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Bifurcation and exact traveling wave solutions for the nonlinear Klein–Gordon equation

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Abstract. This paper focuses on the construction of traveling wave solutions to the nonlinear Klein–Gordon equation by employing the qualitative theory of planar dynamical systems. Based on this theory, we analytically study the existence of periodic, kink (anti-kink), and solitary wave solutions. We then attempt to construct such solutions. For this purpose, we apply a well-known traveling wave solution to convert the nonlinear Klein–Gordon equation into an ordinary differential equation that can be written as a one-dimensional Hamiltonian system. The qualitative theory is applied to investigate and describe phase portraits of the Hamiltonian system. Based on the bifurcation constraints on the system parameters, we integrate the conserved quantities to build new wave solutions that can be classified into periodic, kink (anti-kink), and solitary wave solutions. Some of the obtained solutions are clarified graphically and their connection with the phase orbits is derived.

2020 Mathematics Subject Classifications: 35C07, 34C23, 35Q51, 35B10

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1. Introduction

A wave that moves in a certain direction while maintaining a solid shape is called a traveling wave. Traveling waves typically maintain a constant speed throughout their diffusion. These waves are observed in many fields of science, such as in combustion following a chemical reaction. Recently, there has been considerable interest in traveling wave solutions. In mathematical biology[23], the impulses that occur in nerve fibers can be considered as traveling waves, and the conservation laws related to problems in fluid dynamics describe shock profiles as traveling waves. Nonlinear partial differential equations (NLPDEs)[31], which include time as an independent variable, are helpful in characterizing many nonlinear phenomena that occur in fields such as plasma physics, hydrodynamics, solid-state physics, fluid dynamics, optics, and applied mathematics. Work on NLPDEs is becoming very important following evolutions in nonlinear dynamics.

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blackSeveral aspects of engineering and nonlinear science are affected by solitary wave theory. These types of wave can travel large distances with only slight losses. A key component of solitary wave theory is the investigation of exact traveling wave solutions, particularly exact solitary wave solutions. There have been many novel approaches to problem-solving and different varieties of solitary wave solutions over the past decade. In mathematical physics, soliton dynamics is a novel domain with several applications, especially in optics and optical fibers. Solitons are the main components of data transference over long distances using optical fibers[7, 24–31].

blackVarious methods for producing exact solutions of nonlinear mathematical models using a traveling wave transformation have been developed. This typically involves converting the blackNLPDE into ordinary differential equations.[33, 46, 50, 52]

Existing methods can be split into two categories. The first type of approach creates a limited set of exact solutions by simple computations, whereas the second approach, such the symbolic computation method, requires complex computations, but produces abundant wave solutions. The structure of exact solutions for NLPDEs characterizing natural phenomena is essential in describing these phenomena effectively. Solutions can be generated using the Miura transformation [10], Adomian decomposition method [1, 55], bifurcation theory [6, 14–20, 34, 36, 38, 42–45, 53, 62], Cole–Hopf transformation [47], $G^{'}/G$-expansion method and its variants [4, 37, 46, 51, 59–61], Hirota bilinear form [54], Wronskian technique [39], Darboux transformation [35], modified Khater method, auxiliary equation approach, Khater II method [5], transformed rational function method [40], modified simple equation method [3], extended rational sin–cos and sinh–cosh method [2], extended rational method [57], and methods based on numerical solutions [21, 22, 24, 27, 32, 58].

The present article studies the nonlinear Klein–Gordon (KG) equation, which has the form

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0, \tag{1}$$

where α, β are nonzero constants. Equation (1) appears in many physical problems of nonlinear dispersion [49, 56] and nonlinear meson theory [13, 48]. The theory of planar dynamical systems is employed to explore the dynamical behaviors of the traveling wave solutions for Eq. (1).

In the present work, we construct some new traveling wave solutions for the nonlinear KG equation (1). Bifurcation theory is applied to the traveling wave system corresponding to Eq. (1). Bifurcation theory is useful for constructing wave solutions because we can determine the type of solution before the precise form is identified, which enables us to find all possible wave solutions for the equation under consideration. This motivates us to apply this method in the current work.

The blackreminder of this article is structured in the following steps. In Sec. II, to get the traveling wave system we apply a specific transformation. Moreover, we illustrate the system in the form of a Hamiltonian with one degree of freedom. Section III looks at the resulting bifurcations and phase portraits, before Sec. IV derives exact solutions to the KG equation. In Sec. V, we present some new traveling wave solutions and illustrates them graphically. Finally, Sec. VI summarizes the conclusions to this study.

2. Traveling wave solution

Applying the traveling wave transformation

$$\xi = x - \omega t, \qquad u = \varphi(\xi) \tag{2}$$

to Eq. (1) yields

$$\frac{d^2\varphi}{d\xi^2} + \alpha_0\varphi - \beta_0\varphi^3 = 0, \tag{3}$$

where $\alpha_0 = \frac{\alpha}{\omega^2-1}$, $\beta_0 = \frac{\beta}{\omega^2-1}$ with $\omega \neq 1$. Equation (3) can be expressed as a two-dimensional (2D) dynamical system of the form

$$\frac{d\varphi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \beta_0\varphi^3 - \alpha_0\varphi. \tag{4}$$

This is considered a conservative Hamiltonian system with one degree of freedom, which has the Hamiltonian function

$$H = \frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2 + \frac{\alpha_0}{2}\varphi^2 - \frac{\beta_0}{4}\varphi^4. \tag{5}$$

