EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 1, 2024, 116-123
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Filter of Cyclic B-Algebras

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Abstract. This paper introduces the notion of a B-filter in a B-algebra $(X, *, 0)$ and presents characteristics of its properties : for any $a \in X$, the set $\langle a \rangle_B = \{a^k \ : \ k \in \mathbb{Z}\}$ forms a B-ideal and B-filter of X. Moreover, this paper showns some properties of exponents on B-algebra.

2020 Mathematics Subject Classifications: 06F35, 03G25

Key Words and Phrases: B-algebras; cyclic B-algebras; B-ideal ; B-filter

1. Introduction

J. Neggers and H. S. Kim introduced in [1] the notion of B-algebras and some properties of exponents on its. Furthermore, they investigated the relationship between B-algebras and groups and asked whether a group determines a B-algebra, and conversely. In [5], D. Al-Kadi introduced the notion of B-ideal and then K. E. Belleza and J. P. Vilela in [7] presents the characterizations and properties of B-ideals in a topological B-algebra and introduces the uniform topology on a B-algebra in terms of its B-ideals. Moreover, they have shown that a uniform B-topological space is a topological B-algebra. K. E. Belleza and J. R. Albaracin introduces and characterized the notion of a dual B-algebra, in [6]. Moreover in the year 2022 , K. E. Belleza introduces the dual B-topological space, dual B-ideals and dual B-subalgebras. Also, some properties of a filterbase on a dual B-topological space are provided. In [2], N. C. Gonzaga, Jr and J. P. Vilela introduced the notion of cyclic B-algebras and some of its properties. Moreover the authors had investigated the relationship between the class of cyclic B-algebras and the class of cyclic groups coincide. In [8], K. E. Belleza and J. R. Albaracin introduced the notion of tdB-algebra, presents characteristics and properties of dual B-filters and dual B-subalgebras in a tdB-algebra, and introduces the uniform topology on a dual B-algebra in terms of its dual B-subalgebras.

Specifically, this paper introduces the notion of the B-filter in a B-algebra and we have show that for any $a \in X$, the set $\langle a \rangle_B = \{a^k \ : \ k \in \mathbb{Z}\}$ form a B-ideal and B-filter of X.

DOI: https://doi.org/10.29020/nybg.ejpam.v17i1.4977

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2. Preliminaries

First, we will review some essential notations and definitions of B-algebras and ordinary senses that are needed for this study in this section. Throughout this paper, X will denote the B-algebra $(X, *, 0)$ unless otherwise specified.

Definition 1. *[4] A B-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms :*

$(B1)$ $x * x = 0$,

$(B2)$ $x * 0 = x$,

$(B3)$ $(x * y) * z = x * (z * (0 * y))$ *for all* $x, y, z \in X$.

A B-algebra $(X, *, 0)$ is said to be commutative if $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$ and a nonempty subset S of a X is called a sub-algebra of X if $x * y \in S$ for any $x, y \in S$. Also, the authors of [1] have proved that a B-algebra X is commutative if and only if the equality $x * (x * y) = y$ holds for all $x, y \in X$.

Example 1. *([4],[5]) : Let $X = \{0, 1, 2, 3\}$ and $Y = \{0, 1, 2, 3, 4, 5\}$. Define binary operations $*$ on X and $\odot$ on Y defined by the following two tables respectively :*

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

$\odot$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ is a commutative B-algebra, but $(Y, \odot, 0)$ is a non commutative B-algebra, since $2 * (0 * 5) = 2 * 5 = 4 \neq 3 = 5 * 1 = 5 * (0 * 2)$.

We recall the following axioms for the laws of Exponents for B-algebras.

Theorem 1. *[1] Let $(X, *, 0)$ be a B-algebra. Then the following conditions hold for any $x, y, z \in X$:*

(i) $x = (x * y) * (0 * y)$,

(ii) $y * x = 0 * (x * y)$,

(iii) $0 * (0 * x) = x$,

(iv) $x * (y * z) = (x * (0 * z)) * y$, (*v*) $x * y = x * (0 * (0 * y))$,

(*vi*) $x * y = 0$ *implies* $x = y$,

(*vii*) $x * z = y * z$ *implies* $x = y$ *and*

(*viii*) $0 * x = 0 * y$, *implies* $x = y$.

