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Study on degenerate Stirling numbers

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Abstract. In this paper, we consider various Stirling numbers of both kinds, including the unsigned degenerate Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind, the unsigned degenerate r-Stirling numbers of the first kind and the degenerate r-Stirling numbers of the second kind. The aim of this paper is by using generating functions to further study explicit expressions, some identities and equivalent relations for those Stirling numbers.

2020 Mathematics Subject Classifications: 11B73, 11B83

Key Words and Phrases: unsigned degenerate Stirling numbers of the first kind, unsigned degenerate r-Stirling numbers of the first kind, degenerate Stirling numbers of the second kind, degenerate r-Stirling numbers of the second kind

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1. Introduction

The Stirling number of the second kind ${n \brace k}$ enumerates the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k nonempty disjoint subsets, while the unsigned Stirling number of the first kind ${n \brack k}$ counts the number of permutations of the set $[n]$ having k disjoint cycles. Let r be a positive integer. Then the Stirling numbers of both kinds are generalized as follows. The r-Stirling number of the second kind ${n \brace k}_r$ enumerates the number of partitions of the set $[n]$ into k nonempty disjoint subsets in such a way that $1, 2, \ldots, r$ are in distinct subsets, while the unsigned r-Stirling number of the first kind ${n \brack k}_r$ counts the number of permutations of the set $[n]$ having k disjoint cycles in such a way that $1, 2, \ldots r$ are in distinct cycles.

Carlitz initiated an investigation of degenerate versions of some special numbers and polynomials. Indeed, in [5] he studied degenerate versions of Bernoulli and Euler polynomials, namely the degenerate Bernoulli and degenerate Euler polynomials. In recent years, this exploration for degenerate versions have regained interests of some mathematicians and a lot of interesting results on degenerate versions of many special polynomials and numbers were obtained during the

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course of this quest (see [12, 14, 15] and the references therein). For instance, the unsigned degenerate Stirling number of the first kind ${n \brack k}_\lambda$ and the degenerate Stirling number of the second kind ${n \brace k}_\lambda$ are respectively degenerate versions of ${n \brack k}$ and ${n \brace k}$. Further, the unsigned degenerate r-Stirling number of the first kind ${n \brack k}_{r,\lambda}$ and the degenerate r-Stirling number of the second kind ${n \brace k}_{r,\lambda}$ are respectively degenerate versions of ${n \brack k}_r$ and ${n \brace k}_r$. These investigations about degenerate versions have been carried out by employing such diverse tools as generating functions, combinatorial methods, p-adic analysis, p-adic q-analysis, umbral calculus, probability theory, differential equations, analytic number theory, operator theory, and quantum mechanics.

The aim of this paper is by using generating functions to further study explicit expressions, some identities and equivalent relations for the aforementioned Stirling numbers of both kinds. The outline of this paper is as follows. In Section 1, we recall the degenerate exponentials and degenerate logarithms. We remind the reader of the unsigned Stirling numbers of the first kind and its generalization the unsigned r-Stirling numbers of the first kind, and the Stirling numbers of the second kind and its generalization the r-Stirling numbers of the second kind. Then we recall their degenerate versions, namely the unsigned degenerate Stirling numbers of the first kind and its generalization the unsigned degenerate r-Stirling numbers of the first kind, and the degenerate Stirling numbers of the second kind and its generalization the degenerate r-Stirling numbers of the second kind. Section 2 is the main result of this paper. We find an explicit expression for ${n \brace k}_\lambda$ as a finite sum involving $(l)_{n,\lambda}$ and an equivalent inverse relation expressing $(k)_{n,\lambda}$ in terms of ${n \brace j}_\lambda$ in Theorem 2.1. In Theorem 2.2, we express the finite sum $\sum_{k=1}^{m}(k)_{n,\lambda}H_k$ as a finite sum involving ${n \brace k}_\lambda$. Here H_k are the usual harmonic numbers. In Theorem 2.3, we find an explicit expression of ${n+r \brack k+r}_{r,\lambda}$ as a finite sum involving ${n \brack l}_\lambda$. In Theorem 2.4, we derive finite sum identities involving the unsigned Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind and the generalized falling factorials. An explicit expression for ${n+r \brace k+r}_{r,\lambda}$ is found as a finite sum involving $(l+r)_{n,\lambda}$, as a generalization of the corresponding result. Theorem 2.5 is a generalization of Theorem 2.1, while Theorem 2.6 is that of Theorems 2.2 and 2.4. For the rest of this section, we recall the facts that are needed throughout this paper.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t) = \sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{t^k}{k!}, \quad \text{(see [9, 10, 13, 16])}, \tag{1}$$

