the neighborhood of v is the set $N_G(v) = \{u \in V : uv \in E.\}$. Given $D \subseteq V$, the set $N_G(D) = N(D) = \bigcup_{v \in D} N_G(v)$ and the set $N_G[D] = N[D] = D \bigcup N(D)$ are the *open neighborhood* and the *closed neighborhood* of D respectively. In this paper, we denote $\Delta(G)$ and $\delta(G)$ to be the minimum and maximum degree of G, respectively. We denote P_n, C_n, K_n, T_n for the path graph, cycle graph, complete graph and trees of order n, respectively.

Theorem 1. *[7] A graph G is a cycle graph if and only if every vertex of G is adjacent to two other vertices.*

Definition 1. *[7] A* ***spanning subgraph*** *of a graph G is a subgraph obtained by deleting some edges of G with the same vertex set.*

Example 1. *A cycle C_n is a spanning subgraph of a complete graph K_n.*

The following are the definitions of the binary operations in graphs used in this study: join, corona and cartesian product.

Definition 2. *[14] The* ***join*** *$G + H$ of the two graphs G and H is the graph with vertex set*

$$V(G + H) = V(G) + V(H)$$

and the edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv \ : \ u \in V(G), \ v \in V(H)\}.$$

Definition 3. *[12] The* ***corona*** *$G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H.*

In [2], a subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. That is, $N[S] = V(G)$. The minimum cardinality of the dominating set S of G is called a *domination number* of G and is denoted by $\gamma(G)$. In this case, S is called $\boldsymbol{\gamma}$*-set* of G.

In [16], a dominating set S of G is said to be a *perfect dominating set* of G if every vertex $v \in V(G) \setminus S$ is dominated by exactly one vertex $u \in S$. The minimum cardinality of a perfect dominating set S of G is called a *perfect domination number* of G and is denoted by $\gamma_p(G)$. In this case, we say S a γ_p-set of G.

In [8] and [9], a dominating set S of G is said to be an *equitable dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ with $uv \in E(G)$ such that $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of an equitable dominating set S of G is called an *equitable domination number* of G and is denoted by $\gamma_e(G)$. In this case, we say S a γ_e-set of G.

Caay and Arugay in [4] introduced the notion of perfect equitable domination. A dominating set S of G is said to be a *perfect equitable dominating set* of G if for it is both perfect and equitable dominating set. The minimum cardinality of a perfect equitable dominating set S of G is called a *perfect equitable domination number* of G and is denoted by $\gamma_{pe}(G)$. In this case, we say S a γ_{pe}-set of G.

Finally, in [13], a dominating set $S \subseteq V(G)$ is said to be an *isolate dominating set* of G if there exists $u \in S$ such that $uv \notin E(G)$ for all $v \in S$. The minimum cardinality of an equitable dominating set S of G is called an *isolate domination number* of G and is denoted by $\gamma_0(G)$. In this case, we say S a γ_0-set of G.

Theorem 2. *[13] A dominating set S of G is a minimal dominating set if and only if for every $u \in S$, u is an isolate of $\langle S \rangle$. In particular, $S = \{u\}$ is a γ-set of G if and only if S is a γ_0-set of G.*

Theorem 3. *[16] If $\Delta(G) = n - 1$ for any graph G of order n, then $\gamma_p(G) = 1$. In other words, $S = \{u\}$ is a γ-set of G if and only if S is a γ_p-set of G.*

Theorem 4. *[4] Given a path P_n and cycle C_n, $n \geq 3$, $\gamma_{pe}(P_n) = \gamma_{pe}(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Moreover, in path P_n and cycle C_n, consecutive vertices of γ_{pe}-sets are either adjacent or at a distance 3 apart*

It is natural to ask if what is the relationship of the perfect equitable domination and the isolate domination in graphs in terms of cardinality. The result is negative in general. There is no general way to determine which one is larger. However, the following results will tell about the idea between the perfect equitable versus the perfect equitable isolate and the isolate versus perfect equitable isolate. Following the above definitions, we have the following results which are very obvious.

