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On weakly (τ_1, τ_2)-continuous functions

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Abstract. Our main purpose is to introduce the concept of weakly (τ_1, τ_2)-continuous functions. Moreover, several characterizations of weakly (τ_1, τ_2)-continuous functions are considered.

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Key Words and Phrases: $\tau_1\tau_2$-open set, weakly (τ_1, τ_2)-continuous function

1. Introduction

In 1961, Levine [10] introduced the concept of weakly continuous functions. Moreover, Levine [11] introduced the notion of semi-continuous functions. Neubrunnová [13] showed that semi-continuity is equivalent to quasi-continuity due to Marcus [12]. In 1973, Popa and Stan [17] introduced and studied the concept of weakly quasi-continuous functions. Weak quasi-continuity is implied by both quasi-continuity and weak continuity which are independent of each other. In 1984, Rose [18] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Noiri [14] studied properties of some weak forms of continuity. In 2002, Popa and Noiri [16] introduced the concept of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of weakly (τ, m)-continuous functions. Popa and Noiri [15] introduced and investigated the notion of weakly M-continuous functions as functions from a set satisfying some minimal conditions into a set satisfying some minimal conditions. In 2008, Ekici et al. [8] introduced a new class of functions called weakly λ-continuous functions which is weaker than λ-continuous functions and studied some fundamental properties of weakly λ-continuous functions. In [3], the present author introduced the concept of weakly $\star$-continuous functions and established the relationships between weak $\star$-continuity and $\theta(\star)$-continuity. In 2010, Boonpok [1] introduced and studied the concept of pairwise weakly M-continuous functions in biminimal structure spaces. Viriyapong and Boonpok [20] introduced and investigated the concept of (Λ, sp)-continuous functions.

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Moreover, some characterizations of almost (Λ, s)-continuous functions were presented in [6]. In [5], the authors introduced and studied the notion of weakly (Λ, p)-continuous functions. Laprom et al. [9] studied the concept of $\beta(\tau_1, \tau_2)$-continuity for multifunctions. In addition, some characterizations of almost weak (τ_1, τ_2)-continuity for multifunctions were established in [4]. In this paper, we introduce the concept of weakly (τ_1, τ_2)-continuous functions. Furthermore, several characterizations of weakly (τ_1, τ_2)-continuous functions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-closed* [7] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$*-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$*-closure* [7] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$*-interior* [7] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [19] (resp. $(\tau_1, \tau_2)s$*-open* [2], $(\tau_1, \tau_2)p$*-open* [2], $(\tau_1, \tau_2)\beta$*-open* [2]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$*-closed,* $(\tau_1, \tau_2)s$*-closed,* $(\tau_1, \tau_2)p$*-closed,* $(\tau_1, \tau_2)\beta$*-closed.* Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [19] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* [19] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [19] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open.* The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [19] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

3. Characterizations of weakly (τ_1, τ_2)-continuous functions

In this section, we introduce the notion of weakly (τ_1, τ_2)-continuous functions. Moreover, some characterizations of weakly (τ_1, τ_2)-continuous functions are discussed.

Definition 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\tau_1\tau_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if f has this property at each point of X.*

Theorem 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (τ_1, τ_2)-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus,

$$x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$$

and hence $x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By the hypothesis, $x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence f is weakly (τ_1, τ_2)-continuous at x.

Theorem 2. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (τ_1, τ_2)-continuous if and only if $f^{-1}(V) \subseteq \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is weakly (τ_1, τ_2)-continuous at x, by Theorem 1 we have $x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, we have $x \in f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 1, f is weakly (τ_1, τ_2)-continuous at x. This shows that f is weakly (τ_1, τ_2)-continuous.

