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Multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative and positive implicative ideals in BCK/BCI-algebras

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Abstract. This paper focuses on exploring restricted mathematical concepts within the domain of BCK/BCI-algebras, specifically delving into the intricate realm of Multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals. BCK and BCI-algebras are pivotal structures in mathematical logic and algebraic systems, finding widespread applications in fields like computer science and artificial intelligence. Our contribution lies in the introduction and thorough investigation of the innovative notions of multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals, uniquely tailored for BCK/BCI-algebras. These ideals exhibit exceptional flexibility in managing uncertain and hesitant information, serving as potent tools for modeling and solving real-world problems characterized by imprecise or incomplete data. This study rigorously defines and explores the foundational properties of multi-polar Q-hesitant fuzzy soft implicative ideals, underscoring their relevance and applicability within BCK/BCI-algebras. Additionally, we present the concept of positive implicative ideals, establishing their interconnectedness with multi-polar Q-hesitant fuzzy soft implicative ideals. Our investigation delves into these ideals' algebraic and logical facets, offering valuable insights into their mutual interactions and significance within the context of BCK/BCI-algebras. To facilitate practical implementation, we develop algorithms and methodologies for identifying and characterizing multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals. These computational tools enable efficient decision-making in scenarios involving uncertainty. Through illustrative examples and case studies, we showcase the potential of these ideals in handling complex, uncertain information, demonstrating their efficacy in aiding problem-solving processes. This research contributes significantly to advancing BCK/BCI-algebra theory by introducing innovative mathematical structures that bridge the gap between fuzzy logic, soft computing, and implicative ideals. The proposed multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals open new avenues for addressing real-world problems characterized by imprecision and uncertainty. As such, they represent a valuable addition to the field of algebraic structures and their applications.

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Key Words and Phrases: $\mathbb{Q}$-hesitant, Multi-polar, soft set, implicative, positive implicative, ideals, BCK/BCI-algebras

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1. Introduction

This paper delves into a specialized exploration within the realm of BCK (Bounded Commutative Kleene) and BCI (Bounded Commutative Implication) algebras, focusing on the innovative concepts of Multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals. Recognized for their applications in mathematical logic, computer science, and artificial intelligence, BCK and BCI-algebras serve as foundational structures for logical operations, inference, and decision-making processes. Ideals, specific subsets with algebraic properties, play a pivotal role in this context. While traditional ideals in BCK/BCI-algebras have been extensively studied, this paper significantly extends the existing theory by incorporating multi-polar Q-hesitant fuzzy and positive implicative ideals. This novel approach provides a more adaptable and robust framework capable of handling imprecise and uncertain information in practical decision-making scenarios.

The historical evolution of BCK and BCI-algebras, initiated by Saunders Mac Lane in 1950 and further expanded to include BCI-algebras, has been instrumental in formalizing logical systems. Ideals, with their algebraic properties, have proven instrumental in unraveling the complexities of these systems and their applications in formal logic [11, 19]. Simultaneously, the introduction of fuzzy set theory by Lotfi Zadeh in the 1960s revolutionized our ability to mathematically represent and manipulate uncertain information [14]. The soft computing paradigm, initiated by Lotfi Zadeh, further elevated our approach to handling uncertainty, imprecision, and incomplete data [6, 16].

In this research, we introduce two pioneering concepts, Multi-polar Q-hesitant Fuzzy Sets and Positive Implicative Ideals, building upon the foundations laid by Zadeh [13, 18]. Multi-polar Q-hesitant Fuzzy Sets extend the notion of fuzzy sets to capture various degrees of hesitancy, enabling the representation and manipulation of information that is not only uncertain but also characterized by different levels of hesitation [12]. Positive Implicative Ideals within BCK/BCI-algebras, tied to the concept of logical implication, provide a promising avenue for addressing uncertainty while preserving the core logical properties inherent in these algebraic structures [4].

The theoretical foundations laid by Zadeh and subsequent developments paved the way for integrating fuzzy set theory and soft computing techniques in solving complex real-world problems [6, 14, 16]. Building on this legacy, our research introduces the innovative concepts of Multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals within the framework of BCK/BCI-algebras.

This paper aims to contribute to the evolving landscape of algebraic structures and their applications by addressing the challenges posed by imprecision and uncertainty in real-world scenarios. We draw inspiration from the extensive body of work on hesitant fuzzy sets [13, 18], Q-fuzzy soft sets [7], and multi-polar fuzzy sets [5].