Theorem 5. *If $\gamma_{pe}(G) = k$ for some positive integer k and S is a γ_{pe}-set of G such that $\langle S \rangle$ has an isolated vertex, then $\gamma_{pe0}(G) = k$.*

Theorem 6. *If $\gamma_0(G) = k$ for some positive integer k and S is a γ_0-set of G such that every vertex $v \in V(G) \setminus S$ is dominated by exactly one vertex in S, and $u \in S$, there exists $v \in V(G) \setminus S$ with $uv \in E(G)$ such that $|\deg(u) - \deg(v)| \leq 1$, then $\gamma_{pe0}(G) = k$.*

Theorems 5 and 6 give rise to the definition of the our working definition. They simply tell that a dominating set that is a perfect equitable dominating set and an isolate dominating set, then it is a perfect equitable isolate dominating set.

Definition 4. *A dominating set $S \subseteq V(G)$ is said to be a* ***perfect equitable isolate dominating set (PEID)*** *of G if it is both perfect equitable isolate dominating set. The minimum cardinality of a perfect equitable isolate dominating set S of G is called a perfect equitable isolate domination number of G and is denoted by $\gamma_{pe0}(G)$. In this case, we say S a γ_{pe0}-set of G. Also, if $u \in S$ such that $uv \in E(G)$ for some $v \in V(G) \setminus S$, then either u is said to* ***PEIDly-dominate*** *v, or v is* ***PEIDly-dominated*** *by u.*

Example 2. *Consider the graph in Figure 1. Note that the set $\{u_1, u_5\}$ and $\{u_1, u_8\}$ are γ-sets. For $\{u_4, u_5\}$, note that $N_G(u_4) = \{u_1, u_2, u_3, u_5\}$ and $N_G(u_5) = \{u_4, u_6, u_7, u_8\}$. Thus, $N_G(u_4) \cap N_G(u_5) = \varnothing$. Thus, $\{u_4, u_5\}$ is a γ_p-set. Also, observe that*

$$|\deg(u_4) - \deg(u_1)| \leq 1$$
$$|\deg(u_4) - \deg(u_2)| \leq 1$$
$$|\deg(u_4) - \deg(u_3)| \leq 1$$
$$|\deg(u_5) - \deg(u_6)| \leq 1$$

$$|\deg(u_5) - \deg(u_7)| \leq 1$$
$$|\deg(u_5) - \deg(u_8)| \leq 1.$$

Thus, $\{u_4, u_5\}$ is γ_e-set implying that it is a γ_{pe}-set. However, $u_4u_5 \in E(G)$. This means that $\{u_4, u_5\}$ is not a γ_0, and so it is not a γ_{pe0}-set.

Now for the γ-set $\{u_1, u_8\}$, $N_G(u_1) = \{u_2, u_3, u_4\}$ and $N_G(u_8) = \{u_5, u_6, u_7\}$. Thus, $N_G(u_1) \cap N_G(U_8) = \varnothing$. This means that $\{u_1, u_8\}$ is a γ_p-set. Also, observe that

$$|\deg(u_1) - \deg(u_2)| \leq 1$$
$$|\deg(u_1) - \deg(u_3)| \leq 1$$
$$|\deg(u_1) - \deg(u_4)| \leq 1$$
$$|\deg(u_8) - \deg(u_5)| \leq 1$$
$$|\deg(u_8) - \deg(u_6)| \leq 1$$
$$|\deg(u_8) - \deg(u_7)| \leq 1.$$

Thus, $\{u_1, u_8\}$ is γ_e-set implying that it is a γ_{pe}-set. Also, $u_1u_8 \notin E(G)$ and so $\{u_1, u_8\}$ is γ_0-set. Therefore, $\{u_1, u_8\}$ is a γ_{pe0}-set. Consequently, $\gamma_{pe0}(G) = 2$.