The Hamiltonian function (5) does not depend explicitly on ξ, so it has an energy integral

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2 + \frac{\alpha_0}{2}\varphi^2 - \frac{\beta_0}{4}\varphi^4 = h, \tag{6}$$

where h is an arbitrary parameter that specifies the total energy. Analogous to Hamiltonian mechanics, the first expression in Eq. (6) represents the kinetic energy, whereas the second part,

$$U = \frac{\alpha_0}{2}\varphi^2 - \frac{\beta_0}{4}\varphi^4, \tag{7}$$

is the potential function. The Hamiltonian (5) is a one-dimensional integrable system which physically characterizes the movement of a particle in the plane under the influence of a conservative field with a potential function (7). Thus, Eq. (4) is the Hamiltonian system corresponding to the Hamiltonian function (6).

The solution of the KG equation is equivalent to finding the solution of the Hamiltonian system (4). Including the first equation in Eq. (4) into energy integral (6) and separating the variables, we get the differential shape as follows

$$\frac{\mathrm{d}\varphi}{\sqrt{P_4(\varphi)}} = \mathrm{d}\xi, \tag{8}$$

where $P_4(\varphi)$ is a polynomial in φ of degree 4 , which is given by

$$P_4(\varphi) = 2\left[h - \frac{\alpha_0}{2}\varphi^2 + \frac{\beta_0}{4}\varphi^4\right]. \tag{9}$$

Integrating Eq. (8) requires the range of the parameters α_0, β_0, and h. There are several methods that can be applied to find this range, like the complete discriminant system for the polynomial (9) [41] and bifurcation analysis [6, 14–16, 18–20, 43]. blackBifurcation analysis is more blackapplicable because it affords the zone of those parameters and also determines the sort of solutions before they are constructed by providing the orbits.

3. Bifurcations and phase portraits

In this section, we examine the phase portraits of the traveling wave system using the qualitative theory of differential equations and the bifurcation theory of planar dynamical systems corresponding to the nonlinear KG equation. We now investigate the phase portrait of the KG equation, which is the same as the phase portrait of Eq. (3). Therefore, the equilibrium points should be found. So, we introduce the next theorem.

Theorem 1. *Considering the Hamiltonian system (4) corresponding to the Hamiltonian function (6), we have a maximum of three equilibrium points. Furthermore, if* $\alpha_0\beta_0 \leq 0$*, then* $E_1 = (0, 0)$ *is the unique equilibrium point, whereas if* $\alpha_0\beta_0 > 0$*, there are 3 equilibrium points* $E_1 = (0, 0), E_{2,3} = \left(\pm\sqrt{\frac{\alpha_0}{\beta_0}}, 0\right)$.

black

Proof. The equilibrium points for the Hamiltonian (5) are the critical points for the potential function. Hence, they are $(\varphi_0, 0)$, where φ_0 satisfies

$$\beta_0\varphi^3 - \alpha_0\varphi = 0, \tag{10}$$

which is equivalent to set $\varphi' = y' = 0$ in system (4). Eq.(10) has the solution $\phi_0 = 0, \phi_0 = \pm\sqrt{\frac{\alpha_0}{\beta_0}}$, where the second solution exists if $\alpha_0\beta_0 > 0$. Hence, if $\alpha_0\beta > 0$, there are three equilibrium points E_1 and $E_{2,3}$ while if $\alpha_0\beta_0 < 0$, there is a single equilibrium point E_1.

The determinant of the Jacobian matrix related to the Hamiltonian system (4) is $J(\varphi_0, 0) = -2\alpha_0$. Considering bifurcation theory for planar integrable dynamical systems, the equilibrium point $(\varphi_0,\ 0)$ is a center if $J(\varphi_0, 0) > 0$, a saddle if $J(\varphi_0, 0) < 0$, and a cusp if $J(\varphi_0, 0) = 0$ and its Poincaré index is 0. Using this information, we can characterize the equilibrium points $E_i,\ i = 1, 2, 3$. We now examine the bifurcations and phase forms of the Hamiltonian system (4) for different parameters α_0 and β_0. The energy corresponding to these equilibrium points is

$$h_1 = H(0, 0) = 0, \qquad h_2 = H(\pm\sqrt{\frac{\alpha_0}{\beta_0}}, 0) = \frac{\alpha_0^2}{4\beta_0}. \tag{11}$$

There are two bifurcation curves,

$$L_1 : \alpha_0 = 0, \qquad L_2 : \beta_0 = 0,$$

that split the plane of the parameters $(\alpha_0,\ \beta_0)$ into five regions. These regions are represented as

$$\begin{aligned} R_1 &= \{(\alpha_0,\beta_0)\,|\alpha_0>0,\ \beta_0>0\},\\ R_2 &= \{(\alpha_0,\beta_0)\,|\alpha_0>0,\ \beta_0<0\},\\ R_3 &= \{(\alpha_0,\beta_0)\,|\alpha_0<0,\ \beta_0>0\},\\ R_4 &= \{(\alpha_0,\beta_0)\,|\alpha_0<0,\ \beta_0<0\},\\ R_5 &= \{(\alpha_0,\beta_0)\,|\alpha_0=0,\ \beta_0>0\}\cup\{(\alpha_0,\beta_0)\,|\alpha_0=0,\ \beta_0<0\}. \end{aligned} \tag{12}$$

Considering the abovementioned restrictions on the parameters and employing the MAPLE software to perform the symbolic calculations, we introduce phase portraits for system (4) in the plane $(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}\xi})$.

