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Bayesian regression analysis using median rank set sampling

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Abstract. Bayesian estimation of the linear regression parameter system is considered by deploying Median Rank Set Sampling (MRSS). The full conditional distributions and the associated posterior distribution are obtained. Therefore, based on Markov Chain Monte Carlo simulation, the Bayesian point estimates and credible intervals for the regression parameters are determined. To measure the efficiency of the obtained Bayesian estimates concerning the frequentist estimates we compute the asymptotic relative efficiency of the obtained Bayesian estimates using Markov Chain Monte Carlo simulation.
This study shows that the Bayesian estimation of the simple linear regression parameters under frequentist MRSS is highly beneficial and much superior to the RSS scheme.

2020 Mathematics Subject Classifications: 62C10, 62J20, 62D05

Key Words and Phrases: Median Ranked Set Sampling; Bayes factor; Regression; Bayesian approach

1. Introduction

Regression is the process of investigating the relationship between a dependent variable and some independent variables given a set of historical data. The objective of regression is to discover a model that accurately captures this relationship and minimizes the discrepancy between predicted values from the model and the actual values observed in the data.

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Regression encompasses various techniques, and one common approach is linear regression, which takes the form of:

$$\mathbb{Y} = \beta_0 + \beta_1\mathbb{X}_1 + \beta_2\mathbb{X}_2 + \ldots + \beta_n\mathbb{X}_n + \epsilon \tag{1}$$

In the Equation 1, the dependent variable $\mathbb{Y}$ is related to the independent variables $\mathbb{X}_i$, where $1 \leq i \leq n$. The coefficients β_i represent the respective coefficients for each independent variable, and the term ϵ represents the error. This equation serves as a general representation of linear regression and can be modified to accommodate polynomial regression, as we will explore later in this section.

As previously mentioned, linear regression can be categorized into different types, and one such type is simple linear regression. In simple linear regression, the objective is to establish a model that relates a single dependent variable $\mathbb{Y}$ to a single independent variable $\mathbb{X}$. By substituting $n = 1$ into Equation 1, we obtain the equation specifically for simple linear regression:

$$\mathbb{Y} = \beta_0 + \beta_1\mathbb{X} + \epsilon \tag{2}$$

Note that $\mathbb{X}_1$ was referred to as $\mathbb{X}$ since it is the only independent variable in the data set.

The error in this context is defined as the Euclidean distance between each data point and the model. The objective of simple linear regression is to determine the optimal values of β_0 and β_1 that minimize this error.

Simple linear regression offers several advantages and disadvantages. Some of the advantages include simplicity, high performance, and interpretability. The simplicity arises from the model's relationship between the dependent variable $\mathbb{Y}$ and a single independent variable $\mathbb{X}$. This simplicity contributes to its high performance, making it suitable for real-time systems. Additionally, the model is easily interpretable, with the intercept (β_0) representing the expected value of $\mathbb{Y}$ when $\mathbb{X}$ is zero and the slope coefficient (β_1) indicating the change in $\mathbb{Y}$ for each unit increase in $\mathbb{X}$.

Simple linear regression also has some disadvantages. One such disadvantage is the linearity assumption, which assumes that the relationship between the variables is strictly linear. In reality, observations often exhibit non-linear patterns, making this assumption restrictive and potentially leading to higher mean square error. Another drawback is the sensitivity of simple linear regression to outliers. Outliers can disproportionately influence the model's estimates, leading to distorted results and increased mean square error.

As mentioned earlier, while Equation 1 represents the multi-variable regression, it can also be considered as a general equation of regression. As seen before, Equation 2 is a special case of Equation 1.

