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Generalized Different Types of Mappings in Fuzzy Bitopological Spaces

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Abstract. The principal objective of this research is to present generalized function ideas including: fuzzy generalized continuity, generalized strong continuity, generalized irresoluteness, generalized open and closed mappings. The last part of our study focuses on homomorphisms in fuzzy bitopological spaces. We also explore the relationships between these concepts, their characteristics, compositions, and important theories, along with some relevant counterexamples.

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Key Words and Phrases: Fuzzy bitopology space ($fbts$), fuzzy generalized closed groups ($(i,j) - g\psi - cld$), fuzzy generalized continuous ($(i,j) - g\psi - conts$), fuzzy generalized irresolute ($(i,j) - g\psi - irresolute$), fuzzy generalized strongly continuous ($(i,j) - g\psi - strongly\, conts$), fuzzy generalized open mapping ($(i,j) - g\psi - open\, mapping$), fuzzy generalized closed mapping ($(i,j) - g\psi - closed\, mapping$).

1. Introduction

Our focus in this study is on fuzzy bitopology filed, that was developed of fuzzy topology and presented for the first time in 1965 by scientist Zadeh [14]. After that, some scientists developed the concepts of fuzzy topology by adapting fundamental ideas from general topology to fuzzy topology. For example, in 1968 Chang created several fuzzy concepts [8]. Then, fuzzy bitopological spaces were introduced by Kandil in 1989[4]. And hence, Balasubramanian and Sundaram created generalized fuzzy closed groups in fuzzy topology space in 1997 [11]. In addition, some scientists presented several studies on generalized closed group in fuzzy space [16, 18, 24]. Many studies about mappings in general topological space have been made by scientists, including [6, 12, 23]. As are some scholars also presented different types of studies on functions in fuzzy topology space [5, 7, 10, 13, 15, 17]. Also, there has been research that showed several mapping forms,

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including irresolute and strong functions, such as [20–22]. There is also a lot of research related to the topic of this manuscript but with practical application in real-life scenarios, such as "A comparison of three types of rough fuzzy sets based on two universal sets" [1]. Also, "On Fuzzy Point Applications of Fuzzy Topological Spaces" [2]. Since, generalized closed sets in fuzzy bitopology spaces are essential for incorporating flexibility, granularity, and uncertainty in the study of fuzzy sets and various topological properties, such as continuity. This use has several advantages, it gives a more precise description of the behavior of fuzzy functions. So, this study aims to introduce and explore a range of generalized function concepts within the framework of fuzzy bitopological spaces. More precisely, in the concepts of fuzzy generalized continuity, generalized strong continuity, generalized irresoluteness, as well as generalized open and closed mappings. Furthermore, we investigate the concept of homomorphism in the context of fuzzy bitopological spaces. Our study not only introduces these concepts but also delves into their interrelationships, characteristics, composition, and highlights important theories and counterexamples. Through this comprehensive review, we contribute to a deeper understanding of these fundamental concepts and their applications in the context of fuzzy bitopological spaces. The study is set up as follows: The history, importance, and related research of the subject are examined in section 1 (introduction). In section 2 (preliminaries), we outline a few key antecedent ideas that are relevant to our research. The concept of generalized continuous concepts is presented in section 3 (Types of Fuzzy Generalized Continuous Mappings), which also discusses them in connection to important theorems and distinctive characteristics. However, the types of strong continuity and irresoluteness functions are defined in section 4 (Types of Fuzzy Generalized Strongly Continuity and Irresolute Mapping), which also examines how these definitions relate to the major theories and some significant examples. We also provided crucial definitions and theorems for open and closed mappings in section 5 (Types of Fuzzy Generalized Open and Closed Mappings). In section 6 (Fuzzy Generalized Homomorphism Mapping) we have presented a definition of homomorphism and reviewed the main theories and their relationship to the above functions. Finally, in section 7 (Conclusion), we compile our findings.

2. Preliminaries

In the next section, we mention a few previous concepts which are fundamental to this study.

Definition 1. *[19] Assume I stands for the unit period $[0, 1]$ and X is not a blank, then:*

(1) *a fuzzy set M is referred to a function with domain X and range I, $M(t) \in (0, 1]$ if $t \in M$, as $M(t) = 0$ when $t \notin M$.*

(2) *M is included in L as shown by $M \subseteq L$ if $M(t) \leq L(t)$, while $t \in X$*

(3) *$M \vee L$ is the combination of groups defined as $(M \vee L)(t) = upper\{M(t), L(t)\}$ $\forall$ $t \in X$.*

(4) *$M \wedge L$ is the intersection that defiend as $(M \wedge L)(t) = lower\{M(t), L(t)\}$ $\forall$ $t \in X$.*

(5) *M^c is the completeness that defiend as $(M(t))^c = 1 - M(t)$, $\forall$ $t \in X$.*

The concepts of fuzzy topology as well as fuzzy bitopological spaces are shown below:

Definition 2. *[19] The pair (X, δ) is consider fuzzy topology if the next three conditions holds:*

1. *$0,\ 1 \in \delta$, since $0(t) = 0,\ 1(t) = 1$, as $t \in X$.*

2. *$M \wedge L \in \delta$, $\forall M, L \in \delta$,.*

3. *$\vee_{i \in I} M_i \in \delta$, $\forall (M_{i \in I}) \in \delta$,.*

The pair (X, δ) is referred to as "fuzzy topology space," or "fts" shortly. Also, the parts of δ are known as open fuzzy groups. When $F \in \delta$, thus F^c is regarded as closed fuzzy group, and the set of all closed fuzzy groups denoted by $\mathcal{F}_\delta$.

Definition 3. *[4] A bitopology fuzzy spaces, often known as fbts, (X, δ_1, δ_2) as X is not empty, and δ_1, δ_2 are fuzzy topological spaces on X. during the course of this research, X conducts fuzzy bitopology (X, δ_1, δ_2) and Y takes (Y, σ_1, σ_2) so that $i \neq j, as\, i, j \in \{1, 2\}$*

Definition 4. *[9]. A fuzzy group μ of X is referred to fuzzy point (singleton) iff $\mu(t) = r, (0 < r \leq 1)$ with a specific $t \in X$, $\mu(h) = 0$ with each elements h of X excluding t, and it is indicated by t_r. Sometimes we refer to t_r as a fuzzy point if $0 < r < 1$. Additionally, $S(X)$ refers to the set of each fuzzy points (singletons) included in X.*

One of the fundamental ideas is the continuous and irresolute mapping, which were defined as:

Definition 5. *[19] If t is a function from (X, δ) to (Y, σ).Then t is fuzzy $\delta-$continuous iff $t^{-1}(W) \in \delta, \forall W \in \sigma$.*

Definition 6. *[23] A function $t : (X, \delta) \longrightarrow (Y, \sigma)$ is known as fuzzy $\alpha - irresolute$ when $t^{-1}(W)$ is fuzzy $\alpha-$open of X on all fuzzy $\alpha-$open W of Y.*

One of the fundamental ideas in the research is the generalized fuzzy closed group, that is known as follows:

Definition 7. *[11] K is named generalized fuzzy closed if closure K is subgroup of R, as K is subgroup of R, which is fuzzy open. i.e., K is generalized fuzzy closed when $cl(K) \leq R$, whatever $K \leq R$, R is fuzzy open.*

In the following sections, we divided the work into four parts: fuzzy $(i, j)-$ generalized ψ continuity, $(i, j)-$ generalized ψ strongly continuity and irresolute, $(i, j)-$generalized ψ open and closed mapping and last part is the fuzzy homomorphism. Also, we apply some theorems, some corollaries. As it includes important examples and diagrams to explain the relations via instructors.