where the generalized falling factorials are given by

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1). \tag{2}$$

For $x = 1$, for brevity we write $e_\lambda(t) = e_\lambda^1(t) = \sum_{k=0}^{\infty}(1)_{k,\lambda}\frac{t^k}{k!}$.

As the compositional inverse of $e_\lambda(t)$, the degenerate logarithm is given by

$$\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1), \quad \text{(see [10])}. \tag{3}$$

Note that $\lim_{\lambda\to 0}\log_\lambda(t) = \log t$.

For $n \geq 0$, the unsigned Stirling numbers of the first kind are defined by

$$\langle x\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad \text{(see [1, 3, 4, 8, 17, 18])}, \tag{4}$$

where the rising factorials are given by

$$\langle x\rangle_0 = 1,\ \langle x\rangle_n = x(x+1)\cdots(x+n-1),\ (n \geq 1).$$

The Stirling numbers of the second kind are given by

$$x^n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (x)_k, \quad (n \geq 0), \quad \text{(see [2, 6, 7])}, \tag{5}$$

where the falling factorials are given by

$$(x)_0 = 1,\ (x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1).$$

For $n, r \geq 0$, the unsigned r-Stirling numbers of the first kind are defined by

$$\langle x+r\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k, \quad \text{(see [11, 12, 14])}. \tag{6}$$

The r-Stirling numbers of the second kind are defined by

$$(x+r)^n = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r (x)_k, \quad \text{(see [12, 14])}. \tag{7}$$

For any $\lambda \in \mathbb{R}$, the unsigned degenerate Stirling numbers of the first kind are defined by

$$\langle x\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_\lambda \langle x\rangle_{k,\lambda}, \quad (n \geq 0), \quad \text{(see [10, 14, 15])}, \tag{8}$$

where the generalized rising factorials are given by

$$\langle x\rangle_{0,\lambda} = 1\ \langle x\rangle_{n,\lambda} = x(x+\lambda)\cdots(x+(n-1)\lambda),\ (n \geq 1).$$

In view of (5), the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_\lambda (x)_k, \quad \text{(see [10])}. \tag{9}$$

For $n, r \geq 0$, the unsigned degenerate r-Stirling numbers of the first kind are given by

$$\langle x+r\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} \langle x\rangle_{k,\lambda} \quad (n \geq 0), \quad \text{(see [11, 13--15])}. \tag{10}$$

In [14], the degenerate r-Stirling numbers of the second kind are defined by

$$(x+r)_{n,\lambda} = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} (x)_k, \quad (n \geq 0). \tag{11}$$

2. Some formulas for degenerate Stirling numbers

For (9), we note that

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k=\sum_{n=k}^{\infty}{n \brace k}_\lambda\frac{t^n}{n!},\ (k\ge 0),\quad (\text{see [10, 14]}). \tag{12}$$

By (12), we get

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}=k!{n \brace k}_\lambda,\quad (n,k\ge 0). \tag{13}$$

Theorem 1. *For any nonnegative integers n,k, we have*

$$k!{n \brace k}_\lambda=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j)_{n,\lambda}\iff (k)_{n,\lambda}=\sum_{j=0}^{k}\binom{k}{j}j!{n \brace j}_\lambda. \tag{14}$$