Similar to Simple linear regression, polynomial regression is a regression model that establishes a relationship between a single independent variable $\mathbb{X}$ and the dependent variable $\mathbb{Y}$. In contrast to simple linear regression, polynomial regression offers improved accuracy by incorporating higher-degree polynomials. Additionally, polynomial regression can be viewed as a special case of linear regression. The Equation 3 defines polynomial regression as follows:

$$\mathbb{Y} = \beta_0 + \beta_1\mathbb{X} + \beta_2\mathbb{X}^2 + \ldots + \mathbb{X}^n + \epsilon \tag{3}$$

Since data is being collected from real life scenarios, there is a possibility of error or uncertainly about the validity of the gathered data. That is when Bayesian statistics come into place. Bayesian statistics is a powerful technique that performs predictions based on history of data. It combines the uncertainty of data with the model which allows predictions to be realistic and match the real world event. The equation of Bayes rule is as follows:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \times \mathbb{P}(A)}{\mathbb{P}(B)} \tag{4}$$

Equation 4 deals with causalities. The equation evaluates the probability for event A to happen given that event B has already occurred and been observed. This probability depends on history. History here is seen as prior knowledge represented by $\mathbb{P}(A)$ that is the the probability of the occurrence of event A. The prior probability is also known to be the initial belief that even A would occur.
The marginal likelihood $\mathbb{P}(B)$, also known as evidence, is the overall probability for event B to occur. This is particularly important since Bayes studies the causality relationship between B and A, that is the frequency of B causing A.
The Likelihood of event A causing event B to occur is represented by $\mathbb{P}(B|A)$. Finaly the posterior probability or the most updated probability of event B causing the occurrence of A is given by $\mathbb{P}(A|B)$.
The problem with regression is the fact that it deals with continuous data that could literally be infinite when observed from a continuously active environment. When data exceeds certain size, the model could be vulnerable to over-fitting. We use sampling to make sure regression is performed smoothly and avoid any possible data problems. Sampling is much more efficient than data sample selection from a given population. Sampling also ensures data availability since collecting data for entire population might no be feasible specially for application that involve big data [5].
One of the sampling techniques is the Rank Set Sampling (RSS for short) [22]. RSS is very useful when data sizes are huge or if the evaluation or measurement of the independent variables is computationally expensive. Real time system that uses Regression analysis as an example can make great benefit of RSS sampling since system tasks have deadlines to meet. Rank Set Sampling performs grouping on observations in the available population. Every group has one or more observations. Groups are then ranked and chosen for sampling.
The structure of this article is as follows: Section 1 provides an introduction to the topic. Section 2 presents a review of prior research in the field. Section 3 outlines the proposed model described in this paper. Section 4 introduces the Bayes Estimator and the model selector proposed in this study. Section 5 discusses the Bayes factor of estimators utilized in this article. Section 6 describes the simulation methodology and presents the corresponding results. Finally, Section 8 offers a discussion of the findings and concludes the article [16, 19].

2. Previous Work

The introduction of Ranked Set Sampling (RSS), a relatively new sampling technique for estimating population mean, was originally presented in a paper by McIntyre [22]. RSS involves the following steps: Firstly, a visual inspection is employed to randomly select m^2 sample units from the target population. Secondly, these selected units are allocated into m sets as randomly as possible, with each set comprising m units. Thirdly, an RSS of size m is constructed for analysis by sequentially selecting the smallest ranked unit from the first set, the second smallest ranked unit from the second set, and so on, until the largest ranked unit is chosen from the last set. This process can be repeated r times to obtain a desired sample size of $n = rm$ [26].

Several authors have evaluated the effectiveness of Bayesian approaches in the context of RSS, including Al-Saleh et al. [2], Alodat and Al Sagheer [8], Wolfe [27], Kohlschmidt et al. [20], and Al-saleh and Al-Shrafa [3]. AlOdat et al [6] initially proposed the application of Bayesian statistics to RSS, and Al-Hamide et al. [4] further studied Bayesian inference for the linear regression model, assuming a prior distribution for the regression parameters following the alpha-skew-normal distribution.