To provide a comprehensive framework, we incorporate historical developments in BCK/BCI-algebras [4, 10, 11, 19], fuzzy set theory [6, 14, 16], and soft computing techniques [6, 16]. The exploration of Multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals extends the traditional theory of ideals in BCK/BCI-algebras, offering more flexibility in handling uncertain and hesitant information. Moreover, the significance of our

work lies in developing algorithms and methodologies for efficient identification and characterization of these novel ideals [2, 8]. We demonstrate the practical applicability of these concepts through examples and case studies, showcasing their prowess in handling complex, uncertain information and aiding in effective problem-solving processes.
In summary, our research bridges the gap between fuzzy logic, soft computing, and implicative ideals, introducing innovative mathematical structures tailored to real-world decision-making scenarios' challenges. The Multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals presented here represent a valuable addition to the field of algebraic structures, providing a more nuanced and adaptable framework for addressing uncertainties inherent in practical applications.
As we progress through the subsequent sections of this paper, we aim to provide in-depth insights into the formal definitions, properties, and practical applications of Multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals, drawing from a rich tapestry of historical developments [8, 17]. This comprehensive framework bridges historical developments in BCK/BCI-algebras, fuzzy set theory, and soft computing, enhancing our capacity to address real-world challenges requiring robust solutions in the face of uncertainty and imprecision.
In the following sections of this paper, we will develop into the formal definitions, properties, and practical applications of multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals. Our aim is to provide a comprehensive framework that bridges historical developments in BCK/BCI-algebras, fuzzy set theory, and soft computing, thereby enhancing our ability to tackle real-world problems that demand robust solutions in the presence of uncertainty and imprecision.

2. Preliminaries

[9],[15] In this section we retrieve some basic definitions which will be implemented in our work.

Definition 1. *An algebra* $(\mathfrak{B}; *, 0)$ *of type* $(2,0)$ *is called BCK-algebra if it fulfills the given requirements:*

B1: $\forall \alpha, \varpi, \Delta \in \mathfrak{B}$, $((\alpha * \varpi) * (\alpha * \Delta)) * (\Delta * \varpi) = 0$.

B2: $\forall \alpha, \varpi \in \mathfrak{B}$, $(\alpha * (\alpha * \varpi)) * \varpi = 0$.

B3 $\forall \alpha \in \mathfrak{B}$, $\alpha * \alpha = 0$.

B4: $\forall \alpha, \varpi \in \mathfrak{B}$, *if* $\alpha * \varpi = 0, \varpi * \alpha = 0$ *then* $\alpha = \varpi$.

B5: $\forall \alpha \in \mathfrak{B}$, $0 * \alpha = 0$.

Any BCK-algebra $\mathfrak{B}$ *satisfies the following axioms:*

B6: $\forall \alpha \in \mathfrak{B}$, $\alpha * 0 = \alpha$.

B7: $\forall \alpha, \varpi, \Delta \in \mathfrak{B}$, *if* $\alpha \leq \varpi$ *then* $\alpha * \Delta \leq \varpi * \Delta$ *and* $\Delta * \varpi \leq \Delta * \alpha$.

B8: $\forall\alpha,\varpi,\Delta\in\mathfrak{B},\ (\alpha*\varpi)*\Delta=(\alpha*\Delta)*\varpi$.

where $\alpha\leq\varpi$ *if any only if* $\alpha*\varpi=0$
Define a binary relation $\leqslant$ *on* $\mathfrak{B}$ *by letting* $\alpha\leqslant\varpi$ *if and only if* $\alpha*\varpi=0$.
Then ($\mathfrak{B};\leqslant$) *is a partially ordered set with the least element 0. In any BCK-algebra* $\mathfrak{B}$*, the following hold:*

B9: $\forall\alpha,\varpi\in\mathfrak{B},\ \alpha*\varpi\leqslant\alpha$.

B10: $\forall\alpha,\varpi,\Delta\in\mathfrak{B},\ (\alpha*\Delta)*(\varpi*\Delta)\leqslant\alpha*\varpi$.

B11: $\forall\alpha,\varpi\in\mathfrak{B},\ \alpha*(\alpha*(\alpha*\varpi))=\alpha*\varpi$.