Case I Assume that $(\alpha_0,\beta_0)\in R_1$, and so $\alpha_0\beta_0>0$. The dynamical system (4) has three equilibrium points $E_i,\ i=1,2,3$. Equilibrium point E_1 is a center point because $J(E_1)=\alpha_0>0$. The other two equilibrium points $E_{2,3}$ are saddle points because $J(E_{2,3})=-2\alpha_0<0$. The phase portrait for this case is summarized in Fig. 1(a).

Case II If $(\alpha_0,\beta_0)\in R_2$, then $\alpha_0\beta_0<0$. Consequently, the dynamical system (4) has a unique equilibrium point E_1. Because $J(E_1)=\alpha_0>0$, E_1 is a center point. The phase portrait for the dynamical system (4) in this case is appeared in Fig. 1(b).

Case III If $(\alpha_0,\beta_0)\in R_3$, we have $\alpha_0\beta_0<0$. Thus, the dynamical system (4) has a unique equilibrium point E_1, and this is a saddle point because $J(E_1)=\alpha_0<0$. The phase portrait for the dynamical system (4) in this case is presented in Fig. 1(c).

Case IV For $(\alpha_0,\beta_0)\in R_4$, we have $\alpha_0\beta_0>0$. Thus, the dynamical system (4) has three equilibrium points $E_i,\ i=1,2,3$. E_1 is a saddle point and $E_{2,3}$ are center points. The phase portrait for this case is appeared in Fig. 1(d).

Case V Finally, for $(\alpha_0,\beta_0)\in R_5$, the dynamical system (4) has a unique equilibrium point E_1. This is a saddle point when $\beta_0>0$, as appeared in Fig. 1(e), and a center point when $\beta_0<0$, as appeared in Fig. 1(f).

3.1. Phase portrait description

The phase orbits are energy level curves that take the form

$$C_h=\{(\phi,y)\in\mathbb{R}^2:H=h\}. \tag{13}$$

For $(\alpha_0,\beta_0)\in R_1$, Fig. 1(a) shows that the Hamiltonian system (4) has boundless families of orbits C_h, appeared in green, brown, and black for $h\in]h_2,\infty[\cup]-\infty,0[\cup\{0\}$, respectively. Of the three families of orbits that appeared in blue, two of them are boundless and lie outside the heteroclinic orbit $\{C_h:h=h_2\}$ appeared in black, while