Muttlak [23] introduced Median Ranked Set Sampling (MRSS), which involves quantifying the median of each set from the aforementioned m sets. Within each group, the units are ranked after randomly distributing the chosen m^2 units into m sets of size m. If m is odd, the $\frac{m+1}{2}^{th}$ smallest rank unit (median) is selected from each set, whereas if m is even, the $\frac{m}{2}^{th}$ smallest rank unit is chosen from the first $\frac{m}{2}$ set. This process is repeated r times until a sample of size $n = rm$ is obtained. AlOdat et al. [6] analyzed parameter estimators for a simple linear regression model using the MRSS scheme. Additionally, in his another article AlOdat et al. [7] discussed the large sample properties of the parameter estimators for simple linear regression based on the MRSS design.

In a study by Al-Hadhrami et al. [1], it was demonstrated that Bayesian estimation of the mean of a normal distribution using moving extreme ranked set sampling is more efficient than the frequentist approach of Simple Random Sampling (SRS). Hassan [13] obtained the maximum likelihood and Bayesian estimators of shape and scale parameters of the exponentiated exponential distribution based on SRS and RSS.

Li and Balakrshnan [21] developed the best linear unbiased estimators for parameters of a simple linear regression model using ordered RSS. Haq et al. [12] investigated the best linear unbiased estimators based on double RSS and ordered double ranked set sampling (DRSS) for the simple linear regression model with replicated observations.

Yao et al. [28] recently derived the best linear unbiased estimators for simple linear regression based on moving extremes RSS, which were found to be more efficient than the estimators obtained under SRS.

In a study by Sazak and Ozel [25], the modified maximum likelihood parameter estimation of the regression model using bivariate MRSS was investigated. The obtained estimators were compared with the least squares estimators based on MRSS, as well as with the modified maximum likelihood and least squares estimators based on RSS.

When estimating the population mean under symmetrical unimodal distributions, me-

dian ranked set sampling (MRSS), a variation of RSS, outperforms classical RSS [23]. Numerous studies have been conducted on MRSS and its uses [9, 11]. However, in the case of asymmetric distributions, MRSS can outperform RSS. Moreover, MRSS can improve estimator efficiency since error terms in regression models generally have a normal distribution. In order to develop effective estimators of regression model parameters, the Bayesian regression estimators were examined in this study utilizing a median ranked set sample.

3. Proposed Model

This article focuses on conducting a Bayesian analysis of the simple linear regression model using Ranked Set Sampling (RSS). We investigate the estimation of regression parameters by incorporating a prior distribution. Equation 5 represents the modified simple linear regression model defined by Equation 2, while taking sampling process into consideration.

$$Y_{ij} = \beta_0 + \beta_1 X_{ij} + \epsilon_{ij} \tag{5}$$

where

$$1 \leq i \leq r \text{ and } 1 \leq j \leq n \tag{6}$$

where r is the iteration number of the optimization process, and n is the group size. The group size $n = 2m - 1$ where m is the number of groups. In each group, we have an odd number of samples, that increases linearly with the number of groups.
ϵ_{ij} is the error for group i and data sample j. The normal distribution $\aleph(\mu, \sigma^2)$ with mean $\mu = 0$ and standard deviation of σ is used to calculate the error ϵ. β_0 is the interception and β_1 is the slope and both are unknown parameters that need to be found through the regression process.
Assume that we have a performance of the mean ranked set sampling (MRSS) on the variable of interest and

$$\mathbb{Y}_j(m) = Median(\mathbb{Y}_{j1}, \ldots, \mathbb{Y}_{jn}) \tag{7}$$

where $\mathbb{Y}_j(m)$ is the median of group m in iteration j.
The regression of $Y'^s_{j(m)}$ on X'^s_jcan be written as follows:

$$Y_{j(m)} = B_0 + B_1 X_{j(m)} + \epsilon_{j(m)}, \tag{8}$$

where

$$\epsilon_{j(m)} = Median(\epsilon_{j1}, \ldots, \epsilon_{jn}) \tag{9}$$

It can be noted that $\epsilon_{j(m)'s}$ are independently and identically distribution (iid) errors and have constant variance with the probability density function (pdf):

$$f(\epsilon) = \frac{(2m-1)!}{(m-1)(m-1)!}\Phi(\frac{\epsilon}{\sigma})^{m-1}\times$$