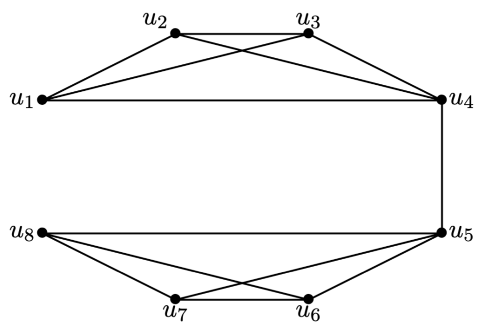

Figure 1: Example of γ_{pe0}-set in a graph G.

The following propositions follow directly from Definition 4.

Proposition 1. *Let S be a γ_0-set of G. Then S is a γ_{pe0}-set if and only if for every $v \in V(G) \setminus S$, $N_G(v) \cap S = \{u\}$ for some $u \in S$ and for every $v \in V(G) \setminus S$, there exists $u \in S$ with $uv \in E(G)$ such that $|\deg(u) - \deg(v)| \leq 1$*

Proposition 2. *If S is γ-set, or a γ_{pe}-set, or a γ_0-set of G with $|S| = 1$, then S is γ_{pe0}-set of G. In particular, $\gamma(G) = \gamma_{pe}(G) = \gamma_0(G) = 1$ if and only if $\gamma_{pe0}(G) = 1$.*

3. PEID in Some Graphs

In this section, we present the results for Equitable Isolate dominations in graphs. The minimality of a perfect equitable isolate dominating set S follows from the paper of [8],

[4] and [16], with the additional property that it acquires at least one vertex in S that is not adjacent to the other vertices in S [13].

Proposition 3. *Given a path P_n and cycle C_n, $n \geq 6$, $\gamma_{pe0}(P_n) = \gamma_{pe0}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof. The proof follows from Theorem 4. □

Corollary 1. *There does not exist γ_{pe0}-set of C_4 and C_5.*

Theorem 7. *Let G be any connected graph of order $n \geq 2$. If $\gamma_{pe0}(G) = 1$, then $\Delta(G) = n-1$. Conversely, if $\Delta(G) = n-1$ and $\delta(G) \geq n-2$, then $\gamma_{pe0}(G) = 1$.*

Proof. Suppose that $\gamma_{pe0}(G) = 1$. Let $S = \{u\}$ be the $\gamma_{pe0} - set$ of G. If G is trivial, then we are done. Assume G is nontrivial. Then every vertex $v \in V(G) \setminus S$ is adjacent to $u \in S$. Hence, $\deg(u) = n-1$. This means that $\Delta(G) = n-1$. Conversely, suppose $\Delta(G) = n-1$ and $\delta(G) \geq n-2$. Then the vertices of G are either of degree $n-1$ or $n-2$. Without loss of generality, take a vertex of degree $n-1$, say $u \in V(G)$. Then u dominates all other vertices of G. Since other vertices of G are either of degree $n-1$ or $n-2$, it follows that for every $v \in V(G)\{u\}$, $|\deg(v) - \deg(u)| \leq 1$. Take $S = \{u\}$ and so it follows that $\{u\}$ is γ_{pe0}-set of G. This proves the claim. □

Corollary 2. *Given a complete graph K_n, $n \geq 3$, $\gamma_{pe0}(K_n) = 1$.*

Proposition 4. *Let $G_{n,m}$ is a complete bipartite graph. If $|n-m| \leq 1$. Then there exists a γ_{pe}-set of G but there does not exist γ_{pe0}-set of G, or there does exists γ_{e0}-set of G, but there does not exist γ_{pe0}-set of G. Moreover, $\gamma_{pe}(G_{n,m}) = 2$ or $\gamma_{pe0}(G_{n,m}) = \min\{n,m\}$.*

Proof. Let P_1 and P_2 be the vertex partitions of a complete bipartite graph G such that $|P_1| = n$ and $|P_2| = m$. Let $u_1 \in P_1$, $i = 1, \cdots, n$ and $v_j \in P_2$, $j = 1, \cdots, m$. Note that u_i dominates v_j for all $u_i \in P_1$ and for all $v_j \in P_2$ with $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Since $|n-m| \leq 1$, $|\deg(u_1) - \deg(v_j)| \leq 1$, for all $u_i \in P_1$ and $v_j \in P_2$.