3. Types of Fuzzy Generalized Continuous Mappings

We define and investigate some concepts of fuzzy generalized continuous mapping which includes fuzzy $(i,j)-g\alpha-$conts, $(i,j)-gs-$conts, $(i,j)-gp-$conts, $(i,j)-g\beta-$conts, and we denote for them by $(i,j)-g\psi-$conts.

Definition 8. *Any subgroup K of fbts (X,δ_1,δ_2) is named as:*

(1) *$(i,j)-$generalized $\psi-$closed (simply, $(i,j)-g\psi-cld$) when $\delta_j-\psi-cl(K)\leq W$, while $K\leq W$, $W\in\delta_i$, as ψ containing the kinds (alpha (α), semi (s), pre (p), and beta (β)).*

(2) *$(i,j)-g\psi-open$ is the complement of the group $(i,j)-g\psi-cld$.*

Remark 1. (1) *A class of each fuzzy $(i,j)-g\psi-$open, $(i,j)-g\psi-cld$ of (X,δ_1,δ_2) is represented by $\mathcal{O}^{fg\psi}_{(i,j)}$, $\mathcal{F}^{fg\psi}_{(i,j)}$, and so forth.*

(2) *The class of each $g\psi-$open, $g\psi-cld$ subgroups of X in relation to δ_i represented by $\mathcal{O}^{fg\psi}_i$, and $\mathcal{F}^{fg\psi}_i$, $i=1,2$.*

In the following, we introduce the most important definitions and theories of the concept of generalized continuous:

Definition 9. *A function $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ is named fuzzy $(i,j)-$generalized $\psi-$continuous (briefly, $(i,j)-g\psi-conts$) when the opposite image of all fuzzy open group of (Y,σ_j) is fuzzy $(i,j)-g\psi-open$ group of (X,δ_1,δ_2).*

By using the complement of the above definition we get the coming remark:

Remark 2. (*i*) *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$. Hence t is fuzzy $(i,j)-g\psi-conts$ iff $\forall$ fuzzy closed group V of (Y,σ_j), $t^{-1}(V)$ is fuzzy $(i,j)-g\psi-cld$ group of X.*

(*ii*) *By setting $\delta_i=\delta_j$, $\sigma_i=\sigma_j$ in Definition 9, we find any fuzzy $(i,j)-g\psi-conts$ is fuzzy $g\psi-conts$.*

Theorem 1. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ is fuzzy $(i,j)-g\psi-conts$. Then any fuzzy point x_r in X with $\sigma_j-Q-nbd$ H of $t(x_r)$, $\exists$ fuzzy $(i,j)-g\psi-Q-nbd$ R of x_r as $t(R)\leq H$.*

Proof. Let $x_r\in I^X$ and $H\in N^Q_j(t(x_r))$. Then $\exists\ W\in\sigma_j$ as $t(x_r)\,q\,W\leq H$, and hence $t^{-1}(W)$ is fuzzy $(i,j)-g\psi-open$ in X with $x_r\,q\,t^{-1}(W)\leq t^{-1}(H)$. If we take $t^{-1}(W)=R$, then $\exists\,R\in N^{g\psi Q}_{(i,j)}(x_r)$, as $R\leq t^{-1}(H)$. So $t(R)\leq H$.

By using the relations via $N^{g\psi}_{(i,j)}$, $N^{g\psi Q}_{(i,j)}$ in [3], and the above theorem we get the next corollary:

Corollary 1. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ be fuzzy $(i,j)-g\psi-conts$. Then $\forall\ x_r\in I^X$ and $\forall\ H\in N_j(t(x_r))$, $\exists\ R\in N^{g\psi}_{(i,j)}(x_r)$ as $t(R)\leq H$.*

Theorem 2. *Suppose* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$*. Then the coming claims are hold:*

(1) *If* t *is fuzzy* $(i, j) - g - conts$*, hence it is* $(i, j) - g\alpha - conts$

(2) *If* t *is fuzzy* $(i, j) - g\alpha - conts$*, hence it is* $(i, j) - gp - conts$ *also* $(i, j) - gs - conts$*.*

(3) *If* t *is fuzzy* $(i, j) - gp - conts$ *or* $(i, j) - gs - conts$*, hence it is* $(i, j) - g\beta - conts$*.*

Proof. It is clear from Definition 9, relationships via each kinds of fuzzy (i, j) generalized neigborhoods in the refrence [3].

Remark 3. *The following diagram explain the relation between statements in the above theorem.*

$$\begin{array}{ccccc} (i,j) - g - conts & \longrightarrow & (i,j) - g\alpha - conts & \longrightarrow & (i,j) - gp - conts \\ & & \downarrow & & \downarrow \\ & & (i,j) - gs - conts & \longrightarrow & (i,j) - g\beta - conts \end{array}$$

Figure 1: Presents the relationships via all varieties of fuzzy $(i, j) - g\psi - conts$.

The examples below demonstrate that the reverse implications of Figure (1) are generally not true:

Example 1. *Suppose* E, F*, and* G *are fuzzy subgroups of* $X = \{a, b\}$*. We determine them as:* $E(a, b) = \{0.5, 0.4\}$*,* $F(a, b) = \{0.7, 0.5\}$*, and* $G(a, b) = \{0.4, 0.4\}$*. Consider the fuzzy bitopology* $\delta_1 = \{0, 1, E\}$ *also* $\delta_2 = \{0, 1, F, G\}$ *on* X*. Suppose* N *with* M *are fuzzy subgroups of* $Y = \{r, h\}$ *defined as:* $N(r, h) = \{0.2, 0.5\}$*,* $M(r, h) = \{0.7, 0.6\}$*. Consider the fuzzy bitopology* $\sigma_1 = \{0, 1, N\}$ *with* $\sigma_2 = \{0, 1, M\}$ *on* Y *and* $t(a) = r, t(b) = h$*. One may notice that* t *is fuzzy* $(1, 2) - g\alpha - conts$*, but not fuzzy* $(1, 2) - g - conts$ *as* $t^{-1}(M^c) \leq E \in \delta_1$ *but* $\delta_2 - cl(t^{-1}(M^c)) \nleq E$*.*

The next example clear that $(1, 2) - gp - conts \nRightarrow (1, 2) - g\alpha - conts$.

Example 2. *Suppose* E, F*, and* G *are fuzzy subgroups of* $X = \{a, b\}$*. We determine them as:* $E(a, b) = \{0.7, 0.5\}$*,* $F(a, b) = \{0.6, 0.8\}$*, and* $G(a, b) = \{0.4, 0.3\}$*. Consider the fuzzy bitopology* $\delta_1 = \{0, 1, E\}$ *also* $\delta_2 = \{0, 1, F, G\}$ *on* X*. Suppose* N *with* M *are fuzzy subgroups of* $Y = \{r, h\}$ *defined as:* $N(r, h) = \{0.2, 0.5\}$*,* $M(r, h) = \{0.8, 0.6\}$*. Consider the fuzzy bitopology* $\sigma_1 = \{0, 1, N\}$ *with* $\sigma_2 = \{0, 1, M\}$ *on* Y *and* $t(a) = r, t(b) = h$*. One may notice that* t *is fuzzy* $(1, 2) - gp - conts$*, but not fuzzy* $(1, 2) - g\alpha - conts$ *as* $t^{-1}(M^c) \leq E \in \delta_1$ *but* $\delta_2 - \alpha - cl(t^{-1}(M^c)) \nleq E$*.*

The next example proves $(1, 2) - gs - conts \nRightarrow (1, 2) - g\alpha - conts$.