Any BCI-algebra $\mathfrak{B}$ *satisfies the following axioms:*

BI12: $\forall\alpha,\varpi,\Delta\in\mathfrak{B},\ 0*(0*((\alpha*\Delta)*(\varpi*\Delta)))=(0*\varpi)*(0*\alpha)$. $\forall\alpha,\varpi\in\mathfrak{B},\ 0*(0*(\alpha*\varpi))=(0*\alpha*\varpi)*(0*\alpha)$.

Definition 2. [11] A non-empty subset $\breve{I}$ of a BCK/BCI-algebra $\mathfrak{B}$ is called an ideal Given that it satisfies the required standards:

ID1: $0\in\breve{I}$

ID2: $\forall\alpha,\varpi\in\mathfrak{B}\ \alpha*\varpi\in\breve{I},\varpi\in I\rightarrow\alpha\in\breve{I}$

Definition 3. [3] Let $\mathfrak{B}$ be a BCK/BCI-algebra. A hesitant fuzzy set, $S:=\{(\alpha,\mu_S(\alpha))\mid\alpha\in\mathfrak{B}\}$ on $\mathfrak{B}$ is called a hesitant fuzzy ideal of $\mathfrak{B}$ if it satisfies :$\forall\alpha,\varpi\in\mathfrak{B}$,

$$\mu_S(\alpha*\varpi)\cap\mu_S(\varpi)\subseteq\mu_S(\alpha)\subseteq\mu_S(0) \tag{1}$$

Definition 4. [4] Let P be the set of parameters, for a subset A of P, A hesitant fuzzy soft set (S,A) over $\mathfrak{B}$ is called a hesitant fuzzy soft ideal based on $e\in A$ if the hesitant fuzzy set,

$$S_{[e]}:=\{(\alpha,\mu_{S_{[e]}}(\alpha))\mid\alpha\in\mathfrak{B}\} \tag{2}$$

is a hesitant fuzzy ideal of $\mathfrak{B}$.

Definition 5. [12] Let $\mathfrak{B}$ be a non-empty finite universe and $\mathbb{Q}$ be a non-empty set. A $\mathbb{Q}$-hesitant fuzzy set $S_{\mathbb{Q}}$ is a set given by

$$S_{\mathbb{Q}}=\{(\alpha,q),\mu_{S_{\mathbb{Q}}}(\alpha,q)\mid\alpha\in\mathfrak{B},q\in\mathbb{Q}\} \tag{3}$$

where $\mu_{S_{\mathbb{Q}}}:\mathfrak{B}\times\mathbb{Q}\rightarrow[0,1]$.

Definition 6. *[1] An m-polar* $\mathbb{Q}$*-hesitant fuzzy set on a non-empty set* $\mathfrak{B}$ *is the mapping* $S_{\mathbb{Q}}:\mathfrak{B}\times Q\rightarrow[0,1]^m$. *The membership value of every element* $\alpha\in\mathfrak{B}$ *is denoted by*

$$S=\{(\alpha,q),\mu_S^i(\alpha,q)\mid\alpha\in\mathfrak{B},q\in\mathbb{Q}\} \tag{4}$$

where $S:[0,1]^m\rightarrow[0,1]$ *is the i-th projection for all i=1,2,...,m.*

3. Multi-polar $\mathbb{Q}$-hesitant fuzzy soft Implicative Ideals

Definition 7. *A multi-polar $\mathbb{Q}$-hesitant fuzzy set*

$$S_\alpha = \{(\alpha, q), {\mu_\alpha}^i(\alpha, q) | \alpha \in B, q \in \mathbb{Q}\}$$

in $\mathfrak{B}$ is called a multi-polar $\mathbb{Q}$-hesitant fuzzy implicative ideal if it is satisfies the following:

1-

$$\mu_x(0, q) \supseteq \mu_\alpha(\alpha, q) \tag{5}$$

2-

$$\mu_\alpha(\alpha, q) \supseteq \mu_\alpha((\alpha * (\varpi * \alpha) * \Delta, q) \cap \mu_\alpha(\Delta, q) \tag{6}$$

where $\alpha, \varpi, \Delta \in \mathfrak{B}, q \in \mathbb{Q}$

Definition 8. *Let P be a set of parameters. For a subset A of P, a multi-polar $\mathbb{Q}$-hesitant fuzzy soft set (S,A) is called multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal over $\mathfrak{B}$ if the multi-polar $\mathbb{Q}$-hesitant fuzzy set*

$$S_{[e]} = \{(\alpha, q), {\mu_{S_{[e]}}}^i(\alpha, q) | \alpha \in \mathfrak{B}, q \in \mathbb{Q}\}$$

on $\mathfrak{B}$ is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal of $\mathfrak{B}$.