$$\Phi(\frac{-\epsilon}{\sigma})^{m-1}\Theta(\frac{\epsilon}{\sigma})\frac{1}{\sigma};$$
$$with - \infty < \epsilon < \infty \quad (10)$$

where Θ is the pdf of a standard normal random variable (i.e $\aleph(0,1)$) and Φ a cumulative standard Normal distribution (CDF).
Utilizing the MRSS scheme, the ordinary least squares estimators of β_0 and β_1 can be easily obtained as follows:

$$\begin{cases} \hat{\beta}_{0M} = \bar{Y}_m - \hat{\beta}_{1M}\bar{x}, \\ \\ \hat{\beta}_{0M} = \frac{\sum_{j=1}^{r}(Y_{j(m)-\bar{Y}_m})(x_j-\bar{x})}{S_{xx}}, \end{cases} \quad (11)$$

where

$$\bar{X} = \frac{1}{r}\sum_{j=1}^{r} x_j, \quad (12)$$

$$\bar{Y}_m = \frac{1}{r}\sum_{j=1}^{r} Y_{j(m)}, \quad (13)$$

and

$$S_{xx} = \sum_{j=1}^{r}(x_j - \bar{x})^2 \quad (14)$$

In this work, the Bayesian estimator of the parameters for the simple linear regression model is obtained employing MRSS and it is compared to SRS setup.

4. Bayes Estimator and Model selector

In this section, we find the Bayes estimator of each parameter in the model defined by Equation 5.

4.1. Bayesian Model

Based on the likelihood function and the prior distribution, a Bayesian model is described, where the likelihood function is the conditional distribution of the response variable, given all the parameters and covariates. We have the following prior distribution of each parameter, (β_0 ,β_1 ,σ^2):

$$m_1 : \begin{cases} \mathbb{Y}_{j(m)} = \beta_0 + \beta_1\mathbb{X}_j + \epsilon_{j(m)}, 1 \le j \le r \\ \beta_0 \sim \aleph(a, b^2) \\ \beta_1 \sim \aleph(\delta, g^2) \\ \sigma^2 \sim IG(\alpha, \beta) \end{cases} \quad (15)$$

Note that, $\aleph(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2 and $\mathrm{IG}(\alpha, \beta)$ is the inverse γ distribution. As far as the prior is concerned, all the hyperparameters a,b^2,δ, g^2, α, and β are considered a priori (conditionally) independent. Here, the fixed effects parameters have weakly informative marginal priors, i.e., in this case, the normal distribution $\aleph$ is centered at 0, with a large variance. Particularly, we let that: a, $\delta \sim \aleph(0, 50)$,$b$,$g \sim$ Uniform(0,10) and $\alpha, \beta \sim \gamma(2, 2)$. Distributions that are flat along the whole real number line are considered weakly informative priors because they don't provide any information.
Let $\pi_j(\theta), j = 1, 2, 3$, be the unknown parameters of the prior distribution and define the marginal density of the random variable $\mathbb{Y}$, we obtain this equation:

$$m_1\underline{y} = \int f_2(\underline{y} \mid \underline{\theta})\partial y, \tag{16}$$

where

$$\theta = (\beta_0, \beta_1, \sigma^2) \tag{17}$$

The distribution of $\mathbb{Y}_{j(m)}$ is given by:

$$f(y_{j(m)}; \theta) = \frac{C_m}{\sigma}\Phi(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})^{m-1}$$

$$\times [1 - \Phi(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})]^{m-1} \tag{18}$$

$$\times \Theta(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})$$

where

$$\theta = (\beta_0, \beta_1, \sigma^2) \tag{19}$$

and

$$C_m = \frac{(2m-1)!}{(m-1)!^{2\cdot}} \tag{20}$$

We assume that $\underline{\theta}$ has the following prior distribution:

$$\pi(\underline{\theta}) = \pi(\beta_0, \beta_1, \sigma^2) = \pi_1(\beta_0)\pi_2(\beta_1)\pi_3(\sigma^2), \tag{21}$$