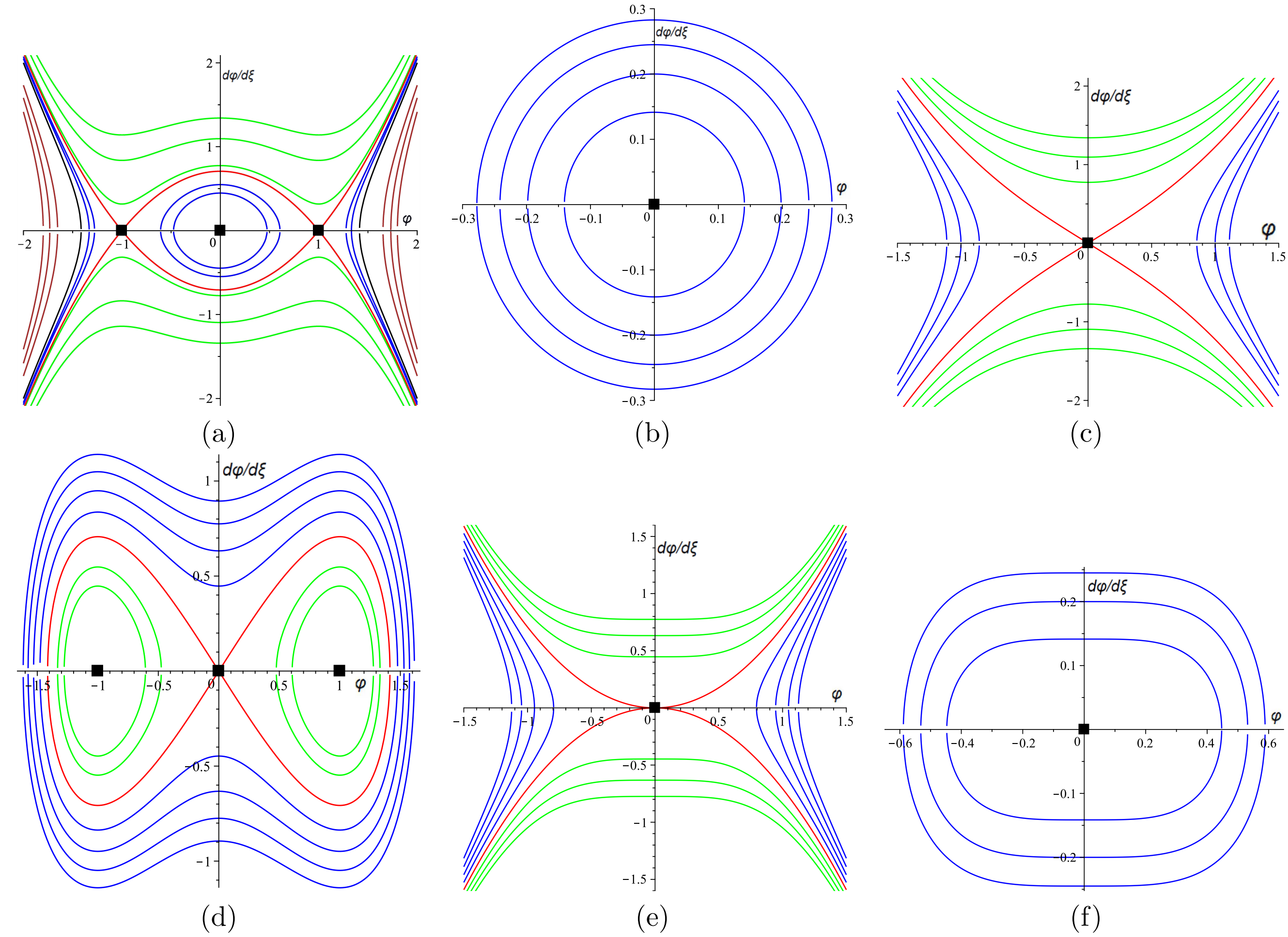


Figure 1: Phase portraits of the one-dimensional Hamiltonian system (4) for different values of α_0, β_0 in the plane $(\varphi, \frac{d\varphi}{d\xi})$. The solid black boxes denote the equilibrium points. (a) $\alpha_0 > 0, \beta_0 > 0$, (b) $\alpha_0 > 0, \beta_0 < 0$, (c) $\alpha_0 < 0, \beta_0 > 0$, (d) $\alpha_0 < 0, \beta_0 < 0$, (e) $\alpha_0 = 0, \beta_0 > 0$, and (f) $\alpha_0 = 0, \beta_0 < 0$.

the remaining one is a bounded periodic orbit that lies inside the heteroclinic orbit. For $(\alpha_0, \beta_0) \in R_2 \cup \{(\alpha_0, \beta_0) \mid \alpha_0 = 0, \beta_0 < 0\}$, the Hamilton system (4) has one family of bounded periodic orbits for $h > 0$, as appeared in Figs. 1(b) and 1(f). If $(\alpha_0, \beta_0) \in R_3 \cup \{(\alpha_0, \beta_0) \mid \alpha_0 = 0, \beta_0 > 0\}$, then all phase orbits are boundless for all amounts of the parameter h, as appeared in Figs. 1(c) and 1(e). If $(\alpha_0, \beta_0) \in R_4$, then all phase orbits of the Hamiltonian system (4) are bounded. For $h > 0$, there is a family of periodic orbits (appeared in blue) that lie outside the homoclinic orbit $C_h : h = 0$ (appeared in black), while for $h \in]h_2, 0[$, there are two families of periodic orbits (appeared in green) that lie inside the homoclinic orbit. black

Lemma 1. *(see, e.g., [8, 9, 41]). Let system* (4) *has a continuous solution* $u = \varphi(x - \omega t) = u(\xi)$ *for* $\xi \in]-\infty, \infty[$ *and assume* $\lim_{\xi\to\infty} u(\xi) = \kappa_1$, *and* $\lim_{\xi\to-\infty} u(\xi) = \kappa_2$

(i) *If $\kappa_1 = \kappa_2$, then the solution $u(\xi)$ is solitary which corresponds to a homoclinic orbit for the system* (4)*.*
(ii) *If $\kappa_1 \neq \kappa_2$, then the solution $u(\xi)$ is a kink (or anti-kink) wave that corresponds to a heteroclinic orbit for the system* (4)*.*
(iii) *If system* (4) *possesses a periodic orbit, then its corresponding solution $u(\xi)$ is also periodic.*
(iv) *If system* (4) *has a closed orbit in the phase portrait evolved by at least two centers and one separatrix layer, then its corresponding solution $u(\xi)$ is a super periodic wave.*

black

Theorem 2. *If the solution of the nonlinear KG equation* (1) *has the form $u(x,t) = \varphi(x - black\omega t)$, then*
(a) it is a periodic wave solution if $(\alpha_0, \beta_0) \in R_1 \cup \{(\alpha_0, \beta_0) \mid \alpha_0 = 0, \beta_0 < 0\}$, or if $(\alpha_0, \beta_0) \in R_2, h \in]0, \infty[$, or if $(\alpha_0, \beta_0) \in R_4, h \in]h_2, 0[\cup]0, \infty[$;
(b) it is a kink (anti-kink) solution if $(\alpha_0, \beta_0) \in R_1, h = h_2$;
(c) it is a solitary wave solution if $(\alpha_0, \beta_0) \in R_4, h = 0$.

black

Proof. Taking into account Lemma 1 and the bifurcation analysis, the theorem is proved directly.

4. Exact traveling wave solutions to the KG equation

Considering the bifurcation constraints on the parameters α_0, β_0, we integrate both sides of Eq. (8) and construct some wave solutions.

4.1. Case of $(\alpha_0, \beta_0) \in R_1$

We first construct exact traveling wave solutions to the KG equation (1) for different energy parameters h.

- On a fixed energy level $h = \frac{\alpha_0^2}{4\beta_0}$, there exists a heteroclinic orbit connecting the two saddle points $E_{2,3}$; see Fig. 1(a). This orbit intersects with the φ axis at the two points $(\pm\frac{\alpha 0}{\beta_0}, 0)$, and so the quartic polynomial (9) takes the form

$$P_4(\varphi) = \frac{\beta_0}{2}\left(\frac{\alpha_0}{\beta_0} - \varphi^2\right)^2. \tag{14}$$

Using Eq. (8), we obtain a kink (or anti-kink) wave solution for Eq. (1) as

$$\varphi(\xi) = \sqrt{\frac{\alpha_0}{\beta_0}} \tanh\left[\sqrt{\frac{\alpha_0}{2}}\xi + c\right], \tag{15}$$

where c is the integration constant.