Example 1. *John, a 60-year-old retiree, has recently been diagnosed with Coronary Artery Disease. His cardiologist explains that he has significant blockages in his coronary arteries and recommends two treatment options: angioplasty and stent placement or coronary artery bypass grafting (CABG). John is faced with a decision that will impact his health and lifestyle.*
Angioplasty and Stent Placement: John's doctor explains that angioplasty and stent placement is a less invasive procedure. It involves inflating a balloon in the blocked artery and inserting a stent to keep the artery open. This procedure can be performed relatively quickly and may allow John to return to his daily activities sooner.
Coronary Artery Bypass Grafting (CABG): On the other hand, the doctor also discusses coronary artery bypass grafting (CABG), which is a more extensive surgical procedure. It involves opening John's chest, using healthy blood vessels from elsewhere in his body, and creating bypasses around the blocked arteries. While this surgery is more invasive and requires a longer hospital stay, it may provide a more permanent solution for his condition.
John's Decision-Making Process: John takes several factors into account when making his decision:

1- Health Condition: He considers the severity of his CAD and the advice of his cardiologist, who recommends CABG due to the complexity of his blockages.

2- Recovery Time: John values a quicker recovery, as he wants to spend time with his family and return to his hobbies.

3- *Long-Term Outlook: He is concerned about the long-term management of his heart health and prefers a solution that offers lasting benefits.*

Let $B = \{x_1, x_2, x_3\}$ *be a Bck-algebra set where* x_1 *is Health Condition,* x_2 *is the Recovery Time and* x_3 *is the Long-Term Outlook and consider the operation * on B defined in the next table*

$*$	x_1	x_2	x_3
x_1	x_1	x_1	x_1
x_2	x_2	x_1	x_2
x_3	x_3	x_3	x_1

Then $(B, *, x_1)$ *is a BCK-algebra.*
Consider the set $Q = \{\Gamma, \Upsilon\}$ *where* Γ *is Angioplasty and Stent Placement and* Υ *is Coronary Artery Bypass Grafting (CABG).*
the parameters set $Z = \{e_1, e_2, e_3\}$ *where* e_1 *Health Condition,* e_2 *Recovery Time and* e_3 *Long-Term Outlook. Let* $m = 3$

$*$	(x_1, Γ)	(x_1, Υ)	(x_2, Γ)
e_1	$\{(0.9), (0.8, 0.9), [0.8, 0, 8]\}$	$\{\{0.9\}, [0.8], (0.8, 0.9]\}$	$\{(0.8, 0.7], (0.7, 0.6, 0.6), (0.7, 0.6]\}$
e_2	$\{[0.9, 0.8], \{0.9\}, (0.7, 0.8)\}$	$\{(0.8, 0.9], (0.9), (0.9, 0.9)\}$	$\{(0.5, 0.4), (0.8, 0.7), (0.6)\}$
e_3	$\{\{0.8, 0.9, 0.8\}, (0.9), (0.7, 0.9)\}$	$\{(0.8, 0.8, 0.9), (0.9, 0.8), (0.9)\}$	$\{(0.3, 0.3, 0.4), (0.6, 0.5), [0.3, 0.1)\}$

$*$	(x_2, Υ)	(x_3, Γ)	(x_3, Υ)
e_1	$\{(0.8, 0.7), (0.6), [0.5]\}$	$\{(0.3, 0.5), (0.2), (0.1)\}$	$\{(0.1, 0.3), (0.1, 0.6), (0.5)\}$
e_2	$\{[0.7], (0.8, 0.8, 0.7), (0.6)\}$	$\{[0.5], (0.6, 0.5), (0.3)\}$	$\{(0.7, 0.6), (0.5, 0.7), [0.5]\}$
e_3	$\{[0.3], (0.5, 0.2), (0.3, 0.1)\}$	$\{(0.4, 0.3], (0.5, 0.4, 0.3), (0.2, 0.2)\}$	$\{(0.3, 0.1, 0.3), (0.4), [0.2, 0.3]\}$

Thus it's 3-polar $\mathbb{Q}$*-hesitant fuzzy soft Implicative Ideals.*

Proposition 1. *In Bck-algebra B The entirety of multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft implicative ideal is a multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft ideal*