Proof. ($\Longrightarrow$) Assume that

$$k!{n \brace k}_\lambda=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j)_{n,\lambda}.$$

Then we have

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}j!{n \brace j}_\lambda&=\sum_{j=0}^{k}\binom{k}{j}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}(l)_{n,\lambda}\\&=\sum_{l=0}^{k}(l)_{n,\lambda}\sum_{j=l}^{k}\binom{k}{j}\binom{j}{l}(-1)^{j-l}\\&=\sum_{l=0}^{k}(l)_{n,\lambda}\binom{k}{l}\sum_{j=0}^{k-l}(-1)^{k-l-j}\binom{k-l}{j}\\&=\sum_{l=0}^{k}(l)_{n,\lambda}\binom{k}{l}(1-1)^{k-l}=(k)_{n,\lambda}.\end{aligned}$$

($\Longleftarrow$) Suppose that

$$(k)_{n,\lambda}=\sum_{j=0}^{k}\binom{k}{j}j!{n \brace j}_\lambda.$$

Then we have

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j)_{n,\lambda}&=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{l=0}^{j}\binom{j}{l}l!{n \brace l}_\lambda\\&=\sum_{l=0}^{k}l!{n \brace l}_\lambda\sum_{j=l}^{k}\binom{k}{j}\binom{j}{l}(-1)^{k-j}\end{aligned}$$

$$= \sum_{l=0}^{k} l! \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_{\lambda} \binom{k}{l} \sum_{j=0}^{k-l} (-1)^{k-l-j} \binom{k-l}{j}$$
$$= \sum_{l=0}^{k} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_{\lambda} \binom{k}{l} l! (1-1)^{k-l} = k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\lambda}.$$

The harmonic numbers are defined by

$$H_0 = 0, \quad H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}, \quad (k \in \mathbb{N}). \tag{15}$$

Theorem 2. *For $n, m \in \mathbb{N}$, we have*

$$\sum_{k=1}^{m} (k)_{n,\lambda} H_k = \sum_{j=1}^{m} j! \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_{\lambda} \binom{m+1}{j+1} \left(H_{m+1} - \frac{1}{j+1} \right).$$

Proof. From Theorem 2.1 and noting that $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{\lambda} = 0$, for $n \geq 1$, we see that

$$\sum_{k=1}^{m} (k)_{n,\lambda} H_k = \sum_{k=1}^{m} H_k \sum_{j=1}^{k} \binom{k}{j} j! \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_{\lambda} \tag{16}$$
$$= \sum_{j=1}^{m} j! \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_{\lambda} \sum_{k=j}^{m} H_k \binom{k}{j}.$$

The forward difference operator $\triangle$ is defined by $\triangle f(x) = f(x+1) - f(x)$. From the definition of the forward difference operator, we get

$$f(x) \triangle g(x) = \triangle \big(f(x) g(x) \big) - \big(\triangle f(x) \big) g(x+1). \tag{17}$$

Thus, by (17), we get

$$\sum_{k=0}^{m-1} f(k) \big(\triangle g(k) \big) = \sum_{k=0}^{m-1} \triangle \big(f(k) g(k) \big) - \sum_{k=0}^{m-1} \big(\triangle f(k) \big) g(k+1) \tag{18}$$
$$= \sum_{k=0}^{m-1} \big(f(k+1) g(k+1) - f(k) g(k) \big) - \sum_{k=0}^{m-1} \big(\triangle f(k) \big) g(k+1)$$
$$= f(m) g(m) - f(0) g(0) - \sum_{k=0}^{m-1} \big(\triangle f(k) \big) g(k+1).$$