- On energy level $h \in]0, \frac{\alpha_0^2}{4\beta_0}[$, there is a group of periodic orbits around the center point $E_1 = (0,0)$. This type of orbit always corresponds to the existence of a periodic traveling wave solution to Eq. (1). As appeared in Fig. 1(a), these orbits cut off the φ axis at four points, $(\pm\varphi_1, 0), (\pm\varphi_2, 0)$. Thus, the polynomial (9) takes the form

$$P_4(\varphi) = \frac{\beta_0}{2}(\varphi^2 - \varphi_1^2)(\varphi^2 - \varphi_2^2). \tag{16}$$

  Using Eqs. (16) and (8), we obtain

$$\varphi(\xi) = \varphi_2 \mathrm{sn}\left(\sqrt{\frac{\beta_0}{2}}\varphi_1(\xi + c), \frac{\varphi_2}{\varphi_1}\right), \tag{17}$$

  where $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$ are the Jacobian elliptic functions [11]. Equation (17) is a periodic traveling wave solution with period $\frac{4}{\varphi_1}\sqrt{\frac{2}{\beta_0}}K(\frac{\varphi_2}{\varphi_1})$, where $K(k)$ is an exact elliptic integral of the first type.

- On the zero energy level $h = 0$, there is a group of orbits that cut off with the φ axis at two points if $|\varphi| \geq \sqrt{\frac{2\alpha_0}{\beta_0}}$; see Fig. 1(a). Setting $h = 0$ in the quartic polynomial (9) and using Eq. (8), we obtain

$$\varphi(\xi) = \sqrt{\frac{2\alpha_0}{\beta_0}}\sec(\sqrt{\alpha_0}\xi + c). \tag{18}$$

- At negative energy levels $h < 0$, there is a group of periodic orbits about the center point $E_1 = (0,0)$, as appeared in Fig. 1(a). These orbits indicate the existence of periodic traveling wave solutions. When $h < 0$, the quartic polynomial takes the form

$$P_4(\varphi) = \sqrt{\frac{\beta_0}{2}}\left[(\varphi^2 - \varphi_1^2)(\varphi^2 + \varphi_2)\right], \tag{19}$$

  where $\pm\varphi_1$ and $\pm i\varphi_2$ are the roots of the polynomial (9), $\varphi_2 > \varphi_1$. Inserting Eq. (19) into Eq. (8), we obtain

$$\varphi(\xi) = \frac{\mathrm{cn}\left(\sqrt{\frac{\beta_0}{2}(\varphi_1^2 + \varphi_2^2)}(\xi + c), \frac{\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}\right)}{\mathrm{dn}\left(\sqrt{\frac{\beta_0}{2}(\varphi_1^2 + \varphi_2^2)}(\xi + c), \frac{\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}\right)} black. \tag{20}$$

  This solution represents a periodic traveling wave solution and is illustrated in Fig. 1(a).

4.2. Case of $(\alpha_0, \beta_0) \in R_2$

For $(\alpha_0, \beta_0) \in R_2$, there are groups of periodic orbits about a center point $E_1 = (0,0)$. These orbits indicate the existence of periodic traveling wave solutions, as appeared in

Fig. 1(b). Because $\alpha_0 > 0, \beta_0 < 0$, the energy constant h should be positive, i.e., $h > 0$, and so the quartic polynomial (9) takes the form

$$P_4(\varphi) = 2\left[h - \frac{\alpha_0}{2}\varphi^2 + \frac{\beta_0}{4}\varphi^4\right] = -\frac{\beta_0}{2}(\varphi_1^2 - \varphi^2)(\varphi_2^2 + \varphi^2). \tag{21}$$

This means that the polynomial (21) has two real roots, with the other roots being imaginary. Inserting Eq. (21) into Eq. (8), we obtain

$$\varphi(\xi) = \frac{\varphi_2}{\sqrt{\varphi_1^2 + \varphi_2^2}}\text{sd}\left(\sqrt{-\frac{\beta_0}{2}(\varphi_1^2 + \varphi_2^2)}(\xi + c), \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}}\right). \tag{22}$$

4.3. Case of $(\alpha_0, \beta_0) \in R_3$

In this case, there is a unique equilibrium (saddle) point. The family of orbits for various energy levels is appeared in Fig. 1(c). We now derive an explicit formulation for the traveling wave solution at these energy levels.

- At the zero energy level, there exists an orbit that passes through the equilibrium point, as appeared in Fig. 1(c). In this case, Eq. (8) gives the following solitary traveling wave solution:

$$\varphi(\xi) = \sqrt{-\alpha_0}\text{sech}\left(\sqrt{\frac{-\alpha_0\beta_0}{2}}\xi + c\right). \tag{23}$$

- At positive energy levels $h > 0$, there exists a group of unbounded orbits, as appeared in Fig. 1(c). Under these conditions on α_0, β_0, and h, the quartic polynomial (9) becomes

$$P_4(\varphi) = \frac{\beta_0}{2}(\varphi_1^2 + \varphi^2)(\varphi_2^2 + \varphi^2). \tag{24}$$