Definition 2. *[5] Let* $(X, *, 0)$ *be a* B*-algebra. A nonempty subset* I *of* X *is called a* B*-ideal of* X *if it satisfies the following conditions for any* $x, y, z \in X$*:*

(*i*) $0 \in I$,

(*ii*) *If* $x * y \in I$ *and* $y \in I$, *then* $x \in I$.

Definition 3. *[6] Let* X *be a non-empty set with a binary operation* $*$ *and a constant* 0*. Then the triple* $(X, *, 0)$ *is a dual* B*-algebra if its satisfies the following axioms for all* $x, y, z \in X$*:*

(*i*) $x * x = 0$,

(*ii*) $0 * x = x$,

(*iii*) $x * (y * z) = ((y * 0) * x) * z$.

Definition 4. *[8] Let* $(X, *, 0)$ *be a dual* B*-algebra. A nonempty subset* F *of* X *is called a dual* B*-filter of* X *if it satisfies the following axioms for all* $x, y, z \in X$*:*

(*i*) $0 \in F$,

(*ii*) *If* $x * y \in F$ *and* $x \in F$, *then* $y \in F$.

There is a B-algebra that is also a dual B-algebra in the following example.

Example 2. *[6] Let* $X = \{0, a, b, c\}$ *and a binary operations* $*$ *on* X *satisfying the following table :*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Example 3. *[8] Let* $X = \{0, a, b, c, d, e\}$ *and a binary operations* $*$ *on* X *satisfies the following table :*

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	b	0	a	d	e	c
b	d	b	0	e	c	d
c	c	d	e	0	a	b
d	d	e	c	b	0	a
e	e	c	d	a	b	0

Then $(X, *, 0)$ *is a dual* B*-algebra. The sets* $F_0 = \{0\}$, $F_2 = \{0, c\}$, $F_3 = \{0, d\}$, $F_4 = \{0, e\}$ *and* $F_5 = \{0, a, b\}$ *are dual* B*-filters of* X *while* $A = \{0, a, e\}$ *is not a dual* B*-filter since* $e * c = a \in A$ *where* $e \in A$ *but* $c \notin A$.

3. Some Axioms of Exponents for B-algebras

In this section, we recall the axioms for a B-algebra $(X, *, 0)$. The paper [1] and [2] introduced the notions of exponents of B-algebra and some of its properties. For any $x, y \in X$ and $n \in \mathbb{Z}^+$, defined the relation:

$$x^n = x^{n-1} * (0 * x) \text{ and } -x = 0 * x$$

where $x^0 = 0$ and $x^1 = x^0 * (0 * x) = 0 * (0 * x) = x$ and denote that expression

$$x * \prod^{n} y = (...((x * y) * y) * ...) * y,$$

where y occurs n times. By convention, $x * \prod^{0} y$ means $x * 0 = x$, so that $x^n = x * (0 * \prod^{n-1} x)$ and $x^{-n} = (-x)^n = -(x)^n = 0 * x^n$ implies that $(x^{-1})^{-n} = (x^{-n})^{-1} = x^n$.

Theorem 2. *[1] Let* $(X, *, 0)$ *be a* B*-algebra ,* $g \in X$ *and* $m, n \in \mathbb{Z}^+$*. Then*

$$g^m * g^n = \begin{cases} g^{m-n} & if\ m \geq n \\ 0 * g^{n-m} & if\ m < n \end{cases}$$

Corollary 1. *[1] Let* $(X, *, 0)$ *be a* B*-algebra. Then the following equalities hold for all* $g \in X$ *and* $m, n \in \mathbb{Z}^+$*:*

(i) $g^m * g^n = g^{m-n}$,

(ii) $g * g^{-n} = g^{n+1}$ *and* $g^{-n} * g = g^{-(n+1)}$, *(iii)* $-g * g^n = g^{-(n+1)}$,

(iv) $g^m * (-g) = g^{(n+1)}$,

(v) $g^m * g^{-n} = g^{(m+n)}$ *and* $g^{-m} * g^n = g^{-(m+n)}$.