Example 3. *Suppose $E, F, G,$ and H are fuzzy subgroups of $X = \{a, b\}$. We determine them as: $E(a, b) = \{0.7, 0.5\}$ $F(a, b) = \{0.5, 0.4\}$, $G(a, b) = \{0.4, 0.3\}$, and $H(a, b) = \{0.5, 0.6\}$. Consider the fuzzy bitopology $\delta_1 = \{0, 1, E\}$ with $\delta_2 = \{0, 1, F, G\}$ on X. Suppose N with M are fuzzy subgroups of $Y = \{r, h\}$ defined as: $N(r, h) = \{0.2, 0.5\}$, $M(r, h) = \{0.5, 0.5\}$. Consider the fuzzy bitopology $\sigma_1 = \{0, 1, N\}$ also $\sigma_2 = \{0, 1, M\}$ on Y and $t(a) = r, t(b) = h$. One may notice that t is fuzzy $(1, 2) - gs - conts$, but not fuzzy $(1, 2) - g\alpha - conts$ since $t^{-1}(M^c) \leq E \in \delta_1$ but $\delta_2 - \alpha - cl(t^{-1}(M^c)) \nleq E$.*

The following example clear that $(1, 2) - g\beta - conts \nRightarrow (1, 2) - gs - conts$.

Example 4. *Suppose $E, F, G,$ and H are fuzzy subgroups of $X = \{a, b\}$. We determine them as: $E(a, b) = \{0.5, 0.7\}$ $F(a, b) = \{0.6, 0.5\}$, $G(a, b) = \{0.4, 0.3\}$, and $H(a, b) = \{0.6, 0.5\}$. Consider the fuzzy bitopology $\delta_1 = \{0, 1, E\}$, $\delta_2 = \{0, 1, F, G\}$ on X. Suppose N, M are fuzzy subgroups of $Y = \{r, h\}$ defined as: $N(r, h) = \{0.2, 0.5\}$, $M(r, h) = \{0.5, 0.5\}$. Consider the fuzzy bitopology $\sigma_1 = \{0, 1, N\}$, $\sigma_2 = \{0, 1, M\}$ on Y and $t(a) = r, t(b) = h$. One may notice that t is fuzzy $(1, 2) - g\beta - conts$, but not $(1, 2) - gs - conts$ as $t^{-1}(M^c) \leq E \in \delta_1$ but $\delta_2 - s - cl(t^{-1}(M^c)) \nleq E$.*

The example follow indicates that $(1, 2) - g\beta - conts \nRightarrow (1, 2) - gp - conts$.

Example 5. *Suppose $E, F, G,$ and H are fuzzy subgroups of $X = \{a, b\}$. We determine them as: $E(a, b) = \{0.5, 0.7\}$ $F(a, b) = \{0.4, 0.6\}$, $G(a, b) = \{0.3, 0.4\}$, and $H(a, b) = \{0.6, 0.5\}$. Consider the fuzzy bitopology $\delta_1 = \{0, 1, E\}$, $\delta_2 = \{0, 1, F, G\}$ on X. Suppose N, M are fuzzy subgroups of $Y = \{r, h\}$ defined as: $N(r, h) = \{0.7, 0.5\}$, $M(r, h) = \{0.5, 0.5\}$. Consider the fuzzy bitopology $\sigma_1 = \{0, 1, N\}$, $\sigma_2 = \{0, 1, M\}$ on Y and $t(a) = r, t(b) = h$. One may notice that t is fuzzy $(1, 2) - g\beta - conts$, but not $(1, 2) - gp - conts$ as $t^{-1}(M^c) \leq E \in \delta_1$ but $\delta_2 - p - cl(t^{-1}(M^c)) \nleq E$.*

Theorem 3. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $\delta_j - \psi - conts$. Thus t is fuzzy $(i, j) - g\psi - conts$.*

Proof. Suppose t is fuzzy $\delta_j - \psi - conts$ and $U \in \mathcal{F}_{\sigma_j}$. Then $t^{-1}(U) \in F\psi C(X, \delta_j)$, and hence $t^{-1}(U)$ is fuzzy $(i, j) - g\psi - cld$ of X. Therefore t is fuzzy $(i, j) - g\psi - conts$.

Remark 4. *The inverse of the above theorem is incorrect. The next example is evidenced that: Suppose E, F are fuzzy subgroups of $X = \{a, b\}$. We determine them as: $E(a, b) = \{0.3, 0.4\}$, $F(a, b) = \{0.3, 0.2\}$. Consider the fuzzy bitopology $\delta_1 = \{0, 1, E\}$, $\delta_2 = \{0, 1, F\}$ on X. Suppose N, M are fuzzy subgroups of $Y = \{r, h\}$ defined as: $N(r, h) = \{0.6, 0.7\}$, $M(r, h) = \{0.3, 0.3\}$. Consider the fuzzy bitopology $\sigma_1 = \{0, 1, N\}$, $\sigma_2 = \{0, 1, M\}$ on Y and $t(a) = r, t(b) = h$. One may notice that t is fuzzy $(1, 2) - g - conts$, so by Theorem 2, t is fuzzy $(1, 2) - g\alpha - conts$ but not $\delta_2 - \alpha - conts$ since $\delta_2 - \alpha - cl(t^{-1}(M^c)) \nleq t^{-1}(M^c)$.*

Corollary 2. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $\delta_j - conts$. Then t is $(i, j) - g\psi - conts$.*

Theorem 4. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is injection mapping. Hence the next statements are equivalent:*

(i) $t((i,j)-g\psi-cl(E)) \leq \sigma_j - cl(t(E)), \forall E \in I^X$.

(ii) $(i,j)-g\psi-cl(t^{-1}(F)) \leq t^{-1}(\sigma_j - cl(F)), \forall F \in I^Y$.

(iii) $t^{-1}(\sigma_j - int(F)) \leq (i,j)-g\psi-int(t^{-1}(F)), \forall F \in I^Y$.

Proof.

$(i) \to (ii)$ Suppose $F \in I^Y$. Then $t^{-1}(F) \in I^X$ by (i) we find $t((i,j)-g\psi-cl(t^{-1}(F))) \leq \sigma_j - cl(t(t^{-1}(F))) \leq \sigma_j - cl(F)$. So, $(i,j)-g\psi-cl(t^{-1}(F)) \leq t^{-1}(\sigma_j - cl(F))$.

$(ii) \to (iii)$ Suppose $F \in I^Y$. By (ii) $(i,j)-g\psi-cl(t^{-1}(F)) \leq t^{-1}(\sigma_j - cl(E))$. After that $(t^{-1}(\sigma_j - cl(E)))^c \leq ((i,j)-g\psi-cl(t^{-1}(E)))^c$, and hence $t^{-1}(\sigma_j - int(E)^c) \leq (i,j)-g\psi-int(t^{-1}(E)^c)$.

$(iii) \to (i)$ Suppose $E \in I^X$. Then $t(E) \in I^Y$ by (iii) $t^{-1}(\sigma_j - int(t(E)) \leq (i,j)-g\psi-int(t^{-1}(t(E))$. As t is injection then, $t^{-1}(\sigma_j - int(t(E)) \leq (i,j)-g\psi-int(E)$. After that $(i,j)-g\psi-cl((E)^c) \leq t^{-1}(\sigma_j - cl(t(E)^c)$. So, $t((i,j)-g\psi-cl((E)^c)) \leq t(t^{-1}(\sigma_j - cl(t(E)^c)) \leq \sigma_j - cl(t(E)^c)$.