Theorem 3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (τ_1, τ_2)-continuous if and only if $\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y. Suppose that

$$\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \nsubseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

There exists $x \in \tau_1\tau_2\text{-Cl}(f^{-1}(V))$, but $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then, $f(x) \notin \sigma_1\sigma_2\text{-Cl}(V)$ and there exists a $\sigma_1\sigma_2$-open set W of Y containing $f(x)$ such that $W \cap V = \emptyset$. Thus, $\sigma_1\sigma_2\text{-Cl}(W) \cap V = \emptyset$. Since f is weakly (τ_1, τ_2)-continuous at x, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(W)$. Therefore, $f(U) \cap V = \emptyset$. Since $x \in \tau_1\tau_2\text{-Cl}(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \emptyset$ and $f(U) \cap V \neq \emptyset$, which is a contradiction. This shows that $\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open in Y. By the hypothesis, $\tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. Thus, $X - \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq X - f^{-1}(V)$ and hence $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 2, f is weakly (τ_1, τ_2)-continuous.

Theorem 4. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f *is weakly* (τ_1, τ_2)*-continuous;*

(2) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ *for every subset* B *of* Y*;*

(6) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): By Theorem 2.

(2) ⇒ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2),

$$\begin{aligned} X - f^{-1}(K) &= f^{-1}(Y - K) \\ &\subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - K))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(K))) \\ &= X - \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))). \end{aligned}$$

Thus, $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(K)$.

(3) ⇒ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Int}(B)$ is $\sigma_1\sigma_2$-closed in Y. By (3), $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) ⇒ (5): Let B be any subset of Y. By (4),

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B)))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))). \end{aligned}$$

(5) ⇒ (6): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then, there exists a $\sigma_1\sigma_2$-open set U of Y containing $f(x)$ such that $U \cap V = \emptyset$. By (5),

$$x \in f^{-1}(U) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(U)))$$

and there exists a $\tau_1\tau_2$-open set G of X containing x such that $f(G) \subseteq \sigma_1\sigma_2\text{-Cl}(U)$. Thus, $G \cap f^{-1}(V) = \emptyset$ and hence $x \notin \tau_1\tau_2\text{-Cl}(f^{-1}(V))$. This shows that

$$\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(6) ⇒ (1): Let $x \in X$ and V be any be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $V = \sigma_1\sigma_2\text{-Int}(V) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$, by (6) we have

$$\begin{aligned} x \in f^{-1}(V) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \end{aligned}$$

$$\subseteq X - \tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$$
$$= \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

There exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$; hence $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, f is weakly (τ_1, τ_2)-continuous at x. This shows that f is weakly (τ_1, τ_2)-continuous.

Theorem 5. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(3) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(4) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) ⇒ (2): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, $\sigma_1\sigma_2$-Int(K) is $\sigma_1\sigma_2$-open in Y, by Theorem 4 (6) we have

$$\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K))) = f^{-1}(K).$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, we have

$$\sigma_1\sigma_2\text{-Cl}(V) \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$$

and hence $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed. By (2),

$$\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(3) ⇒ (4): This is obvious.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, we have V is $(\sigma_1, \sigma_2)s$-open in Y. By (4), $\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq \tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ and by Theorem 4 (6), f is weakly (τ_1, τ_2)-continuous.

Theorem 6. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, we have

$$\sigma_1\sigma_2\text{-Cl}(V) \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y. Thus, by Theorem 5 (2),

$$\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (3),

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - \tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))). \end{aligned}$$

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y. Thus by (4) and Theorem 4 (2), f is weakly (τ_1, τ_2)-continuous.