Let $f(k) = H_k$ and $g(k) = \binom{k}{j+1}$. Then we have

$$\begin{aligned} \triangle g(k) &= g(k+1) - g(k) = \binom{k+1}{j+1} - \binom{k}{j+1} = \binom{k}{j}, \\ \triangle f(x) &= f(k+1) - f(k) = H_{k+1} - H_k = \frac{1}{k+1}. \end{aligned} \tag{19}$$

From (18) and (19), we note taht

$$\begin{aligned}\sum_{k=0}^{m-1}\binom{k}{j}H_k &= \sum_{k=0}^{m-1}\big(\triangle g(k)\big)f(k) = f(m)g(m) - \sum_{k=0}^{m-1}\big(\triangle f(k)\big)g(k+1) \\ &= H_m\binom{m}{j+1} - \sum_{k=0}^{m-1}\frac{1}{k+1}\binom{k+1}{j+1} = H_m\binom{m}{j+1} - \frac{1}{j+1}\sum_{k=0}^{m-1}\binom{k}{j} \\ &= H_m\binom{m}{j+1} - \frac{1}{j+1}\sum_{k=0}^{m-1}\left[\binom{k+1}{j+1} - \binom{k}{j+1}\right] \\ &= H_m\binom{m}{j+1} - \frac{1}{j+1}\binom{m}{j+1} = \binom{m}{j+1}\left(H_m - \frac{1}{j+1}\right).\end{aligned} \tag{20}$$

Thus, by (16) and (20), we get

$$\sum_{k=1}^{m}(k)_{n,\lambda}H_k = \sum_{j=1}^{m} j!\left\{{n \atop j}\right\}_\lambda \sum_{k=j}^{m} H_k\binom{k}{j} = \sum_{j=1}^{m} j!\left\{{n \atop j}\right\}_\lambda \binom{m+1}{j+1}\left(H_{m+1} - \frac{1}{j+1}\right).$$

From (10), we note that

$$\begin{aligned}\left(\frac{1}{1-t}\right)^r\left(\frac{1}{1-t}\right)^x &= \sum_{n=0}^{\infty}\langle x+r\rangle_n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n+r \\ k+r\end{bmatrix}_{r,\lambda}\langle x\rangle_{k,\lambda}\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}\begin{bmatrix} n+r \\ k+r\end{bmatrix}_{r,\lambda}\frac{t^n}{n!}\right)\langle x\rangle_{k,\lambda}.\end{aligned} \tag{21}$$

On the other hand, by (1) and (3), we get

$$\begin{aligned}\left(\frac{1}{1-t}\right)^r\left(\frac{1}{1-t}\right)^x &= e_\lambda^{-x}\Big(\log_\lambda(1-t)\Big)\left(\frac{1}{1-t}\right)^r = \sum_{k=0}^{\infty}\frac{(-\log_\lambda(1-t))^k}{k!}\left(\frac{1}{1-t}\right)^r\langle x\rangle_{k,\lambda} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^k\left(\frac{1}{1-t}\right)^r\langle x\rangle_{k,\lambda}.\end{aligned} \tag{22}$$

By (21) and (22), we get

$$\frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^k\left(\frac{1}{1-t}\right)^r = \sum_{n=k}^{\infty}\begin{bmatrix} n+r \\ k+r\end{bmatrix}_{r,\lambda}\frac{t^n}{n!}, \quad (k\geq 0). \tag{23}$$

From (8) or (23), we see that

$$\frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^k = \sum_{n=k}^{\infty}\begin{bmatrix} n \\ k\end{bmatrix}_{\lambda}\frac{t^n}{n!}. \tag{24}$$

Theorem 3. *For any nonnegative integers n,k, we have*

$$\begin{bmatrix} n+1 \\ k+1\end{bmatrix}_{1,\lambda} = \sum_{l=k}^{n}\binom{l}{k}\begin{bmatrix} n \\ l\end{bmatrix}_{\lambda}\langle 1\rangle_{l-k,\lambda}.$$