Theorem 5. *Suppose $t:(X,\delta_1,\delta_2) \to (Y,\sigma_1,\sigma_2)$ is fuzzy $(i,j)-g\psi-conts$. Hence $t((i,j)-g\psi-cl(E)) \leq \sigma_j - cl(t(E))$, for each $E \in I^X$.*

Proof. Assume t is fuzzy $(i,j)-g\psi-conts$ and $E \in I^X$. Then $E \leq t^{-1}(\sigma_j - cl(t(E)))$. As $\sigma_j - cl(t(E)) \in F\mathcal{C}(Y,\sigma_j)$, t is fuzzy $(i,j)-g\psi-conts$, thus $t^{-1}(\sigma_j - cl(t(E)))$ is fuzzy $(i,j)-g\psi-cld$ of X, so $(i,j)-g\psi-cl(E) \leq t^{-1}(\sigma_j - cl(t(E)))$. So, $t((i,j)-g\psi-cl(E)) \leq \sigma_j - cl(t(E))$.

Remark 5. (1) *If t is an injective and fuzzy $(i,j)-g\psi-conts$ mapping. Then any statement of Theorem 4 is hold in Theorem 5 since they are equivalent.*

(2) *The converse of Theorem 5, Corollary 5 is incorrect because if we have a fuzzy set $F \in \mathcal{F}_{\sigma_j}$, then by (ii) for example in Theorem 4 we find $(i,j)-g\psi-cl(t^{-1}(F)) = t^{-1}((F))$ but $t^{-1}((F))$ is not fuzzy $(i,j)-g\psi-cld$ group in X.*

Theorem 6. *Suppose $t:(X,\delta_1,\delta_2) \to (Y,\sigma_1,\sigma_2)$, $g:(Y,\sigma_1,\sigma_2) \to (Z,\eta_1,\eta_2)$ are fuzzy $(i,j)-g\psi-conts$, hence $g \circ t:(X,\delta_1,\delta_2) \to (Z,\eta_1,\eta_2)$ is not fuzzy $(i,j)-g\psi-conts$.*

We show that if the functions t with g are fuzzy $(i,j)-g\alpha-conts$ by the coming example:

Example 6. *Suppose E, F, and G are fuzzy subgroups of $X=\{a,b\}$ known as: $E(a,b)=\{0.3,0.4\}$, $F(a,b)=\{0.4,0.5\}$, $G(a,b)=\{0.4,0.4\}$. Consider the fuzzy bitopology $\delta_1=\{0,1,E\}$ also $\delta_2=\{0,1,F,G\}$ on X. Suppose N with M are fuzzy subgroups of $Y=\{r,h\}$ defined as: $N(r,h)=\{0.5,0.4\}$, $M(r,h)=\{0.5,0.5\}$. Consider the fuzzy bitopology $\sigma_1=\{0,1,N\}$, $\sigma_2=\{0,1,M\}$ on Y, and $t(a)=r, t(b)=h$.*

Suppose L and K are fuzzy subgroups of $Z = \{a, b\}$ defined as: $L(a, b) = \{0.2, 0.3\}$, $K(a, b) = \{0.7, 0.5\}$. Consider the fuzzy bitopology $\eta_1 = \{0, 1, L\}$ and $\eta_2 = \{0, 1, K\}$ on Z, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ such that $g(a) = a, g(b) = b$. one may notice that t, g are fuzzy $(1, 2) - g - conts$, and hence by Theorem 2 they are $(i, j) - g\alpha - conts$, but $g \circ t$ is not fuzzy $(1, 2) - g\alpha - conts$ since $(g \circ t)^{-1}(K^c) \leq E \leq \delta_1$, but $\delta_2 - \alpha - cl((g \circ t)^{-1}(K^c)) \nleq E$.

Theorem 7. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is $\delta_j - \psi - conts$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $(i, j) - g\psi - conts$, and any fuzzy $(i, j) - g\psi - cld$ of Y is fuzzy open of (Y, σ_i). Thus $g \circ t : (X, \delta_1, \delta_2) \to (Z, \eta_1, \eta_2)$ is $(i, j) - g\psi - conts$.*

Proof. Assume W is fuzzy closed of (Z, η_j). As g is fuzzy $(i, j) - g\psi - conts$, then $g^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ of Y. So by hypotheses $g^{-1}(W)$ is fuzzy open of (Y, σ_i), so $g^{-1}(W)$ is fuzzy $\delta_j - \psi - cld$ of (Y, σ_j). As t is fuzzy $\delta_j - \psi - conts$, then $t^{-1}(g^{-1}(W))$ is fuzzy $\delta_j - \psi - cld$ of X, thus $(g \circ t)^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ of X. Therefore $g \circ t$ is fuzzy $(i, j) - g\psi - conts$.

Theorem 8. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $(i, j) - g\psi - conts$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $\sigma_j - conts$. Therefore $g \circ t : (X, \delta_1, \delta_2) \to (Z, \eta_1, \eta_2)$ is $(i, j) - g\psi - conts$.*

Proof. Assume W is fuzzy closed of (Z, η_j). As g is fuzzy $\sigma_j - conts$, then $g^{-1}(W)$ is fuzzy closed of (Y, σ_j). As t is fuzzy $(i, j) - g\psi - conts$, then $t^{-1}(g^{-1}(W))$ is fuzzy $(i, j) - g\psi - cld$ of X. Therefore $g \circ t$ is fuzzy $(i, j) - g\psi - conts$.

Theorem 9. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $\delta_j - \psi - conts$ and $\delta_i - open$ mapping. Then any fuzzy $(i, j) - g\psi - cld$ group F of Y, $t^{-1}(F)$ is $(i, j) - g\psi - cld$ of X.*

Proof. Assume F is fuzzy $(i, j) - g\psi - cld$ group of Y, W is fuzzy open of (X, δ_i) including $t^{-1}(F)$. Since t is fuzzy $\delta_i - open$, then $t(W)$ is fuzzy open of (Y, σ_i). As $F \leq t(W)$, F is fuzzy $(i, j) - g\psi - cld$ of Y, then $\sigma_j - \psi - cl(F) \leq t(W)$ which implies $t^{-1}(\sigma_j - \psi - cl(F)) \leq W$. Since t is fuzzy $\delta_j - \psi - conts$, hence $t^{-1}(\sigma_j - \psi - cl(F)) = \delta_j - \psi - cl(t^{-1}(\sigma_j \psi - cl(F)))$. Then $\delta_j - \psi - cl(t^{-1}(F)) \leq t^{-1}(\sigma_j - \psi - cl(F)) \leq W$. So, $t^{-1}(F)$ is fuzzy $(i, j) - g\psi - cld$ of X.

4. Types of Fuzzy Generalized Strongly Continuity and Irresolute Mapping

In the coming part, we define fuzzy generalized strong continuity and irresolute mapping of generalized closed sets and we denote by $(i, j) - g\psi-$strongly conts, and $(i, j) - g\psi-$irresolute respectively after that we study some properties and theorem for them and presented counter examples too.

Definition 10. *A function $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is claimed:*

(1) *Fuzzy $(i, j)-$generalized $\psi-$strongly continuous (briefly, $(i, j) - g\psi-$strongly conts) when $t^{-1}(W)$ is fuzzy $\delta_j -$ closed group of X for all W is $(i, j) - g\psi - cld$ of Y.*

(2) *Fuzzy* $(i,j)-$ *generalized* $\psi-$ *irresolute mapping (briefly,* $(i,j)-g\psi-$ *irresolute) when* $t^{-1}(W)$ *is fuzzy* $(i,j)-g\psi-cld$ *of* X *for all* W *is* $(i,j)-g\psi-cld$ *of* Y.