Corollary 2. *[2] Let* $(X, *, 0)$ *be a* B*-algebra,* $g \in X$ *and* $m, n \in \mathbb{Z}$*. Then* $g^m * g^n = g^{m-n}$.

4. On Exponents and cyclic B-Algebras

We shall give some elementary properties of cyclic B-algebras. Recall that for a B-algebra $(X, *, 0)$ (see [2]) if there is an $a \in X$ such that

$$\langle a \rangle_B = \{a^k \ : \ k \in \mathbb{Z}\} = X,$$

then X is called a cyclic B-algebra generated by a. Also, the authors of [2] have proved that every cyclic B-algebra is commutative.

Theorem 3. *Let* $(X, *, 0)$ *be a* B*-algebra and* $x, y \in X$ *with* $n \in \mathbb{Z}^+$*, then*

$$0 * (x * y)^n = (y * x)^n$$

Proof. Cleary $0 * (x * y) = (0 * (0 * y) * x) = y * x$. *Thus, the equality holds for* $n = 1$ *Next, suppose that* $0 * (x * y)^n = (y * x)^n$ *for any* $n > 1$.
So $0 * (x * y)^{n+1} = 0 * \{(x * y) * (0 * \prod^{n}(x * y))\}$

$$= 0 * \{(x * y) * [[...[[0 * \underbrace{(x * y)] * (x * y)] * ...] * (x * y)]}_{(x*y)\ occurs\ n\ times}\}$$

$$= 0 * \{[(x * y) * [0 * (x * y)]] * [[...[[0 * \{\underbrace{(x * y)] * (x * y)] * ...] * (x * y)]}_{(x*y)\ occurs\ n-1\ times}\}$$

$$= 0 * \{[(x * y) * (y * x) * [[...[[0 * \underbrace{(x * y)] * (x * y)] * ...] * (x * y)]}_{(x*y)\ occurs\ n-1\ times}\}$$

$$= 0 * \{[[...[(x * y) * \underbrace{(y * x)] * ...] * (y * x)] * (y * x)}_{(y*x)\ occurs\ n-1\ times}] * [0 * (x * y)]\}$$

$$= 0 * \{[[[...[[0 * \underbrace{(y * x)] * ...] * (y * x)] * (y * x)}_{(y*x)\ occurs\ n-1\ times} * [0 * (x * y)] * 0]\}$$

$$= (y * x) * [[...[[0 * \underbrace{(y * x)] * (y * x)]] * ...] * (y * x)] * (y * x)}_{(y*x)\ occurs\ n\ times}$$

$$= (y * x) * (0 * \prod^{n}(y * x))$$

$= (y * x)^{n+1}$

Therefore, $0 * (x * y)^n = (y * x)^n$ *for any* $n \in \mathbb{Z}^+$. □

Theorem 4. *Let* $(X, *, 0)$ *be a* B*-algebra and* $x, y \in X$ *with* $n \in \mathbb{Z}^+$*, then*

$$0 * (0 * x)^n = 0 * (0 * x^n)$$

Proof. Since $0 * x^n = x^{-n}$ *in [2],*

$$\begin{aligned} 0 * (0 * x)^n &= (0 * x)^{-n} \\ &= (x^{-1})^{-n} \\ &= (x^{-n})^{-1} \\ &= 0 * (x^{-n}) \\ &= 0 * (0 * x^n). \end{aligned}$$

□

Theorem 5. *Let* $(X, *, 0)$ *be a* B*-algebra and* $a \in X$*, then* $\langle a \rangle_B$ *is a* B*-ideal of* X.

Proof. Clearly, $0 = a^0 \in \langle a \rangle_B$. *Next, let* $x * y \in \langle a \rangle_B$ *and* $y \in \langle a \rangle_B$*, then* $x * y = a^k$ *and* $y = a^r$ *for some* $k, r \in Z$. *So,* $y * x = 0 * (x * y) = 0 * a^k = a^0 * a^k = a^{0-k} = a^{-k} \in < a >$ *and hance by [1],* $x = y * (y * x) = a^r * a^{-k} = a^{r-(-k)} = a^{r+k} \in \langle a \rangle_B$. *Therefore,* $\langle a \rangle_B$ *is an ideal of* X. □

In 2023, K. E. Belleza, J. R. Albaracin [8] introduced the concept of dual B-filters of dual B-algebra. Thus, we can have the following definition:

Definition 5. *Let* $(X, *, 0)$ *be a* B*-algebra. A nonempty subset* F *of* X *is called a* B*-filter of* X *if it satisfies the following axioms for all* $x, y \in X$:

(*i*) $0 \in F$,

(*ii*) *If* $x * y \in F$ *and* $x \in F$*, then* $y \in F$.