where

$$\pi_1(\beta_0) = \frac{1}{\sqrt{2\pi b}} Exp\{-\frac{1}{2b^2}(\beta_0 - a)^2\} \tag{22}$$

and

$$-\infty < \beta_0 < \infty \tag{23}$$

$$\pi_2(\beta_1) = \frac{1}{\sqrt{2\pi b}} Exp\{-\frac{1}{2g^2}(\beta_1 - \delta)^2\} \tag{24}$$

and

$$-\infty < \beta_1 < \infty \tag{25}$$

$$\pi_3(\sigma^2) = \frac{1}{\Gamma(\alpha)(\sigma^2)^{\alpha+1}} Exp\{-\frac{1}{\sigma^2\beta}\} \tag{26}$$

and $\sigma^2 > 0$. Equation 22, 24 and 26 are our prior distributions of the three independent parameters of the vector θ.
If $\underline{y}_m(y_{1(m)}, y_{2(m)}, ..., y_{r(m)},)$, then $\underline{y}_m$ has the following joint pdf:

$$\begin{aligned} f_{\underline{y}_m}(\underline{y}_m;\underline{\theta}) = \frac{C_m^r}{\sigma^r} \prod_{j=1}^{r} [\Phi(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})^{m-1}] \\ \times [\Phi(\frac{-y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})^{m-1} \\ \times \Theta(\frac{-y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})]. \end{aligned} \tag{27}$$

The joint pdf is formed as a product of individual probabilities because we have assumed the observations are independently drawn from the underlying distribution. The term inside the product notation corresponds to the probability density of each individual observation $y_j(m)$. Each of these is distributed according to a modified normal distribution, with the parameters of the distribution depending on β_0, β_1, and x_j (value of the predictor variable for the jth group). The term $\Phi(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma})^{m-1}$ accounts for the cumulative likelihood of each residual from the model raised to the power of $(m-1)$. Similarly, $\Phi(\frac{-y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma})^{m-1}$ captures the remaining part of cumulative distribution from the observed value, also raised to the power of $(m-1)$. $\Theta(\frac{-y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma})$ is the pdf of the standardized residuals. This term models the density of the residuals around the model fit.
The posterior pdf of θ given $\underline{\mathbb{Y}}_m = \underline{y}_m$ is proportional to the product of the likelihood function, $f_{\underline{\mathbb{Y}}_m}(\underline{\mathbb{Y}}_m;\theta)$, and the prior distribution $\pi(\underline{\theta})$. After some manipulation we can write the posterior pdf as below:

$$\begin{aligned} \pi(\theta \mid \underline{\mathbb{Y}}_m)\, \alpha\, f_{\underline{\mathbb{Y}}_m}(\underline{\mathbb{Y}}_m;\theta)\pi(\underline{\theta}) \\ \alpha \frac{1}{\sigma^r}[\prod_{j=1}^{r} E_j] \frac{1}{(\sigma^2)^{\alpha+1}} [Exp\{-\frac{1}{2b^2}(\beta_0 - a)^2\} \\ \times Exp\{-\frac{1}{2g^2}(\beta_1 - \delta)^2\} \times Exp\{-\frac{1}{\sigma^2\beta}\}], \end{aligned} \tag{28}$$

where

$$E_j = [\Phi(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma})\Phi\times$$

$$(\frac{\beta_0+\beta_1 x_j - y_{j(m)}}{\sigma})]^{m-1} \times \theta(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma}) \quad (29)$$