Theorem 10. *Suppose* $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$. *Thus, the next claims are accurate:*

(1) *If* t *is fuzzy* $(i,j)-g\psi-strongly\ conts$, *thus it is* $\delta_j-conts$.

(2) *If* t *is* $(i,j)-g\psi-strongly\ conts$, *thus it is* $(i,j)-g\psi-irresolute$.

(3) *If* t *is* $(i,j)-g\psi-irresolute$, *thus it is* $(i,j)-g\psi-conts$.

Proof.

(1) Assume t is $(i,j)-g\psi-strongly\ conts$, $W\in\mathcal{F}_{\sigma_j}$. Since W is fuzzy closed of (Y,σ_j), then W is $(i,j)-g\psi-cld$ group of Y. As t is fuzzy $(i,j)-g\psi-strongly\ conts$, thus $t^{-1}(W)\in\mathcal{F}_{\delta_j}$. Therefore t is $\delta_j-conts$.

(2) Assume t is $(i,j)-g\psi-strongly\ conts$ and W is $(i,j)-g\psi-cld$ group of Y. Thus $t^{-1}(W)\in\mathcal{F}_{\delta_j}$, so $t^{-1}(W)$ is $(i,j)-g\psi-cld$ group of X. Consequently, t is $(i,j)-g\psi-irresolute$ mapping.

(3) Assume t is $(i,j)-g\psi-irresolute$, $W\in\mathcal{F}_{\sigma_j}$. Since W is fuzzy closed of (Y,σ_j), then W is $(i,j)-g\psi-cld$ of Y. As t is $(i,j)-g\psi-irresolute$, thus $t^{-1}(W)$ is $(i,j)-g\psi-cld$ of X. So, t is $(i,j)-g\psi-conts$.

Remark 6. *The next diagram explaining the relation in each statements in the above theorem:*

$$\begin{array}{ccc}(i,j)-g\psi-irresolute & \longrightarrow & (i,j)-g\psi-conts\\ \uparrow & & \uparrow\\ (i,j)-g\psi-strongly\ conts & \longrightarrow & \delta_j-conts\end{array}$$

Figure 2: Presents the relationships via all varieties of fuzzy $(i,j)-g\psi-mappings$.

The coming examples clear the opposite implications of Figure (2) are generally not true and also clear that the fuzzy $\delta_j-conts$ and $(i,j)-g\psi-irresolute$ are independent as we explain that for type ψ is fuzzy $\alpha-$open.

Example 7. *Suppose* $E,B,G,$ *and* H *are subgroups of* $X=\{a,b\}$ *defined as:* $E(a,b)=\{0.5,0.4\}$, $F(a,b)=\{0.7,0.5\}$, $G(a,b)=\{0.4,0.3\}$. *Consider the fuzzy bitopology* $\delta_1=\{0,1,E\}$, $\delta_2=\{0,1,F,G\}$ *on* X. *Suppose* N, M *are fuzzy subgroups of* $Y=\{r,h\}$ *defined as follows:* $N(r,h)=\{0.3,0.1\}$, $M(r,h)=\{0.7,0.6\}$. *Consider the fuzzy bitopology* $\sigma_1=\{0,1,N\}$, $\sigma_2=\{0,1,M\}$ *on* Y *and* $t(a)=r, t(b)=h$. *One may notice that* M^c *is fuzzy* $\sigma_2-closed$ *of* Y *and* $t^{-1}(M^c)$ *is fuzzy* $(1,2)-g\alpha-cld$ *of* X, *so* t *is* $(1,2)-g\alpha-conts$, *but not* $\delta_2-conts$ *since* $\delta_2-cl(t^{-1}(M^c))$ *not closed of* X. *Also, we find* t *is* $(i,j)-g\psi-irresolute$ *but not fuzzy* $(1,2)-g\alpha-strongly\,conts$ *since* M^c *is fuzzy* $(1,2)-g\alpha-cld$ *of* Y *but* t^{-1} *is not closed of* X.

Theorem 11. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ is fuzzy $(i,j)-g\psi-irresolute$ mapping, with all $(i,j)-g\psi-cld$ of X is fuzzy open in (X,δ_i). Hence t is $\delta_j-\psi-conts$ mapping.*

Proof. Assume $W\in\mathcal{F}_{\sigma_j}$. Then W is $(i,j)-g\psi-cld$ of Y. As t is $(i,j)-g\psi-irresolute$, then $t^{-1}(W)$ is $(i,j)-g\psi-cld$ of X. Thus by hypotheses $t^{-1}(W)\in F\mathcal{O}(X,\delta_i)$, so $t^{-1}(W)\in F\psi\mathcal{C}(X,\delta_j)$. Theretore t is fuzzy $\delta_j-\psi-conts$ mapping.

Theorem 12. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ is fuzzy $\delta_j-\psi-irresolute$ mapping, and every fuzzy $(i,j)-g\psi-cld$ of Y is fuzzy open of (Y,σ_i). Hence t is $(i,j)-g\psi-irresolute$ mapping.*

Proof. Suppose W is $(i,j)-g\psi-cld$ group of Y. Then by hypotheses $W\in(X,\delta_i)$, and hence $W\in F\psi\mathcal{C}(Y,\sigma_j)$. As t is $\delta_j-\psi-irresolute$, then $t^{-1}(W)\in F\psi\mathcal{C}(X,\delta_j)$, and hence $t^{-1}(W)$ is fuzzy $(i,j)-g\psi-cld$ of X. Hence, t is fuzzy $(i,j)-g\psi-irresolute$ mapping.

Theorem 13. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$. Thus, the next claims are accurate:*

(1) *If t is $(i,j)-g\beta-strongly\,conts$, so it is $(i,j)-gs-strongly\,conts$ also $(i,j)-gp-strongly\,conts$.*

(2) *If t is $(i,j)-gs-strongly\,conts$ or $(i,j)-gp-strongly\,conts$, so it is $(i,j)-g\alpha-strongly\,conts$.*

(3) *If t is $(i,j)-g\alpha-strongly\,conts$, so it is $(i,j)-g-strongly\,conts$.*

Proof. It is clear by uses Theorems in [3].

Remark 7. *The following diagram explaining the relation in each statements in the above theorem:*

$$\begin{tikzcd}
(i,j)-g\beta-strongly\ conts \arrow[r] \arrow[d] & (i,j)-gs-strongly\ conts \arrow[d] \\
(i,j)-gp-strongly\ conts \arrow[r] & (i,j)-g\alpha-strongly\ conts \arrow[d] \\
 & (i,j)-g-strongly\ conts
\end{tikzcd}$$

Figure 3: Presents the relationships via all varieties of fuzzy $(i,j)-g\psi-strongly\,conts$.

The converses of the above relations are not valid in general and this is based on the relationships between the $(i,j)-g\psi-cld$ groups that were explained with examples in Reference No [3].

Theorem 14. *Suppose* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy* $(i, j)-g\psi-irresolute$ *mapping. After that* $t((i, j) - g\psi - cl(E)) \leq \sigma_j - \psi - cl(t(E))$, $\forall\, E \in I^X$.

Proof. Assume t is $(i, j)-g\psi-irresolute$, $E \in I^X$. Thus $E \leq t^{-1}(\sigma_j-\psi-cl(t(E)))$, and hence $\sigma_j - \psi - cl(t(E))$ is fuzzy $(i, j) - g\psi - cld$ of Y. As t is fuzzy $(i, j) - g\psi - irresolute$, so $t^{-1}(\sigma_j - \psi - cl(t(E)))$ is fuzzy $(i, j) - g\psi - cld$ of X, thus $(i, j) - g\psi - cl(E) \leq t^{-1}(\sigma_j - \psi - cl(t(E)))$. Therefore $t((i, j) - g\psi - cl(E)) \leq \sigma_j - \psi - cl(t(E))$.