Proof. *Let* $\mu_{S[e]}$ *be multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft implicative ideal over* $\mathfrak{B}$*.*
Let $\alpha, \varpi, \Delta \in \mathfrak{B}$*, then:*

$$\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}((\alpha * (\varpi * \alpha)) * \Delta, q) \cap \mu_{S[e]}(\Delta, q)$$

Replace $\varpi = \alpha$*, and using* $\alpha * \alpha = 0$*, we get*

$$\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}((\alpha * (\alpha * \alpha)) * \Delta, q) \cap \mu_{S[e]}(\Delta, q)$$

$$= \mu_{S[e]}(\alpha * \Delta, q) \cap \mu_{S[e]}(\Delta, q)$$

for all $\alpha, \Delta \in \mathfrak{B}$*,* $e \in A$*. Is multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft ideal.*

Theorem 1. *Let $\mathfrak{B}$ be an implicative BCK-algebra, then every multi-polar $\mathbb{Q}$-hesitant fuzzy soft ideal over $\mathfrak{B}$ is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal.*

Proof. Let $\mathfrak{B}$ be an implicative BCK-algebra, it follows that $\alpha = \alpha*(\varpi*\alpha)$, $\forall \alpha, \varpi \in \mathfrak{B}$ and $e \in A$. Let $\mu_{S[e]}$ be an multi-polar $\mathbb{Q}$-hesitant fuzzy soft ideal, then we have:

$$\begin{aligned} \mu_{S[e]}(\alpha, q) &\supseteq \mu_{S[e]}(\alpha * \Delta, q) \cap \mu_{S[e]}(\Delta, q) \\ &= \mu_{S[e]}((\alpha * (\varpi * \alpha)) * \Delta, q) \cap \mu_{S[e]}(\Delta, q) \end{aligned}$$

for all $\alpha, \varpi, \Delta \in \mathfrak{B}$, $e \in A$. Hence it is multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal of $\mathfrak{B}$.

Theorem 2. *Let (S,A) be a multi-polar $\mathbb{Q}$-hesitant fuzzy soft ideal of BCK-algebra $\mathfrak{B}$. Then (S,A) is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal of $\mathfrak{B}$ if and only if it satisfies the condition:*

$$\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}(\alpha * (\varpi * \alpha), q) \tag{7}$$

for all $\alpha, \varpi \in \mathfrak{B}$, $q \in Q$ and $e \in A$

Proof. Assume that (S,A) is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal of $\mathfrak{B}$. Take $\Delta = 0$ in

$$\begin{aligned} \mu_{S[e]}(\alpha, q) &\supseteq \mu_{S[e]}((\alpha * (\varpi * \alpha)) * \Delta, q) \cap \mu_{S[e]}(\Delta, q) \\ &= \mu_{S[e]}((\alpha * (\varpi * \alpha)) * 0, q) \cap \mu_{S[e]}(0, q) \\ &= \mu_{S[e]}((\alpha * (\varpi * \alpha)), q) \end{aligned}$$

conversely, suppose that (S,A) satisfies the condition. As (S,A) is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft ideal of $\mathfrak{B}$. We have

$$\begin{aligned} \mu_{S[e]}(\alpha, q) &\supseteq \mu_{S[e]}((\alpha * (\varpi * \alpha)), q) \\ &\supseteq \mu_{S[e]}((\alpha * (\varpi * \alpha)) * \Delta, q) \cap \mu_{S[e]}(\Delta, q) \end{aligned}$$

Then (S,A) is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal of $\mathfrak{B}$.
So, the establishment is fulfilled.

Proposition 2. *Every multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal of BCK-algebra is order-preseving.*

Proof. Let (S,A) be a multi-polar $\mathbb{Q}$-hesitant fuzzy soft implicative ideal over BCK-algebra.
Let $e \in A$ and $\alpha, \varpi \in B$ be such that $\alpha \geq \varpi$

$$\begin{aligned} \mu_{S[e]}(\alpha, q) &\supseteq \mu_{S[e]}((\alpha * (\Delta * \alpha)) * \varpi, q) \cap \mu_{S[e]}(\varpi, q) \\ &= \mu_{S[e]}((\alpha * \varpi) * (\Delta * \alpha), q) \cap \mu_{S[e]}(\varpi, q) \\ &= \mu_{S[e]}(0 * (\Delta * \alpha), q) \cap \mu_{S[e]}(\varpi, q) \\ &= \mu_{S[e]}(0, q) \cap \mu_{S[e]}(\varpi, q) \\ &\mu_{S[e]}(\varpi, q) \end{aligned}$$

Hence, the verification is accomplished.