  This means that there are four purely imaginary roots. Inserting Eq. (24) into Eq. (8), we obtain

$$\varphi(\xi) = -\varphi_1\frac{\text{sn}\left(\sqrt{\frac{\beta_0}{2}}\varphi_2(\xi + c), \sqrt{1 - \frac{\varphi_1^2}{\varphi_2^2}}\right)}{\text{cn}\left(\sqrt{\frac{\beta_0}{2}}\varphi_2(\xi + c), \sqrt{1 - \frac{\varphi_1^2}{\varphi_2^2}}\right)}. \tag{25}$$

- At negative energy levels, there exists a group of orbits that cut off the φ axis, as appeared in Fig. 1(c). Thus, the polynomial (9) takes the form

$$P_4(\varphi) = \frac{\beta_0}{2}(\varphi^2 - \varphi_1^2)(\varphi^2 + \varphi_2^2). \tag{26}$$

Inserting Eq. (26) into Eq. (8), the parametric representation of the traveling wave solution for Eq. (1) is

$$\varphi(\xi) = \frac{\operatorname{cn}\left(\sqrt{\frac{\beta 0}{2}(\varphi_1^2+\varphi_2^2)}(\xi+c), \frac{\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}\right)}{\operatorname{dn}^2\left(\sqrt{\frac{\beta 0}{2}(\varphi_1^2+\varphi_2^2)}(\xi+c), \frac{\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}\right)}. \tag{27}$$

4.4. Case of $(\alpha_0, \beta_0) \in R_4$

Under these conditions, the phase portrait for the dynamical system (4) has three equilibrium points, in which E_1 is a saddle and $E_{2,3}$ are centers. We now study the traveling wave solutions according to the value of the energy constant h.

- At the zero energy level, there is a certain orbit that repeatedly passes through the saddle point. This is a separatrix closed loop (homoclinic loop). The existence of such an orbit indicates a solitary wave solution for Eq. (1). Figure 1(d) clearly shows that polynomial (9) has 4 real roots, and could be written in the form

$$P_4(\varphi) = -\frac{\beta_0}{2}\varphi^2\left[\frac{2\alpha_0}{\beta_0} - \varphi^2\right]. \tag{28}$$

Using Eqs. (28) and (8), we obtain

$$\varphi(\xi) = \sqrt{\frac{2\alpha_0}{\beta_0}}\operatorname{sech}\sqrt{-\alpha_0}(\xi+c). \tag{29}$$

- At negative energy levels, there exists a group of periodic orbits around the two centers, as appeared in Fig. 1(d). These orbits correspond to periodic traveling wave solutions. Such orbits cut off the φ axis at four points, and so the quartic polynomial (9) has four real roots, $\pm\varphi_1, \pm\varphi_2$. Thus, the polynomial (8) admits the form

$$P_4(\varphi) = -\frac{\beta_0}{2}(\varphi_1^2 - \varphi^2)(\varphi^2 - \varphi_2^2). \tag{30}$$

Using Eqs. (30) and (8), we obtain the periodic traveling wave solution

$$\varphi(\xi) = \frac{\varphi_2}{\operatorname{dn}\left(\sqrt{\frac{-\beta_0}{2}}\varphi_1(\xi+c), \sqrt{1-\frac{\varphi_2^2}{\varphi_1^2}}\right)}. \tag{31}$$

- At positive energy levels, there exists a group of orbits that cut off the φ axis at two points. Thus, the polynomial (9) has 2 real roots $\pm\varphi_1$ due to the symmetry about the $\varphi = 0$ axis [see Fig. 1(d)]. In this case, the polynomial (9) takes the form

$$P_4(\varphi) = \frac{-\beta_0}{2}(\varphi_2^2 + \varphi^2)(\varphi_1^2 - \varphi^2). \tag{32}$$

Employing Eqs. (8) and (32), we obtain

$$\varphi(\xi) = \frac{\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}\operatorname{sd}\left(\sqrt{\frac{-\beta_0}{2}(\varphi_1^2+\varphi_2^2)}(\xi+c), -\frac{\varphi_1}{\sqrt{\varphi_1^2+\varphi_2^2}}\right). \tag{33}$$

4.5. Case of $(\alpha_0, \beta_0) \in R_5$

We set α_0 in the dynamical system (4) and study the following two cases:
Case 1: $\beta_0 > 0$. The phase portrait for the dynamical system (4) is appeared in Fig. 1(e). We study the types of traveling wave solutions for Eq. (1) according to the value of the energy constant h.

- At the zero energy level, i.e., $h = 0$, there exist groups of orbits passing out of the equilibrium point E_1 [see Fig. 1(e)]. Under these conditions, Eq. (8) gives a kink (or anti-kink) traveling wave solution for Eq. (1):

$$\varphi(\xi) = \frac{1}{\sqrt{\frac{\beta_0}{2}}(c - \xi)}. \tag{34}$$

- At negative energy levels, i.e., $h < 0$, there is a group of trajectories that cut off the φ axis at two points, say $(\pm\varphi_1, 0)$. The polynomial (9) takes the form [see Fig. 1(e)]

$$P_4(\varphi) = \frac{\beta_0}{2}(\varphi^2 - \varphi_1^2)(\varphi^2 + \varphi_1^2). \tag{35}$$

  Inserting Eq. (35) into Eq. (8), we obtain

$$\varphi(\xi) = \frac{\varphi_1}{\mathrm{cn}\left(\varphi_1\sqrt{\beta_0}(\xi + c), \frac{1}{\sqrt{2}}\right)}. \tag{36}$$