Example 4. *Consider the* B*-algebra* $X = \{0, 1, 2, 3\}$ *from Example 1. The sets* $F_1 = \{0\}$, $F_2 = \{0, 2\}$ *are* B*-filters of* X *while* $F_3 = \{0, 3\}$ *is not a* B*-filter of* X*, since* $0 * 1 = 3 \in F_3$ *where* $0 \in F_3$ *but* $1 \notin F_3$.

Consider $(Y, \odot, 0)$*, the sets* $F_4 = \{0\}$, $F_5 = \{0, 3\}$, $F_6 = \{0, 4\}$, $F_7 = \{0, 5\}$*, are* B*-filters of* Y.

Theorem 6. *Let* $(X, *, 0)$ *be a* B*-algebra and* $a \in X$*, then* $\langle a \rangle_B$ *is a* B*-filter of* X.

Proof. Clearly $0 = a^0 \in \langle a \rangle_B$. *Next, let* $x * y \in \langle a \rangle_B$ *and* $x \in \langle a \rangle_B$ *, then* $x * y = a^k$ *and* $x = a^r$ *for some* $k, r \in \mathbb{Z}$ *and hence by theory 3.2 in [1],*

$$y = x * (x * y) = a^r * a^k = a^{r-k} \in \langle a \rangle_B$$

Therefore, $\langle a \rangle_B$ *is a filter of* X. □

Theorem 7. *Let $(X, *, 0)$ be a B-algebra and $a \in X$ with $0 * a = a$, then $\langle a \rangle_B = \{0, a\}$ form a B-filter of X.*

Proof. Let $a \in X$ with $0 * a = a$. Consider

$$\begin{aligned}
&a^0 = 0,\\
&a^1 = a,\\
&a^2 = a * a = 0,\\
&a^3 = a^2 * a = 0 * a = a,\\
&a^4 = a^3 * a = a * 0 = 0,\\
&.\\
&.\\
&.
\end{aligned}$$

This implies that $\langle a \rangle_B = \{0, a\}$.

Next, Let $x, y \in X$, $F = \{0, a\}$ *with* $0 * a = a$, $x * y \in F$ *and* $x \in F$.

Case 1 : *If* $x * y = 0$ *and* $x = 0$, *then* $x * y = 0 * y = 0 = y * y$ *by theorem 1 (vii),* $y = 0 \in F$.

Case 2 : *If* $x * y = 0$ *and* $x = a$, *then* $x * y = a * y = 0 = y * y$ *by theorem 1 (vi),* $y = a \in F$.

Case 3 : *If* $x * y = a$ *and* $x = 0$, *then* $x * y = 0 * y = a$ *by assumption* $a = 0 * a$ *and theorem 1 (viii),* $y = a \in F$.

Case 4 : *If* $x * y = a$ *and* $x = a$, *then* $x * y = a * y = a$ *by theorem 1,* $a = a * y = 0 * (y * a) = 0 * (0 * a) = a$ *implies that* $(y * a) = (0 * a)$ *and hence* $y = 0 \in F$.

Therefore, $\langle a \rangle_B = F = \{0, a\}$ *is a B-filter of* X. □

5. Conclusion

In this paper shown some properties of exponents on B-algebra and we introduces the notion of B-filter on a B-algebra and presented together with some of its properties on a cyclic B-algebra, that is for any an element a in a B-algebra $(X, *, 0)$, we show that the set $\langle a \rangle_B = \{a^k \ : \ k \in \mathbb{Z}\}$ form a B-ideal and B-filter of X. Moreover, if $0 * a = a$, we obtain $\langle a \rangle_B = \{0, a\}$ form a B-filter of X.

Acknowledgements

This research project was financially supported by Thailand Science Research and Innovation (TSRI).

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