Note that $u_iu_j \notin E(G)$ for all $i \neq j$ and u_i and u_j are in P_1. Thus, P_1 is a γ_{e0}-set of G. However, every $v_j \in P_2$, $j = 1, \cdots, m$ is dominated by all vertices of P_1, it follows that P_1 is not γ_p-set and so it is not a γ_{pe0}-set of G. Moreover, $\gamma_{e0}(G) = \min\{n,m\}$.

Now suppose we pick one $u_i \in P_1$ and one $v_j \in P_2$. Then every vertices in P_2 is dominated by u_i for some i and every vertices in P_1 is dominated by v_j for some j. Thus, $\{u_i, v_j\}$ is a γ_{pe}-set for some i and j. However, $u_iv_j \in E(G)$ and so $\{u_i, v_j\}$ is not a γ_0-set. Hence, $\{u_i, v_j\}$ is not a γ_{pe0}-set. Moreover, $\gamma_{pe}(G) = 2$.

This proves the claim. □

Theorem 8. *Let $G = G_{P_1,\cdots,P_k}$ be a k-partite graph. Then G has a γ_{pe0}-set if there exists a vertex partition P_j with $|P_j| = 1$ and $|P_k| \leq 2$, for all $i \neq j$. Moreover, $\gamma_{pe0}(G) = 1$.*

Proof. Without loss of generality, let P_1 be such partition with $|P_1| = 1$. Then for partitions P_i with $i \neq 1$, either $|P_i| = 1$ or $|P_i| = 2$. Let $u \in P_1$. Then $\deg(u) \leq 2k$. Also, $\deg(v) \leq 2k+1$ for all $v \neq u$. Thus, $|\deg(v) - \deg(u)| \leq 1$. This means that $\{u\}$ is a γ-set. Since u dominates all vertices of G, $\{u\}$ is also a γ_p-set and so it is a γ_{pe}-set. By Theorem 2, $\{u\}$ is a γ_0-set of G. Therefore, $\{u\}$ is a γ_{pe0}-set of G. Consequently, $\gamma_{pe0}(G) = 1$. □

Theorem 9. *There does not exist γ_{pe0}-set of $G_{P_1,\cdots P_k}$ for any non-trivial partition P_i, $i = 1, \cdots, k$.*

Proof. Suppose on the contrary that there exists a γ_{pe0}-set S of $G = G_{P_1,\cdots,P_k}$, and let $u \in S$ such that $u \in P_k$ for some kth vertex-partition of G. Then u dominates v_i for all $v_i \notin P_k$. Since P_k is non-trivial, there exists $u_j \in P_k$ with $u_j \neq u$ such that u does not dominate u_j. Thus, either $u_j \in S$ or $u_j \notin S$.

If $u_j \in S$, then u_j must dominate v_j for all $v_j \notin P_k$. This is a contradiction to being γ_{pe0}-set since v_j is dominated by u, for all $v_j \notin P_k$. If $u_j \notin S$, then there must be $v_s \notin P_k$ such that $u_jv_s \in E(G)$. But v_s is adjacent to some $v_t \notin P_k$ which are also dominated by u. This is also a contradiction to being γ_{pe0}-set.

Therefore, there does not exist γ_{pe0}-set of $G_{P_1,\cdots P_k}$ for any non-trivial partition P_i, $i = 1, \cdots, k$. □

4. PEID in the Join of Graphs

The following proposition is an obvious result.