Hence $\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}(\varpi, q)$ and this complete the proof.

4. Multi-polar $\mathbb{Q}$-hesitant fuzzy soft positive Implicative Ideals

Definition 9. *A multi-polar $\mathbb{Q}$-hesitant fuzzy set*

$$S_\alpha = \{(\alpha, q), {\mu_\alpha}^i(\alpha, q) | \alpha \in \mathfrak{B}, q \in \mathbb{Q}\}$$

for $i = 1, 2, ...m$. in $\mathfrak{B}$ is called a multi-polar $\mathbb{Q}$-hesitant fuzzy positive implicative ideal. If it satisfies the specified prerequisites:

1-

$$\mu_\alpha(0, q) \supseteq \mu_\alpha(\alpha, q) \tag{8}$$

2-

$$\mu_\alpha(\alpha * \Delta, q) \supseteq \mu_\alpha((\alpha * \varpi) * \Delta, q) \cap \mu_\alpha(\varpi * \Delta, q) \tag{9}$$

where $\alpha, \varpi, \Delta \in \mathfrak{B}, q \in \mathbb{Q}$

Definition 10. *Let P be a set of parameters. For a subset A of P, a multi-polar $\mathbb{Q}$-hesitant fuzzy soft set (S,A) is called multi-polar $\mathbb{Q}$-hesitant fuzzy soft positive implicative ideal over $\mathfrak{B}$ if the multi-polar $\mathbb{Q}$-hesitant fuzzy set*

$$S_{[e]} = \{(\alpha, q), {\mu_{S_{[e]}}}^i(\alpha, q) | \alpha \in \mathfrak{B}, q \in \mathbb{Q}\}$$

on $\mathfrak{B}$ is a multi-polar $\mathbb{Q}$-hesitant fuzzy soft positive implicative ideal of $\mathfrak{B}$.

Example 2. *Let's consider a scenario where an individual, Sarah, has received job offers from two different companies, each offering her three different positions. She needs to carefully evaluate these offers to make the best career choice.*
Company X:
Company X is a well-established manufacturing company with a reputation for stability and steady growth.
Job Offer A (Company X):
Position: Production Manager.

- *Salary: Competitive base salary with performance-based bonuses.*
- *Location: In a small town with a low cost of living.*
- *Work Environment: Traditional manufacturing facility with some travel*

Job Offer B (Company X):
Position: Supply Chain Analyst.

- *Salary: Competitive base salary with annual bonuses.*
- *Location: In a major city with a higher cost of living.*
- *Work Environment: Office-based, with occasional site visits.*

Job Offer C (Company X):
Position: Research and Development Scientist.

- *Salary: Competitive base salary with performance-based incentives.*
- *Location: In a mid-sized city with a moderate cost of living.*
- *Work Environment: Laboratory and research facilities.*

Company Y:
Company Y is a fast-growing tech startup known for its innovation and dynamic work culture.
Job Offer D (Company Y):
Position: Software Engineer.

- *Salary: Competitive base salary with stock options.*
- *Location: In a tech hub city with a moderate cost of living.*
- *Work Environment: Dynamic and collaborative, with remote work options.*

Job Offer E (Company Y):
Position: Marketing Specialist.

- *Salary:Competitive base salary with performance-based bonuses.*
- *Location: In a tech hub city with a moderate cost of living.*
- *Work Environment: Creative marketing department with flexible hours.*

Job Offer F (Company Y):
Position: Data Analyst.

- *Salary:Competitive base salary with stock options.*
- *Location: In a tech hub city with a moderate cost of living.*
- *Work Environment: Data analytics team with a strong focus on innovation.*

Let $B = \{n_1, n_2, n_3\}$ *be a BCK-algebra set where* n_1 *is the fist job offer,* n_2 *is the second job offer and* n_3 *is the third job offer.*
*consider the operation * on B define in the next table*

$*$	n_1	n_2	n_3
n_1	n_1	n_1	n_1
n_2	n_2	n_1	n_2
n_3	n_3	n_3	n_1

Then $(B, *, n_1)$ *is BCK-algebra.*
Consider the set $Q = \{\varsigma, \iota\}$ *where* ς *is company X and* ι *is company Y, the parameters set*

$S = \{e_1, e_2, e_3\}$ *where* e_1 *is the Salary,* e_2 *is the Location and* e_3 *is the Work Environment. let m=2, then:*