Theorem 7. *For a function* $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(5) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(6) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(7) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let B be any subset of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$. Then, we have $f(x) \notin \sigma_1\sigma_2$-Cl$(B)$ and there exists a $\sigma_1\sigma_2$-open set U of Y containing $f(x)$ such that $U \cap B = \emptyset$. Therefore, $\sigma_1\sigma_2\text{-Cl}(U) \cap \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)) = \emptyset$. Since f is weakly (τ_1, τ_2)-continuous at x, there exists a $\tau_1\tau_2$-open set W of X containing x such that $f(W) \subseteq \sigma_1\sigma_2$-Cl$(U)$. Thus, $W \cap f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B))) = \emptyset$ and hence

$$x \notin \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))).$$

This shows that $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(2) ⇒ (3): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (2), we have

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= f^{-1}(K).\end{aligned}$$

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)r$-closed in Y. By (3), $\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Since $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open in Y, by (4) we have

$$\begin{aligned}X - \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= \tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - f^{-1}(V)\end{aligned}$$

and hence $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (5), $x \in f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Put $W = \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, W is $\tau_1\tau_2$-open set of X containing x such that $f(W) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, f is weakly (τ_1, τ_2)-continuous at x. This shows that f is weakly (τ_1, τ_2)-continuous.

(1) ⇒ (6): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y and $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Then, $f(x) \notin \sigma_1\sigma_2\text{-Cl}(V)$ and there exists a $\sigma_1\sigma_2$-open set G of Y containing $f(x)$ such that $G \cap V = \emptyset$. Since V is $(\sigma_1, \sigma_2)p$-open, we have

$$\begin{aligned}V \cap \sigma_1\sigma_2\text{-Cl}(G) &\subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cap \sigma_1\sigma_2\text{-Cl}(G) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)) \cap G) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V) \cap G)) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V \cap G))) \\ &\subseteq \sigma_1\sigma_2\text{-Cl}(V \cap G) = \emptyset.\end{aligned}$$

Since f is weakly (τ_1, τ_2)-continuous at x, there exists a $\tau_1\tau_2$-open set W of X containing x such that $f(W) \subseteq \sigma_1\sigma_2\text{-Cl}(G)$. Thus, $f(W) \cap V = \emptyset$ and hence $W \cap f^{-1}(V) = \emptyset$. Therefore, $x \notin \tau_1\tau_2\text{-Cl}(f^{-1}(V)$. This shows that $\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(6) ⇒ (7): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-open and hence $Y - \sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)p$-open in Y. Then by (6), we have

$$\begin{aligned}X - \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= \tau_1\tau_2\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))\end{aligned}$$

$$\subseteq X - f^{-1}(V)$$

and hence $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since V is $(\sigma_1, \sigma_2)p$-open in Y and by (7), $x \in f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Put $U = \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, U is a $\tau_1\tau_2$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, f is weakly (τ_1, τ_2)-continuous at x. This shows that f is weakly (τ_1, τ_2)-continuous.

Theorem 8. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (τ_1, τ_2)-continuous;

(2) $f(\tau_1\tau_2\text{-Cl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(f(A))$ for every subset A of X;

(3) $\tau_1\tau_2\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X. Suppose that $x \in \tau_1\tau_2\text{-Cl}(A)$ and G is any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(G)$. Since $x \in \tau_1\tau_2\text{-Cl}(A)$, we have $U \cap A \neq \emptyset$. It follows that $\emptyset \neq f(U) \cap f(A) \subseteq \sigma_1\sigma_2\text{-Cl}(G) \cap f(A)$. Thus, $f(x) \in (\sigma_1, \sigma_2)\theta\text{-Cl}(f(A))$ and hence $f(\tau_1\tau_2\text{-Cl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then,

$$f(\tau_1\tau_2\text{-Cl}(f^{-1}(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(f(f^{-1}(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$$

and hence $\tau_1\tau_2\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$.

(3) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since

$$\sigma_1\sigma_2\text{-Cl}(V) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset,$$

$f(x) \notin (\sigma_1, \sigma_2)\theta\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))$ and hence $x \notin f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. By (3), $x \notin \tau_1\tau_2\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \cap f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$; hence $f(U) \cap (Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$. Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence f is weakly (τ_1, τ_2)-continuous at x. This shows that f is weakly (τ_1, τ_2)-continuous.

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