In general, for $r \geq 0$, we have

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \sum_{l=k}^{n} \binom{l}{k} \begin{bmatrix} n \\ l \end{bmatrix}_{\lambda} \langle r \rangle_{l-k,\lambda}.$$

Proof. From (23), we note that

$$\frac{1}{1-t}\frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k} = \sum_{n=k}^{\infty} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1,\lambda} \frac{t^n}{n!}. \tag{25}$$

On the other hand, by (3) and (24), we get

$$\begin{aligned} \frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k}\frac{1}{1-t} &= \frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k}\sum_{l=0}^{\infty}\frac{\langle 1\rangle_{k,\lambda}}{l!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{l} \qquad (26)\\ = \sum_{l=0}^{\infty}\frac{\langle 1\rangle_{l,\lambda}(k+1)!}{l!k!}\frac{1}{(k+1)!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k+l} &= \sum_{l=0}^{\infty}\frac{\langle 1\rangle_{k,\lambda}(k+l)!}{k!l!}\sum_{n=k+l}^{\infty}\begin{bmatrix} n \\ k+l \end{bmatrix}_{\lambda}\frac{t^n}{n!}\\ = \sum_{l=0}^{\infty}\binom{k+l}{k}\langle 1\rangle_{l,\lambda}\sum_{n=k+l}^{\infty}\begin{bmatrix} n \\ k+l \end{bmatrix}_{\lambda}\frac{t^n}{n!} &= \sum_{l=k}^{\infty}\binom{l}{k}\langle 1\rangle_{l-k,\lambda}\sum_{n=l}^{\infty}\begin{bmatrix} n \\ l \end{bmatrix}_{\lambda}\frac{t^n}{n!}\\ = \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{l}{k}\begin{bmatrix} n \\ l \end{bmatrix}_{\lambda}\langle 1\rangle_{l-k,\lambda}\right)\frac{t^n}{n!}. \end{aligned}$$

Thus, by (25) and (26), we get

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{\lambda} = \sum_{l=k}^{n} \binom{l}{k} \begin{bmatrix} n \\ l \end{bmatrix}_{\lambda} \langle 1 \rangle_{l-k,\lambda}. \tag{27}$$

More generally, for any $r \geq 0$, we have

$$\begin{aligned} \sum_{n=k}^{\infty}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda}\frac{t^n}{n!} &= e_{\lambda}^{-r}\big(\log_{\lambda}(1-t)\big)\frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k} \qquad (28)\\ &= \sum_{l=0}^{\infty}\langle r\rangle_{l,\lambda}\frac{1}{l!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{l}\frac{1}{k!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k}\\ &= \sum_{l=0}^{\infty}\langle r\rangle_{l,\lambda}\binom{k+l}{l}\frac{1}{(k+l)!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{k+l}\\ &= \sum_{l=k}^{\infty}\langle r\rangle_{l-k,\lambda}\binom{l}{k}\frac{1}{l!}\left(\log_{-\lambda}\left(\frac{1}{1-t}\right)\right)^{l}\\ &= \sum_{l=k}^{\infty}\langle r\rangle_{l-k,\lambda}\binom{l}{k}\sum_{n=l}^{\infty}\begin{bmatrix} n \\ l \end{bmatrix}_{\lambda}\frac{t^n}{n!}\\ &= \sum_{n=k}^{\infty}\left(\sum_{l=k}^{n}\binom{l}{k}\begin{bmatrix} n \\ l \end{bmatrix}_{\lambda}\langle r\rangle_{l-k,\lambda}\right)\frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients on both sides of (28), we get

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \sum_{l=k}^{n} \binom{l}{k} \begin{bmatrix} n \\ l \end{bmatrix}_{\lambda} \langle r \rangle_{l-k,\lambda}, \quad (n,k \geq 0).$$