Corollary 3. *Suppose* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy* $(i, j) - g\psi - strongly\ conts$ *mapping. Hence* $t((i, j) - g\psi - cl(E)) \leq \sigma_j - \psi - cl(t(E))$, $\forall\, E \in I^X$.

Theorem 15. *Suppose* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy* $(i, j) - g\psi - irresolute$ *(resp,* $(i, j)-g\psi-strongly\ conts$*) mapping. Thus* $t((i, j)-g\psi-cl(E)) \leq \sigma_j-cl(t(E))$, $\forall\, E \in I^X$.

Proof. It is clear from Theorem 10 and Theorem 5.

Remark 8. *From Remark 5 and Theorem 10 we find when* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ *is an injective and fuzzy* $(i, j) - g\psi - irresolute$ *(resp,* $(i, j) - g\psi - strongly\ conts$*) mapping. Hence any statement of Theorem 4 is hold in Theorem 15.*

Theorem 16. *Suppose* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$. *Hence the next cliams are true:*

(1) $g \circ t$ *is fuzzy* $(i, j) - g\psi - strongly\, conts$ *if* t, g *are fuzzy* $(i, j) - g\psi - strongly\, conts$.

(2) $g \circ t$ *is fuzzy* $(i, j) - g\psi - irresolute$ *if* t, g *are fuzzy* $(i, j) - g\psi - irresolute$.

(3) $g \circ t$ *is fuzzy* $(i, j) - g\psi - conts$ *if* g *is* $(i, j) - g\psi - conts$, t *is* $(i, j) - g\psi - irresolute$.

Proof.

(1) Assume W is fuzzy $(i, j) - g\psi - cld$ of Z. As g is fuzzy $(i, j) - g\psi - strongly\, conts$, hence $g^{-1}(W)$ is fuzzy closed of Y, so $g^{-1}(W)$ is $(i, j) - g\psi - cld$ group of Y. As t is $(i, j) - g\psi - strongly\, conts$, then $t^{-1}(g^{-1}(W)) = (g \circ t)^{-1}(W)$ is fuzzy closed of X. Therefore $g \circ t$ is fuzzy $(i, j) - g\psi - strongly\, conts$.

(2) Assume W is fuzzy $(i, j) - g\psi - cld$ of Z. As g is fuzzy $(i, j) - g\psi - irresolute$, hence $g^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ of Y. As t is $(i, j) - g\psi - irresolute$, then $t^{-1}(g^{-1}(W)) = (g \circ t)^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ of X. Therefore $g \circ t$ is fuzzy $(i, j) - g\psi - irresolute$.

(3) Suppose $W \in \mathcal{F}_{\eta_j}$. Since g is fuzzy $(i, j) - g\psi - conts$, then $g^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ of Y. As t is fuzzy $(i, j) - g\psi - irresolute$, thus $t^{-1}(g^{-1}(W)) = (g \circ t)^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ of X. Therefore $g \circ t$ is fuzzy $(i, j) - g\psi - conts$.

Theorem 17. *Suppose* $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ *is fuzzy* $\delta_j - \psi - conts$ *and* $\delta_i - open$ *mapping (resp,* $\delta_i - closed$*). Hence* t *is fuzzy* $(i, j) - g\psi - irresolute$ *mapping.*

Proof. It is clear from Theorem 9 and Definition 10.

Theorem 18. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$. Consequently, the claims below are equivalent:*

(i) *t is fuzzy $(i, j) - g\psi - irresolute$ mapping.*

(ii) *The converse of all fuzzy $(i, j) - g\psi - open$ of Y is $(i, j) - g\psi - open$ of X.*

Proof. It's clear by taking the complement of Definition 10.

Theorem 19. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$. Consequently, the next claims are equivalent:*

(i) *t is fuzzy $(i, j) - g\psi - strongly\, conts$.*

(ii) *The inverse of every fuzzy $(i, j) - g\psi - open$ group of Y is $\delta_j - open$ group of X.*

Proof. It is clear by taking the complement of Definition 10.

Theorem 20. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$. Hence the next claims are true:*

(1) *$g \circ t$ is fuzzy $\delta_j - conts$ when g is fuzzy $(i, j) - g\psi - conts$ and t is $(i, j) - g\psi - strongly\ conts$.*

(2) *$g \circ t$ is fuzzy $(i, j) - g\psi - strongly\ conts$ when g is $(i, j) - g\psi - strongly\ conts$ and t is $\delta_j - conts$.*

(3) *$g \circ t$ is fuzzy $(i, j) - g\psi - strongly\ conts$ when g is $(i, j) - g\psi - irresolute$, t is $(i, j) - g\psi - strongly\ conts$.*

Proof.

(1) Assume $W \in \mathcal{F}_{\eta_j}$. As g is $(i, j) - g\psi - conts$, thus $g^{-1}(W)$ is $(i, j) - g\psi - cld$ group of Y. Since t is fuzzy $(i, j) - g\psi - strongly\ conts$, then $t^{-1}(g^{-1}(W)) = (g \circ t)^{-1}(W)$ is fuzzy $\delta_j - closed$ of X. Therefore $g \circ t$ is fuzzy $\delta_j - conts$.

(2) Assume W is fuzzy $(i, j) - g\psi - cld$ group of Z. As g is fuzzy $(i, j) - g\psi - strongly\ conts$, thus $g^{-1}(W)$ is fuzzy closed (Y, σ_j). As t is fuzzy $\delta_j - conts$, then $t^{-1}(g^{-1}(W)) = (g \circ t)^{-1}(W) \in \mathcal{F}_{\delta_j}$. So, $g \circ t$ is fuzzy $(i, j) - g\psi - strongly\ conts$.

(3) Assume W is fuzzy $(i, j) - g\psi - cld$ group of Z. As g is fuzzy $(i, j) - g\psi - irresolute$, thus $g^{-1}(W)$ is fuzzy $(i, j) - g\psi - cld$ group of Y. As t is fuzzy $(i, j) - g\psi - strongly\ conts$, hence $t^{-1}(g^{-1}(W)) = (g \circ t)^{-1}(W) \in \mathcal{F}_{\delta_j}$. So, $g \circ t$ is fuzzy $(i, j) - g\psi - strongly\ conts$.

5. Types of Fuzzy Generalized Open and Closed Mappings

In the third part, we introduce some concepts for fuzzy $(i,j)-$generalized $\psi-$open and closed mapping then we study some properties and important theorems.

Definition 11. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$. Thus t is referred to as:*

(1) *fuzzy $(i,j)-generalized\,\psi-open$ mapping (briefly, $(i,j)-g\psi-open$) when $t(V)$ is fuzzy $(i,j)-g\psi-open$ in Y for any V is $(i,j)-g\psi-open$ of X.*

(2) *fuzzy $(i,j)-generalized\,\psi-closed$ mapping (briefly, $(i,j)-g\psi-closed$) when $t(V)$ is fuzzy $(i,j)-g\psi-cld$ of Y for any V is $(i,j)-g\psi-cld$ of X.*

Theorem 21. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ is fuzzy σ_j-open function. Thus $t(K)$ is fuzzy $(i,j)-g\psi-open$ in Y for any $K\in(x,\delta_j)$.*

Proof. Assume t is σ_j-open, $K\in\mathcal{O}_{\delta_j}$. So $t(K)\in(Y,\sigma_j)$, and hence by relation via open group and $(i,j)-g\psi-open$ group which was clear in refrence [3], we find $t(K)$ is fuzzy $(i,j)-g\psi-open$ in Y.