The posterior pdf is then compactly represented as the product of three components give as:

$$\pi(\beta_0,\beta_1,\sigma^2 \mid \underline{y}_m)\,\alpha\,\frac{1}{(\sigma^2)^{\frac{r}{2}+\alpha+1}} \exp(\frac{-\sum_{j=1}^{r}(y_{j(m)}-\beta_0-\beta_1 x_j)^2}{2\sigma^2} - \frac{(\beta_0-a)^2}{2b^2} - \frac{(\beta_1-\delta)}{2g^2} - \frac{1}{\sigma^2\beta})x \prod_{j=1}^{r}\Phi(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma})^{m-1} \Phi(\frac{-y_{j(m)}+\beta_0+\beta_1 x_j}{\sigma})^{m-1}. \quad (30)$$

were $\frac{1}{(\sigma^2)^{\frac{r}{2}+\alpha+1}}$ is a normalization term, the second term is the exponential function which has four components in the exponent, each representing a component of the prior distributions. Lastly, the third term is the product over all residuals of two standard normal cumulative distribution functions (CDFs) evaluated at the standardized residuals, each raised to the power of (m-1).
For facilitating the derivation of posterior distributions we first reformulate the sum of residuals using some simple algebraic manipulations as:

$$\sum_{j=1}^{r}(y_{j(m)}-\beta_0-\beta_1 x_j)^2 = \sum_{j=1}^{r}(y_{j(m)}-\beta_1 x_j-\bar{y}_m+\beta_1\bar{x})^2 + r(\bar{y}-\beta_1\bar{x}-\beta_0)^2 \quad (31)$$

then the marginal conditional distribution of β_0 given β_1, σ^2 and $\underline{y_m}$ is:
$\pi_1(\beta_0 \mid \beta_1, \sigma^2, \underline{y}_m)$

$$\alpha \times Exp\{\frac{-r(\beta_0-\bar{y}+\beta_1\bar{x})^2}{2\sigma^2} - \frac{(\beta_0-a)^2}{2b^2} \times \prod_{j=1}^{r}\Phi(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma})^{m-1} \times \Phi(\frac{-y_{j(m)}+\beta_0+\beta_1 x_j}{\sigma})^{m-1}, \alpha Exp\{-\frac{\beta_0^2}{2}(\frac{r}{\sigma^2}+\frac{1}{b^2})+$$

$$\beta_0(r\frac{\beta_1\bar{x}-\bar{y_m}}{\sigma^2}+\frac{a}{b^2})\}$$
$$\times\prod_{j=1}^{r}\Phi(\frac{y_{j(m)}-\beta_0-\beta_1x_j}{\sigma})^{m-1}$$
$$\times\Phi(\frac{-y_{j(m)}+\beta_0+\beta_1x_j}{\sigma})^{m-1}, \tag{32}$$

This marginal posterior distribution represents our updated beliefs about the parameter β_0, given the observed data and the specified values for the other parameters. After some manipulations we compactly write it as:
$\pi_1(\beta_0 \mid \beta_1, \sigma^2, \underline{y}_m)$

$$\alpha Exp\{-\frac{(\frac{r}{\sigma^2}+\frac{1}{\underline{2}})}{2}$$
$$\times(\beta_0-\frac{r\frac{\beta_1\bar{x}-\bar{y}_m}{\sigma^2}+\frac{a}{b^2}}{\frac{r}{\sigma^2}+\frac{1}{b^2}})^2\}$$
$$\times\prod_{j=1}^{r}\Phi(\frac{y_{j(m)}-\beta_0-\beta_1x_j}{\sigma})^{m-1}$$
$$\times\Phi(\frac{-y_{j(m)}+\beta_0+\beta_1x_j}{\sigma})^{m-1}. \tag{33}$$

We can easily identify two terms; the first one is the Gaussian likelihood function with the required adjustment for the prior belief about the parameter β_0; second term used to capture the cumulative distribution of the residuals. Overall, this expression models how the residuals are distributed around the model fit and combines this with prior information and likelihood derived from the data to give a posterior belief about β_0.