Note that

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1} = \lim_{\lambda \to 0} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1,\lambda} = \lim_{\lambda \to 0} \sum_{l=k}^{n} \binom{l}{k} \begin{bmatrix} n \\ l \end{bmatrix}_{\lambda} \langle 1 \rangle_{l-k,\lambda} = \sum_{l=k}^{n} \binom{l}{k} \begin{bmatrix} n \\ l \end{bmatrix},$$

and

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r} = \lim_{\lambda \to 0} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\lambda} = \sum_{l=k}^{n} \binom{l}{k} \begin{bmatrix} n \\ l \end{bmatrix} r^{l-k}.$$

Theorem 4. *For $n,m \in \mathbb{N}$, we have*

$$\sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} (k)_{n,\lambda} = \sum_{j=1}^{m} j! \begin{Bmatrix} n \\ j \end{Bmatrix}_{\lambda} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix}_{1}.$$

Proof. By (14), we get

$$\sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} (k)_{n,\lambda} = \sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \sum_{j=1}^{k} \binom{k}{j} j! \begin{Bmatrix} n \\ j \end{Bmatrix}_{\lambda} \tag{29}$$

$$= \sum_{j=1}^{m} j! \begin{Bmatrix} n \\ j \end{Bmatrix}_{\lambda} \sum_{k=j}^{m} \binom{k}{j} \begin{bmatrix} m \\ k \end{bmatrix} = \sum_{j=1}^{m} j! \begin{Bmatrix} n \\ j \end{Bmatrix}_{\lambda} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix}_{1}. \tag{30}$$

From (11), we note that

$$\frac{1}{k!}\big(e_{\lambda}(t)-1\big)^{k} e_{\lambda}^{r}(t) = \sum_{n=k}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} \frac{t^{n}}{n!}, \quad (k,r \geq 0). \tag{31}$$

By (31), we get

$$\sum_{n=k}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} \frac{t^{n}}{n!} = \frac{1}{k!}\big(e_{\lambda}(t)-1\big)^{k} e_{\lambda}^{r}(t) = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e_{\lambda}^{l+r}(t) \tag{32}$$

$$= \sum_{n=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l+r)_{n,\lambda} \frac{t^{n}}{n!}.$$

Thus, by comparing the coefficients on both sides of (32), we get

$$k! \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l+r)_{n,\lambda}, \quad (n \geq k). \tag{33}$$

Theorem 5. *For $n,r \geq 0$, we have*

$$k! \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_{r,\lambda} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l+r)_{n,\lambda} \iff (k+r)_{n,\lambda} = \sum_{l=0}^{k} \binom{k}{l} l! \begin{Bmatrix} n+r \\ l+r \end{Bmatrix}_{r,\lambda}.$$

Proof. This can be shown just as the proof of Theorem 2.1.

Theorem 6. *For $n, m \in \mathbb{N}$, we have*

$$\sum_{k=1}^{m} (k+r)_{n,\lambda} H_k = \sum_{j=1}^{m} j! \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_{r,\lambda} \binom{m+1}{j+1} \left(H_{m+1} - \frac{1}{j+1} \right),$$

and

$$\sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} (k+r)_{n,\lambda} = \sum_{j=1}^{m} j! \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_{r,\lambda} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix}_1 .$$

Proof. By using (33), this can be proved just as the proofs of Theorem 2.2 and Theorem 2.4. The details are left to the reader.

3. Conclusion

Various degenerate Stirling numbers of both kinds appear very frequently when we study degenerate versions of many special numbers and polynomials. In this paper, we investigated several degenerate Stirling numbers like the unsigned degenerate Stirling numbers of the first kind, the degenerate Stirling numbers of the second kind, the unsigned degenerate r-Stirling numbers of the first kind and the degenerate r-Stirling numbers of the second kind. We found some identities, explicit expressions and some equivalent relations among them. It is one of our future projects to continue to explore degenerate versions of some special numbers and polynomials and their applications to statistics, physics, science, engineering, and social sciences as well as to mathematics.

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