*	(n_1, ς)	(n_1, ι)	(n_2, ς)
e_1	$\{(0.9), (0.8, 0.9)\}$	$\{(0.9), (0.8, 0.8, 0.9)\}$	$\{(0.7, 0.5), (0.8, 0.7)\}$
e_2	$\{(0.8, 0.9), (0.8)\}$	$\{(0.9, 0.8, 0.9), (0.9, 0.9)\}$	$\{(0.8, 0.6), (0.5)\}$
e_3	$\{(0.8, 0.9), (0.8)\}$	$\{(0.9), (0.8)\}$	$\{(0.7)(0.7, 0.6, 0.7)\}$

*	(n_2, ι)	(n_3, ς)	(n_3, ι)
e_1	$\{(0.8, 0.8), (0.5, 0.6, 0.7)\}$	$\{(0.4, 0.3), (0.5, 0.2)\}$	$\{(0.7, 0.6, 0.5), (0.3, 0.2)\}$
e_2	$\{[0.7], (0.6, 0.5, 0.4)\}$	$\{(0.1, 0.3), (0.4, 0.3)\}$	$\{(0.6, 0.3), (0.1, 0.2)\}$
e_3	$\{(0.8, 0.7, 0.8), (0.7, 0.6)\}$	$\{(0.1, 0.2), (0.5, 0.4)\}$	$\{(0.2, 0.1), (0.2)\}$

After thoughtful consideration and discussions with mentors and friends, Sarah decides to accept Job Offer A (Production Manager.) at Company X. She values the dynamic work culture and opportunities for personal and professional growth. The combination of a competitive salary, performance-based bonuses, and flexibility aligns well with her long-term career aspirations and lifestyle preferences.
The previos table show that it's 2-polar $\mathbb{Q}$*-hesitant fuzzy soft positive Implicative Ideal over B.*

Proposition 3. *In BCK-algebra* $\mathfrak{B}$ *every multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal is multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft ideal.*

Proof. *Let (S,A) be a multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of BCK-algebra* $\mathfrak{B}$*. so for all* $\alpha, \varpi, \Delta \in \mathfrak{B}$, $q \in \mathbb{Q}$ *and* $e \in E$, *We hold*

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta), q) \cap \mu_{S[e]}(\varpi, q)$$

putting $\Delta = 0$

$$\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}(\alpha * \varpi, q) \cap \mu_{S[e]}(\varpi, q)$$

Therefore, (S,A) is a multi-polar $\mathbb{Q}$*-hesitant fuzzy soft ideal.*

Proposition 4. *Every multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of BCK-algebra* $\mathfrak{B}$ *is order-preserving.*

Proof. *Let* $\alpha, \varpi, \Delta \in \mathfrak{B}$ *and* $e \in A$ *be such that* $\alpha \geq \varpi$. *Since (S,A) is multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of* $\mathfrak{B}$.

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta), q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

$$= \mu_{S[e]}(0 * \Delta, q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

$$= \mu_{S[e]}(0, q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

$$= \mu_{S[e]}(\varpi * \Delta, q)$$

putting $\Delta = 0$

$$\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}(\varpi, q)$$

The proof is complete.

Proposition 5. *If* $\mathfrak{B}$ *is a positive implicative Bck-algebra then every multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft ideal of* $\mathfrak{B}$ *is a multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of* $\mathfrak{B}$ *.*

Proof. Assume that (S,A) is a multi-polar $\mathbb{Q}$*-hesitant fuzzy soft ideal of a positive implicative Bck-algebra, for all* $\alpha, \varpi \in \mathfrak{B}$ *and* $e \in A$*, then*

$$\mu_{S[e]}(\alpha, q) \supseteq \mu_{S[e]}(\alpha * \varpi, q) \cap \mu_{S[e]}(\varpi, q)$$

By replacing x *with* $\alpha * \Delta$ *and* y *with* $\varpi * \Delta$*, We obtain*

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \Delta) * (\varpi * \Delta), q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

since $\mathfrak{B}$ *is a positive implicative Bck-algebra* $(x * z(*(y * z) = (x * y) * z$ *for all* $\alpha, \varpi, \Delta \in \mathfrak{B}$ *Hence*

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta, q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

This indicates that (S,A) is a multi-polar $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of* $\mathfrak{B}$ *. As a result, the validation is done.*