Proposition 5. *There does not exist a γ_{pe0}-set of the following graphs below:*

i. *Wheel graph,* $W_n = K_1 + C_{n-1}, \quad n \geq 6$

ii. *Star graph,* $S_n = K_1 + \overline{K_{n-1}}, \quad n \geq 4$

iii. *Fan graph,* $F_n = K_1 + P_{n-1}, \quad n \geq 5$

iv. *Friendship graph,* $Fr_n = K_1 + nP_2, \quad n \geq 2$

v. *Windmill graph* $W_m^n = K_1 + C_{n-1}, \quad n \geq 2, m \geq 3$

Theorem 10. *Let G and H be any graphs of order n and m, respectively, with $\gamma_{pe}(G) = 1$ or $\gamma_{pe}(H) = 1$. Then $\gamma_{pe0}(G + H) = 1$ if and only if either $S_1 = \{u\}$ is a γ_{pe0}-set of G and $\deg(v) \geq m - 2$ for all $v \in V(H)$, or $S_2 = \{x\}$ is a γ_{pe0}-set of H and $\deg(y) \geq n - 2$ for all $y \in V(G)$.*

Proof. Let $\gamma_{pe0}(G + H) = 1$. By Theorem 7, $\Delta(G + H) = (n + m) - 1$. Suppose $S = \{u\} \subseteq V(G)$ be a γ_{pe0}-set of $G + H$. This means that for every $v \in V(G + H)$ with $v \neq u$, we have

$$1 \geq |\deg(u) - \deg(v)|$$

$$\geq |(n+m) - 1 - \deg(v)|$$
$$\geq |n+m-1| - |\deg(v)|.$$

Thus, $\deg(v) \geq (m+n) - 1 - 1 = (m+n) - 2$. This means that $\deg(v) \geq m-2$ on H for all $v \in V(H)$. Similarly, if $S = \{x\} \subseteq V(H)$ is a γ_{pe0}-set of $G+H$, then $\deg(y) \geq n-2$ on G for all $y \in V(G)$.

Conversely, suppose $S_1 = \{u\}$ is a γ_{pe0}-set of G and $\deg(v) \geq m-2$ for all $v \in V(H)$. Since $S_1 = \{u\}$ is a γ_{pe0}-set of G, by Theorem 7, $\Delta(G) = n-1$. This means that $\deg(u) = n-1+m$ in $G+H$. Also, $\deg(v) \geq m-2$ for every $v \in V(H)$ implies that $\deg(v) \geq m-2+n$ in $G+H$. Thus,

$$|\deg(u) - \deg(v)| \leq |(n-1+m) - (m-2-n)| = 1.$$

Hence, $S = \{u\}$ is a γ_{pe0}-set of $G+H$ implying $\gamma_{pe0}(G+H) = 1$. The same argument with the other case. □

The next corollary is a very obvious result as a consequence of Theorem 10.

Corollary 3. *Let G and H be any graphs of degree n and m, respectively. If $\Delta(G) = n-1$ and $\delta(G) \geq n-2$, and $\deg(u) \geq m-2$ for all $u \in V(H)$. Then $\gamma_{pe0}(G+H) = 1$.*

Theorem 11. *Let S_1 and S_2 be the minimal nontrivial γ_{pe0}-sets of G and H, respectively. That is, $|S_1| \neq 1$ and $|S_2| \neq 1$. Then $S_1 \cup S_2$ is a not a γ_{pe0}-set of $G+H$ but a γ_e-set of $G+H$.*

Proof. Let S_1 and S_2 be the minimal nontrivial γ_{pe0}-sets of G and H, respectively. Then for every $u \in V(G) \setminus S_1$, there exists exactly $v \in S_1$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$, and there exists $v_i \in S_1$ such that $v_i v \notin E(G)$ for some $v \in S_1$. Similarly, for every $x \in V(H) \setminus S_2$, there exists exactly $y \in S_2$ such that $xy \in E(G)$ and $|\deg(x) - \deg(y)| \leq 1$, and there exists $y_j \in S_2$ such that $y_j y \notin E(G)$ for some $y \in S_1$. Then $S_1 \cup S_2 := \{v_i, y_j,\ v_i \in S_1,\ y_j \in S_2, \text{ for some } i, j\} \subseteq V(G+H)$. Thus, for all $w \in V(G+H) \setminus (S_1 \cup S_2)$, there exists $z \in S_1 \cup S_2$ such that $wz \in E(G+H)$ and $|\deg(w) - \deg(z)| \leq 1$. Hence, $S_1 \cup S_2$ is a γ_e-set of $G+H$.