- At positive energy levels $h > 0$, there are groups of orbits in which there is no intersection with the ϕ axis. Hence, the polynomial (9) has no real roots and has the shape [see Fig. 1(e)]

$$\varphi(\xi) = \frac{\beta_0}{2}(\varphi^2 + \varphi_1^2)^2. \tag{37}$$

  Inserting Eq. (37) into Eq. (8), we obtain

$$\varphi(\xi) = \varphi_1 \tan\left(\sqrt{\frac{\beta_0}{2}}(\xi + c)\right). \tag{38}$$

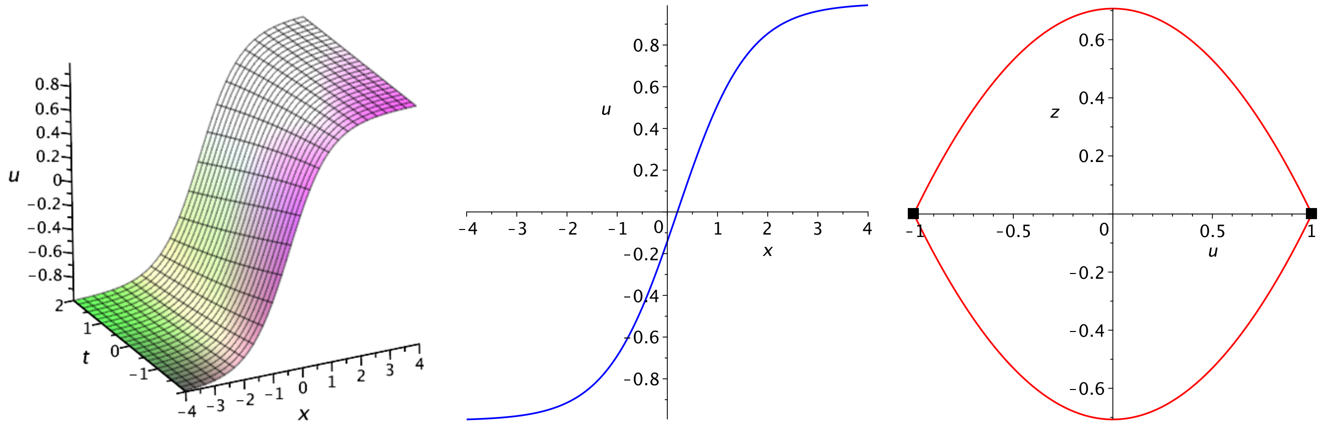
Case 2: $\beta_0 < 0$. On a positive level $h > 0$, the phase form for the dynamical system (4) is as appeared in Fig. 1(f). There exists a group of periodic orbits about the equilibrium point E_1 that cut off with the φ axis at two points, say $(\pm\varphi, 0)$. These orbits indicate the existence of periodic traveling wave solutions. The quartic polynomial (9) takes the form

$$P_4(\varphi) = -\frac{\beta_0}{2}(\varphi_1^2 - \varphi^2)(\varphi_1^2 + \varphi^2). \tag{39}$$

Inserting Eq. (39) into Eq. (8), we obtain the following periodic traveling wave solution:

$$\varphi(\xi) = \frac{1}{\sqrt{2}}\mathrm{sd}\left(\varphi_1\sqrt{-\beta_0}(\xi + c), -\frac{1}{\sqrt{2}}\right). \tag{40}$$

black In the comparison with previous works such as [12, 63], the majority of obtained solutions in the current work are expressed as Jacobi-elliptic functions. Therefore, our results are new.



(a) (b) (c)

Figure 2: Graphical representation of the kink solution (15) for $\alpha_0 = \beta_0 = 1, h = 0.25$. (a) 3D representation, (b) 2D representation, and (c) phase orbit.

5. Graphical representation

black This section presents a graphical illustration of some of the newly obtained solutions for the nonlinear KG equation by introducing three-dimensional (3D) and 2D representations of the solutions. In addition, we provide graphical evidence that the type of solution agrees with the type of phase orbit.

- For $h = 0.25, \alpha_0 = \beta_0 = 1$, the dynamical system (4) has heteroclinic phase orbits, as summarized by Fig. 2(c). Based on Theorem 2, this implies that the KG equation (1) has a kink wave solution, which is described by Figs. 2(a) and 2(b).
- The dynamical system (4) has a homoclinic phase orbit when $\alpha_0 = \beta_0 = -1, h = 0$, as illustrated by Fig. 3(c). Consequently, the KG equation (1) has a solitary wave solution in the form of Eq. (29). The 3D and 2D representations of this solution are appeared in Figs. 3(a) and 3(b).
- The dynamical system (4) possesses a periodic phase orbit when $\alpha_0 = \beta_0 = 1, h = 0.125$, as outlined by Fig. 4(c). Based on Theorem 2, the KG equation (1) has a periodic wave solution in the form of Eq. (17). The 3D and 2D representations of this solution are outlined in Figs. 4(b) and 4(c).