Theorem 22. *Suppose $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$, $g:(Y,\sigma_1,\sigma_2)\to(Z,\eta_1,\eta_2)$. Then the claims below are true:*

(1) *$g\circ t$ is fuzzy $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$) if t, g are $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$) mapping.*

(2) *t is fuzzy $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$) if $g\circ t$ is $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$), g is $(i,j)-g\psi-irresolute$ and injective mapping.*

(3) *t is fuzzy δ_j-open (resp, $\delta_j-closed$) if $g\circ t$ is $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$), g is $(i,j)-g\psi-strongly\,conts$ and injective mapping.*

(4) *g is fuzzy $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$) if $g\circ t$ is $(i,j)-g\psi-open$ (resp, $(i,j)-g\psi-closed$), t is $(i,j)-g\psi-irresolute$ and surjective mapping.*

(5) *g is fuzzy σ_j-open (resp, $\sigma_j-closed$) if $g\circ t$ is δ_j-open (resp, $\delta_j-closed$), t is $(i,j)-g\psi-strongly\,conts$ and surjective mapping.*

Proof.

(1) Assume K is fuzzy $(i,j)-g\psi-open$ group of X. As t is $(i,j)-g\psi-open$ mapping, thus $t(K)$ is $(i,j)-g\psi-open$ group of Y. As g is $(i,j)-g\psi-open$ mapping, hence $g(t(K))$ is $(i,j)-g\psi-open$ group of Z. So $(g\circ t)(K)$ is $(i,j)-g\psi-open$ of Z. So, $g\circ t$ is $(i,j)-g\psi-open$ mapping.

(2) Assume K is $(i,j)-g\psi-open$ group of X. As $g\circ t$ is $(i,j)-g\psi-open$ mapping, then $g(t(K))$ is $(i,j)-g\psi-open$ group of Z. As g is $(i,j)-g\psi-irresolute$ mapping, injective, then $t(K)$ is $(i,j)-g\psi-open$ group of Y. So, t is fuzzy $(i,j)-g\psi-open$ mapping.

(3) Assume $K \in \delta_j$. After that K considers $(i,j) - g\psi - open$ group of X. As $g \circ t$ is $(i,j) - g\psi - open$ mapping, thus $g(t(K))$ is $(i,j) - g\psi - open$ group of Z. As g is $(i,j) - g\psi - stongly\, conts$ mapping with injective, then $t(K) \in \sigma_j$. Therefore t is fuzzy $\delta_j - open$ mapping.

(4) Assume W is fuzzy $(i,j) - g\psi - open$ group of Y.As t is $(i,j) - g\psi - irresolute$ mapping, thus $t^{-1}(W)$ is $(i,j) - g\psi - open$ group of X. As $g \circ t$ is $(i,j) - g\psi - open$ mapping, t is surjective, then $g(W)$ is $(i,j) - g\psi - open$ group of Z. So, g is fuzzy $(i,j) - g\psi - open$ mapping.

(5) Assume $W \in \sigma_j$. Then W is $(i,j) - g\psi - open$ group of Y. As t is $(i,j) - g\psi - stongly\, conts$ mapping, then $t^{-1}(W) \in \delta_j$. As $g \circ t$ is $\delta_j - open$, with t is surjective, thus $g(W) \in \eta_j$. So, g is $\sigma_j - open$ mapping

Corollary 4. (1) *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $(i,j) - g\psi - strongly\ conts$, injective mapping, also $g \circ t$ is $(i,j) - g\psi - open\, (resp,$ $(i,j) - g\psi - closed)$. Thus, t is $(i,j) - g\psi - open\, (resp,\ (i,j) - g\psi - closed)$ mapping.*

(2) *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is $(i,j) - g\psi - strongly\, conts$, surjective function, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$, also $g \circ t$ is $(i,j) - g\psi - open\, (resp,\ (i,j) - g\psi - closed)$. Thus g is $(i,j) - g\psi - open\, (resp,\ (i,j) - g\psi - closed)$ mapping.*

(3) *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $(i,j) - g\beta - strongly\, conts$, injective mapping, also $g \circ t$ is $(i,j) - g\psi - open\, (resp,\ (i,j) - g\psi - closed)$. Thus t is $\delta_j - open\, (resp,\ \delta_j - closed)$ mapping.*

Theorem 23. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $(i,j) - g\psi - open\ (resp, (i,j) - g\psi - closed)$ mapping. Thus all W is fuzzy subgroup of Y, and $K \in \mathcal{F}_{\delta_j}$ $(resp, K \in \delta_j)$ including $t^{-1}(W)$, $\exists$ W is fuzzy $(i,j) - g\psi - cld\ (resp, (i,j) - g\psi - open)$ of Y including W as $t^{-1}(W) \leq K$.*

Proof. Assume t is $(i,j) - g\psi - open$, $K \in \mathcal{F}_{\delta_j}$ as $t^{-1}(W) \leq K$, as $W \in I^Y$. So $K^c \leq (t^{-1}(W))^c = t^{-1}(W^c)$. Since t is fuzzy $(i,j) - g\psi - open$, also K^c is fuzzy $(i,j) - g\psi - open$ group of X, hence $t(K^c)$ is fuzzy $(i,j) - g\psi - open$ of Y and $t(K^c) \leq W^c$, and hence $W \leq (t(K^c))^c$ if we chose $W = (t(K^c))^c$, thus $\exists\, W$ is fuzzy $(i,j) - g\psi - cld$ group of Y including W as $t^{-1}(W) = K$.

Theorem 24. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is bijective mapping. Hence the next claims are equivalent:*

(i) t is fuzzy $(i,j) - g\psi - open$ mapping.

(ii) t is fuzzy $(i,j) - g\psi - closed$ mapping.

Proof.

$(i) \to (ii)$ Suppose t is $(i,j)-g\psi-open$ mapping, K is $(i,j)-g\psi-cld$ of X. Thus K^c is $(i,j)-g\psi-open$ of X. As t is $(i,j)-g\psi-open$ mapping, so $t(K^c)$ is $(i,j)-g\psi-open$ of Y. As t is bijective, then $t(X)=Y$, hence $Y-t(K)=(t(K))^c$ is $(i,j)-g\psi-open$ of Y, then $t(K)$ is $(i,j)-g\psi-cld$ of Y. So, t is $(i,j)-g\psi-closed$ mapping.

$(ii) \to (i)$ Suppose t is $(i,j)-g\psi-closed$ mapping with K is $(i,j)-g\psi-open$ of X. Thus K^c is $(i,j)-g\psi-cld$ of X. As t is $(i,j)-g\psi-closed$ mapping, so $t(K^c)$ is $(i,j)-g\psi-cld$ of Y. As t is bijective, so $t(X)=Y$, and hence $Y-t(K)=(t(K))^c$ is $(i,j)-g\psi-cld$ of Y, then $t(K)$ is $(i,j)-g\psi-open$ of Y. So, t is $(i,j)-g\psi-open$ mapping.

Theorem 25. *Suppose* $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ *is fuzzy* $(i,j)-g\psi-open$ *(resp,* $(i,j)-g\psi-closed$*) mapping. Thus for any* $R\in I^X$, $t(\delta_j-\psi-int(R))\leq(i,j)-g\psi-int(t(R))$.

Proof. Suppose t is $(i,j)-g\psi-open$ mapping with R is subgroup of X. Since $t(\delta_j-\psi-int(R))\leq t(R)$ and $\delta_j-\psi-int(R)$ considers $(i,j)-g\psi-open$ group of X. Then $t(\delta_j-\psi-int(R))\leq(i,j)-g\psi-int(t(R))$.