Now if $v_i \in S_1$ is an isolated vertex of S_1, then $v_i u_j \in E(G+H)$ for all $u_j \in S_2$, and $v_k u_j \in E(G+H)$, for all $v_k \in S_1$ with $v_i \neq v_k$. This means that v_i is no longer isolated. Since v_i is arbitrary, this holds for all isolated dominating vertices. Hence, $S_1 \cup S_2$ is not γ_{e0}-set of $G+H$. Moreover, for every $u \in V(G) \setminus S_1$, there exists exactly one $v \in S_1$ such that $uv_1 \in E(G)$ However, u is adjacent to vertices of H. This means that u is adjacent to some $w_j \in S_2$. Hence, $S_1 \cup S_2$ is not γ_{p0}-set of $G+H$.

This proves the claim. □

Remark 1. *$S_1 \cup S_2$ is a γ_e-set of $G+H$ of Theorem 11 is not necessarily minimal.*

5. PEID in the Corona of Graphs

Theorem 12. *There does not exist a γ_{pe0}-set of $G \circ H$ for any non-trivial graphs G and H.*

Proof. Suppose on the contrary that there exists a γ_{pe0}-set S of $G \circ H$. Then either $S \subseteq V(G)$ or $S \subseteq V(H)$ or $S \subseteq V(G+H)$. Suppose $S \subseteq V(G)$ and let $u \in S$. Then $u \in V(G)$. This means that the degree of u in $G+H$ is equal to the degree of u in G plus the cardinality of H. Since G is nontrivial, $\deg(u) \geq 1$ in G. Thus, $\deg(u) \geq 1+m$ in $G+H$. But every vertex $v \in V(H)$ has at most $m-1$ degree. Hence, it follows that $|\deg(u) - \deg(v)| \geq 1$, a contradiction. Similarly, assume $S \subseteq V(H)$ and let $w \in S$. Then $w \in V(H)$. This means that the degree of w in $G+H$ is equal to the degree of w in H plus the cardinality of G. Since H is nontrivial, $\deg(w) \geq 1$ in H. Thus, $\deg(w) \geq 1+n$ in $G+H$. But every vertex $z \in V(G)$ has at most $n-1$ degree. Hence, it follows that $|\deg(w) - \deg(z)| \geq 1$, a contradiction. Lastly, suppose $S \subseteq V(G+H)$. Then there exist $u_1, u_2 \in S$ such that $u_1 \in V(G)$ and $u_2 \in V(H)$. Since every vertices in H are adjacent to u_1, this means that there are vertices in H adjacent to both u_1 and u_2, a contradiction. Hence, all of the cases lead to contradiction. Therefore, there does not exist a γ_{pe0}-set of $G \circ H$ for any non-trivial graphs G and H. □

Theorem 13. *Let G and H be any graphs having γ_{pe0}-sets. Then $G \circ H$ does not have γ_{pe0}-set, but $G+H$ has a γ_e-set.*

Proof. Suppose $u \in S^1 \subseteq V(G)$. Then $uv_i \in E(G \circ H)$ for all $v_i \in V(H) \setminus S^2$, where S^2 is a γ_{pe0}-set of H. But $v_iv_j \in E(G \circ H)$ for some $v_j \in S^2$, $i \neq j$. This means that $S^1 \cup S^2$ is no longer γ_p-set. Moreover, by Theorem 12, S^1 and S^2 are no longer γ_{pe0}-sets. Consequently, S^1, S^2 and $S^1 \cup S^2$ are no longer γ_0-sets since the elements are adjacents. Lastly, since every vertices in $V(G \circ H) \setminus (S^1 \cup S^2)$, it follows that $S^1 \cup S^2$ is γ_e-set. This proves the claim. □

Remark 2. *The γ_e-set of $G \circ H$ of Theorem 13 is not necessarily minimal.*

Proposition 6. *Let G be a trivial graph. Then $\gamma_{pe0}(G \circ K_n) = \gamma_{pe0}(K_n \circ G) = 1$.*

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