The conditional distribution for β_1 given β_0, σ^2 and $\underline{y_m}$ is:

$$\begin{aligned}
&\pi_2(\beta_1 \mid \beta_0, \sigma^2, \underline{y}_m) \\
&\quad \alpha \quad \exp\left\{\frac{-\sum_{j=1}^{r}(y_{j(m)} - \beta_0 - \beta_1 x_j)^2}{2\sigma^2}\right\} \\
&\quad \times \exp\left\{-\frac{(\beta_1 - \delta)^2}{2g^2}\right\} \\
&\times \prod_{j=1}^{r}\left[\Phi\left(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma}\right)^{m-1}\right. \\
&\qquad \left.\times\ \Phi\left(\frac{-y_{j(m)} + \beta_0 + \beta_1 x_j}{\sigma}\right)^{m-1}\right] \\
&\quad \alpha \quad \exp\left\{\frac{-2\sum_{j=1}^{r} x_j(y_{j(m)} - \beta_0)\beta_1}{2\sigma^2}\right\} \\
&\quad \times \exp\left\{-\frac{\beta_1^2 \sum_{j=1}^{r} x_j^2}{2\sigma^2}\right\} \\
&\quad \times \exp\left\{-\frac{\beta_1^2}{2g^2} + \frac{\delta\beta_1}{g^2}\right\} \\
&\times \prod_{j=1}^{r}\left[\Phi\left(\frac{y_{j(m)} - \beta_0 - \beta_1 x_j}{\sigma}\right)^{m-1}\right. \\
&\qquad \left.\times\ \Phi\left(\frac{-y_{j(m)} + \beta_0 + \beta_1 x_j}{\sigma}\right)^{m-1}\right].
\end{aligned} \tag{34}$$

By some simplification,

$$\begin{aligned}\pi_2(\beta_1 \mid \beta_0, \sigma^2, \underline{y}_m) \quad \alpha \quad & \exp\left\{-\frac{\beta_2^2}{2}\left(\frac{\sum_{j=1}^r x_j^2}{\sigma^2}+\frac{1}{g^2}\right)\right. \\ & \left.+\left(\frac{\beta_1^2\sum_{j=1}^r x_j^2}{\sigma^2}+\frac{\delta}{g^2}\right)\beta_1\right\} \\ & \times\prod_{j=1}^r\left[\Phi\left(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma}\right)^{m-1}\right. \\ & \left.\times\Phi\left(\frac{-y_{j(m)}+\beta_0+\beta_1 x_j}{\sigma}\right)^{m-1}\right] \\ \alpha \quad & \exp\left\{-\frac{\frac{\sum_{j=1}^r x_j^2}{\sigma^2}+\frac{1}{g^2}}{2}\right. \\ & \left.+\left[\beta_1-\frac{\frac{\sum_{j=1}^r x_j(y_{j(m)}-\beta_0)}{\sigma^2}+\frac{\delta}{g^2}}{\frac{\sum_{j=1}^r x_j^2}{\sigma^2}+\frac{1}{g^2}}\right]^2\right\} \\ & \times\prod_{j=1}^r\left[\Phi\left(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma}\right)^{m-1}\right. \\ & \left.\times\Phi\left(\frac{-y_{j(m)}+\beta_0+\beta_1 x_j}{\sigma}\right)^{m-1}\right].\end{aligned} \tag{35}$$

and finally we have that:

$$\begin{aligned}\pi_2(\beta_1 \mid \beta_0, \sigma^2, \underline{y}_m) \quad \alpha \quad & \exp\left\{-\frac{\frac{\sum_{j=1}^r x_j^2}{\sigma^2}+\frac{1}{g^2}}{2}\right. \\ & \left.+\left[\beta_1-\frac{\frac{\sum_{j=1}^r x_j(y_{j(m)}-\beta_0)}{\sigma^2}+\frac{\sigma^2\delta}{g^2}}{\sum_{j=1}^r x_j^2+\frac{\sigma^2}{g^2}}\right]^2\right\} \\ & \times\prod_{j=1}^r\left[\Phi\left(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma}\right)^{m-1}\right. \\ & \left.\times\Phi\left(\frac{-y_{j(m)}+\beta_0+\beta_1 x_j}{\sigma}\right)^{m-1}\right].\end{aligned} \tag{36}$$