6. Fuzzy Generalized Homomorphism Mapping

Finally, we investigate some theorems for the fuzzy generalized homomorphism and we denote by $g\psi-$homomorphism.

Definition 12. *Suppose* $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$. *Hence* t *is known as fuzzy* $g\psi-$*homomorphism if and only if the claims below are true:*

(1) *t is bijective.*

(2) *t is fuzzy* $(i,j)-g\psi-conts$.

(3) t^{-1} *is fuzzy* $(i,j)-g\psi-conts$.

Remark 9. *Suppose* $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$, $g:(Y,\sigma_1,\sigma_2)\to(Z,\eta_1,\eta_2)$ *both of them are fuzzy* $g\psi-$*homomorphism mapping. Then* $g\circ t$ *is not fuzzy* $g\psi-$*homomorphism because* $g\circ t$ *is not fuzzy* $(i,j)-g\psi-conts$. *Refer to Theorem 6.*

Theorem 26. *Suppose* $t:(X,\delta_1,\delta_2)\to(Y,\sigma_1,\sigma_2)$ *is fuzzy* $g\psi-$*homomorphism, fuzzy* $(i,j)-g\psi-irresolute$ *mapping, with* $g:(Y,\sigma_1,\sigma_2)\to(Z,\eta_1,\eta_2)$ *is fuzzy* $g\psi-$*homomorphism mapping. Thus* $g\circ t$ *is fuzzy* $g\psi-$*homomorphism.*

Proof. Suppose $R\in\mathcal{F}_{\eta_j}$. Hence $g^{-1}(R)$ is $(i,j)-g\psi-cld$ of Y. As t is fuzzy $(i,j)-g\psi-irresolute$ mapping, then $t^{-1}(g^{-1}(R))$ is $(i,j)-g\psi-cld$ of X. So $(g\circ t)(R)$ is fuzzy $(i,j)-g\psi-conts$.
Assume R is $(i,j)-g\psi-open$ of X. As t^{-1} is $(i,j)-g\psi-conts$, then $(t^{-1})^{-1}(R)=t(R)\in\mathcal{F}_{\sigma_j}$, and hence $t(R)$ is $(i,j)-g\psi-cld$ of Y. Then $(g^{-1})^{-1}t(R)=g(t(R))\in\mathcal{F}_{\eta_j}$,

thus $((g \circ t)^{-1})^{-1}(R) = (g \circ t)(R) = g(t(R))$ is fuzzy closed of (Z, η_j). So $(g \circ t)^{-1}$ is tuzzy $(i, j) - g\psi - conts$.
Then, assume $(g \circ t)(R_1) = (g \circ t)(R_2)$. Then $g(t(R_1)) = g(t(R_2))$, as t is injective, then $g(R_1) = g(R_2)$. As g is injective, then $R_1 = R_2$. Therefore $g \circ t$ is injective mapping. Also, assume $c \in Z$. As g is surjective, hence $\exists\ b \in Y$ such that $g(b) = c$. As t is also surjective, hence for any $b \in Y\ \exists\ a \in X$ as $t(a) = b$, and hence for any $c \in Z\ \exists\ a \in X$ such that $(g \circ t)(a) = g(t(a)) = g(b) = c$. Thus $g \circ t$ is surjection. Hence $g \circ t$ is bijective. So, $g \circ t$ is fuzzy $g\psi-$homomorphism.

Corollary 5. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $g\psi-$homomorphism, $(i, j) - g\psi - strongly\ conts$, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $g\psi-$homomorphism mapping. Then $g \circ t$ is fuzzy $g\psi-$homomorphism.*

Proof. Since $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is fuzzy $(i, j) - g\psi - strongly\ conts$, so by Theorem 10 t is fuzzy $(i, j) - g\psi - irresolute$ mapping, and hence by Theorem 26 we get the desired.

Theorem 27. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is bijective function, t^{-1} is $(i, j) - g\psi - conts$ mapping. Then t is $(i, j) - g\psi - open$ (resp, $(i, j) - g\psi - closed$) mapping.*

Proof. Assume t^{-1} is fuzzy $(i, j) - g\psi - conts$, R is fuzzy $(i, j) - g\psi - open$ of X. Then $t^{-1}(R) \in \sigma_j$, hence $t^{-1}(R)$ is fuzzy $(i, j) - g\psi - open$ of Y. As t is bijection, then $(t^{-1})^{-1}(R) = t(R)$. So $t(R)$ is fuzzy $(i, j) - g\psi - open$ of Y. So, t is fuzzy $(i, j) - g\psi - open$ mapping.

Corollary 6. *Suppose $t : (X, \delta_1, \delta_2) \to (Y, \sigma_1, \sigma_2)$ is bijective function, $g\psi-$homomorphism. Then t is $(i, j) - g\psi - open$ (resp, $(i, j) - g\psi - closed$) mapping.*

Proof. Assume t is fuzzy $g\psi-$homomorphism. Then by Definition 12 we find t^{-1} is $(i, j) - g\psi - conts$ mapping, hence by Theorem 27 achieved what we want to be proved.

In the following table, we summarize the composition process among all types of functions: $(i, j)-g\psi-conts$, $(i, j)-g\psi-stronglyconts$, $(i, j)-g\psi-irresolute$, $(i, j)-g\psi-open$, $\delta_j - \psi - conts$, $\delta_j - open$, $\sigma_j - conts$, and $\sigma_j - open$, where zero indicates that there is no result of the outcome while 1 indicates that the composition process is possible and has a result.

$g \circ t$	$(i,j)-g\psi-conts$	$\delta_i-\psi-conts$	$(i,j)-g\psi-irresolute$	$(i,j)-g\psi-strongly\ conts$	$(i,j)-g\psi-open$	δ_j-open
$(i,j)-g\varphi-conts$	0	0	1	1	0	0
$\sigma_i-conts$	1	1	1	1	0	0
$(i,j)-g\psi-irresolute$	0	0	1	1	0	0
$(i,j)-g\psi-strongly\ conts$	1	1	1	1	0	0
$(i,j)-g\psi-open$	0	0	0	0	1	0
σ_j-open	0	0	0	0	0	1

Table 1: Composition process between all types of mappings.

Remark 10. (1) *To clarify further, for example, if g is $(i,j)-g\psi-open$ and t is $(i,j)-g\psi-irresolute$, we notice from the table that the result of the composition process is equal to 0 because of the difference in the effect of domains, as t is moved from Y to X, but g is moved from Y to Z, there is no connection between X and Z, so the result equals 0, while we find that if g, t are $(i,j)-g\psi-irresolute$, thus $g \circ t = 1$, because $g \circ t$ has a result and its $(i,j)-g\psi-irresolute$ too, see Theorem 16.*

(2) *We would like to note that the composition process collecting t, g is not equal to the process of collecting g, t. i.e. $g \circ t \neq t \circ g$.*

(3) *A detailed explanation of the resultant of the composition, t if δ_j-open and g if $(i,j)-g\psi-open$. The reslut $g \circ t = 0$, but $(g \circ t)(K)$ is $(i,j)-g\psi-open$ group of Z, when K is open group of X.*

7. Conclusion

In this research, we have delved into the structures of various functions within generalized closed groups in the context of bitopological fuzzy spaces. Additionally, we have explored the interconnections among these functions. Subsequently, we have scrutinized fundamental theorems and distinctive characteristics associated with these concepts. Through this in-depth analysis, we contribute to a better comprehension of these key ideas in the context of fuzzy bitopological spaces. This work also opens up new horizons for the future study of these functions in other fields such as fuzzy sets like gamma, theta, or regular set, also for more than two topologies.

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