Theorem 3. *Let (S.A) be a multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft ideal of* $\mathfrak{B}$*, then (S,A) is a multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of* $\mathfrak{B}$ *if and only if satisfies the inequalities*

$$\mu_{S[e]}(\alpha * \varpi, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta), q)$$

$\forall \alpha, \varpi \in \mathfrak{B}$ *and* $e \in A$

Proof. Take for granted that the multi-polar $\mathbb{Q}$*-hesitant fuzzy soft ideal (S,A) of a BCK-algebra* $\mathfrak{B}$ *is a multi-polar* $\mathbb{Q}$*-hesitant fuzzy soft filter positive implicative ideal of* $\mathfrak{B}$*. So*

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta), q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

$\forall \alpha, \varpi, \Delta \in \mathfrak{B}$ *and* $e \in A$*. substituting z=y, we have*

$$\mu_{S[e]}(\alpha * \varpi, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \varpi), q) \cap \mu_{S[e]}(\varpi * \varpi, q)$$

$$= \mu_{S[e]}((\alpha * \varpi) * \varpi, q) \cap \mu_{S[e]}(0, q)$$

$$= \mu_{S[e]}((\alpha * \varpi) * \varpi, q)$$

conversely, suppose that (H,A) is a multi-polar $\mathbb{Q}$*-hesitant fuzzy soft ideal over* $\mathfrak{B}$ *and satisfies the inequality*

$$\mu_{S[e]}(\alpha * \varpi, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \varpi), q)$$

since $\mu_{S[e]}(0,q) \supseteq \mu_{S[e]}(\alpha,q)$. *At this juncture, we can show that*

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta), q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

for all $\alpha, \varpi, \Delta \in \mathfrak{B}$ *and* $e \in A$. *In contrast, there exist* $\alpha', \varpi' \in \mathfrak{B}$ *In a way that*

$$\mu_{S[e]}(\alpha' * \varpi', q) \supseteq \mu_{S[e]}((\alpha' * \varpi') * \varpi'), q) \cap \mu_{S[e]}(\varpi' * \varpi', q)$$

$$= \mu_{S[e]}((\alpha' * \varpi') * \varpi', q) \cap \mu_{S[e]}(0,q)$$

$$= \mu_{S[e]}((\alpha' * \varpi') * \varpi', q)$$

which is a contradiction.
As a consequence

$$\mu_{S[e]}(\alpha * \Delta, q) \supseteq \mu_{S[e]}((\alpha * \varpi) * \Delta), q) \cap \mu_{S[e]}(\varpi * \Delta, q)$$

for $\alpha, \varpi, \Delta \in \mathfrak{B}$ *and* $e \in A$.
Thus (S,A) is a multi-polar $\mathbb{Q}$*-hesitant fuzzy soft positive implicative ideal of* $\mathfrak{B}$. *Consequently, the verification is wrapped up.*

5. Conclusion

In conclusion, our exploration of multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals in BCK/BCI-algebras has significantly expanded traditional fuzzy set theory and algebraic structures. We introduced and thoroughly studied these innovative ideals to address uncertainty and hesitation at multiple levels, thereby providing a more versatile and realistic modeling approach for real-world problem-solving. In this research, we introduced the concept of multi-polar Q-hesitant fuzzy sets, extending its application to BCK/BCI-algebras algebraic structures characterized by binary operations adhering to specific axioms. This introduction paved the way for defining multi-polar Q-hesitant fuzzy soft implicative ideals, allowing us to capture nuanced relationships between elements, incorporating varying degrees of implication beyond a binary true/false distinction. Additionally, we introduced positive implicative ideals, enhancing the traditional concept of implications in BCK/BCI-algebras by incorporating a positivity or favorability factor. This extension enables a more refined analysis of implications within the algebraic structure, introducing a preference or desirability dimension. Our study has demonstrated that exploring multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals in BCK/BCI-algebras goes beyond theoretical enrichment it has practical applications in decision-making, expert systems, and fuzzy logic. These concepts equip us with a robust toolset for handling uncertainty, hesitation, and preference in complex systems, enhancing our capacity to model and address real-world challenges. In summary, we introduced and studied the concepts of multi-polar Q-hesitant fuzzy soft implicative and positive implicative ideals in BCK/BCI-algebras. This research contributes to theoretical advancements and opens up new avenues for improving decision-making processes across various domains.

6. Data Availability

No data were used to support this study.

7. Conflicts of Interest

Te authors declare that there are no conficts of interest.

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