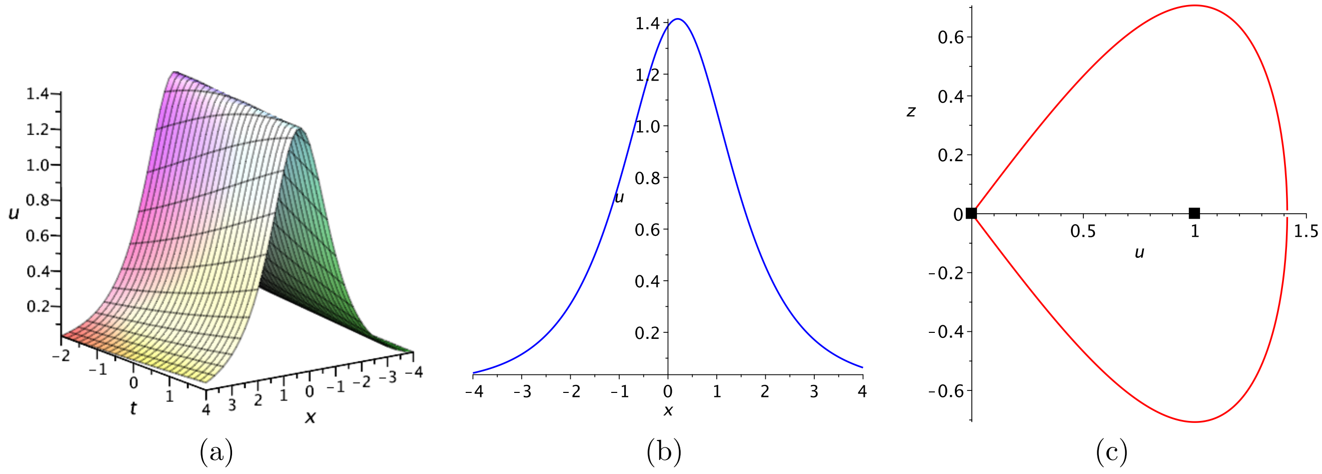


Figure 3: Graphical representation of the solitary wave solution (29) for $\alpha_0 = \beta_0 = -1, h = 0$. (a) 3D representation, (b) 2D representation, and (c) phase orbit.

6. Conclusion

This work has focused on building several traveling wave solutions of the nonlinear KG equation. A familiar wave transformation was used for the KG equation to convert it into an ordinary differential equation in the form of a one-dimensional Hamiltonian system describing the motion of a unit mass particle under the effect of a conservative potential. The qualitative theory for planar dynamical systems was used to study the bifurcations and phase portraits of this system. blackThe application of the qualitative theory for planar dynamical systems has the following advantages over other methods:
(a) It enables us to find the required range of the parameters h, α_0, β_0 that is necessary to integrate both sides of the differential form (8).
(b) It helps us to define the shape of solutions before constructing them because it links the shape of solution with the phase orbit. For example, the existence of periodic, homoclinic, and heteroclinic orbits in the phase plane indicate the presence of periodic, solitary, and kink (or anti-kink) solutions, as outlined by Theorem 2.
(c) It illustrates the reliance of the solutions on the initial conditions over the parameter h, which is determined by the initial conditions. For clarification, if $(\alpha_0, \beta_0) \in R_1$, there are entirely other solutions from the point of view of the mathematical and physical researchers. For example, when $h = \frac{\alpha_0^2}{4\beta_0}$, there is a kink solution, whereas if $h \in]0, \frac{\alpha_0^2}{4\beta_0}[$, there is a periodic solution.

This study has derived some new periodic, kink (anti-kink), and solitary wave solutions. Some of these solutions have been outlined graphically in connection with the corresponding phase orbits.

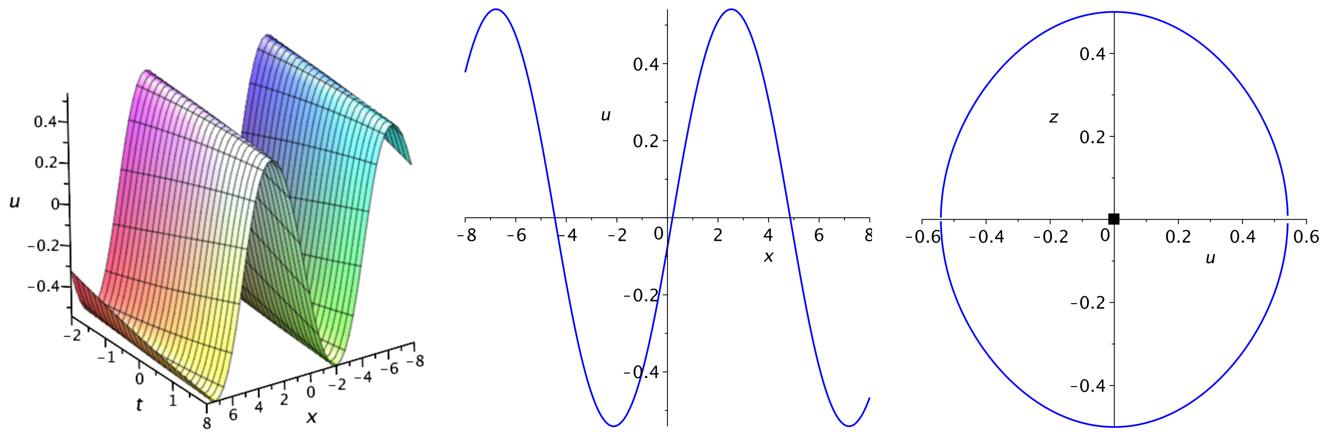


Figure 4: Graphical representation of the periodic wave solution (17) for $\alpha_0 = \beta_0 = 1, h = 0.125$. (a) 3D representation, (b) 2D representation, and (c) phase orbit.

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Competing Interests

The authors declare that they have no competing interests.

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