The conditional distribution of σ^2 given β_0, β_1 and $\underline{y_m}$ is given in the simplified compact

form as:

$$\begin{aligned}\pi_3(\sigma^2 \mid \beta_0, \beta_1, \underline{y}_m) \quad \alpha \quad & \frac{1}{(\sigma^2)^{\frac{r}{2}+\alpha+1}} \\ & \times \exp\left\{\frac{\frac{-\sum_{j=1}^{r}(y_{j(m)}-\beta_0-\beta_1 x_j)^2}{2}+\frac{1}{\beta}}{\sigma^2}\right\} \\ & \times \prod_{j=1}^{r}\left[\Phi\left(\frac{y_{j(m)}-\beta_0-\beta_1 x_j}{\sigma}\right)^{m-1}\right. \\ & \left.\times\Phi\left(\frac{-y_{j(m)}+\beta_0+\beta_1 x_j}{\sigma}\right)^{m-1}\right].\end{aligned} \tag{37}$$

5. Bayes Factor of Estimators in MRSS

The Bayes factor is the ratio of the marginal densities of the two cases. The Bayes factor is given by:

$$B_{ji} = \frac{m_j(y)}{m_i(y)} = \frac{\int f_j(\underline{y} \mid \underline{\theta})\pi_j(\theta)d\theta}{\int f_i(\underline{y} \mid \underline{\theta})\pi_i(\theta)d\theta} \tag{38}$$

Here, this ratio evaluates the modification of the odds of $m_j(y)$ against $m_i(y)$ due to the observation and can naturally be compared to 1, although an exact comparison scale can only be based upon a loss function. And the Bayes factor depends on prior information. It measures the strength of evidence for a model in a way that takes into account both the goodness of fit and the complexity of the model. However, it requires the computation of multi-dimensional integrals, which can be quite difficult in practice, especially for complex models with many parameters. For this reason, it's often approximated using techniques like Markov chain Monte Carlo (MCMC) methods.
We can simplify the Bays factor for the estimator depending on their posterior density as:

$$\begin{aligned}\hat{\beta_{0b}} = & \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta_0}{\sigma^r}[\prod_{j=1}^{r}E_m]\times\frac{1}{(\sigma^2)^{\alpha+1}}\times}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\prod_{j=1}^{r}E_m]d\beta_0 d\beta_1 d\sigma^2} \\ & \times\left[\exp\left(-\frac{1}{2b^2}(\beta_0-a)^2\right)\right. \\ & \left.\times\exp\left(-\frac{1}{2g^2}(\beta_1-\delta)^2\right)\times\exp\left(-\frac{1}{\sigma^2\beta}\right)\right]d\beta_0 d\beta_1 d\sigma^2\end{aligned} \tag{39}$$

In this particular equation, the expectation of β_0 ($\hat{\beta_{0b}}$) under the posterior distribution is being calculated, which serves as the Bayesian point estimate for this parameter. This estimate takes into account both the likelihood of the data given the parameters and the

prior beliefs about the parameters.

$$\begin{aligned}\hat{\beta_{1b}} =& \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\beta_1}{\sigma^r}[\prod_{j=1}^{r}E_m]\times\frac{1}{(\sigma^2)^{\alpha+1}}\times}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\prod_{j=1}^{r}E_m]d\beta_0 d\beta_1 d\sigma^2} \\ &\times\left[\exp\left(-\frac{1}{2b^2}(\beta_0-a)^2\right)\right. \\ &\left.\times\exp\left(-\frac{1}{2g^2}(\beta_1-\delta)^2\right)\times\exp\left(-\frac{1}{\sigma^2\beta}\right)\right]d\beta_0 d\beta_1 d\sigma^2\end{aligned} \tag{40}$$

$$\begin{aligned}\hat{\sigma_b^2} =& \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma^2}{\sigma^r}[\prod_{j=1}^{r}E_m]\times\frac{1}{(\sigma^2)^{\alpha+1}}\times}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\prod_{j=1}^{r}E_m]d\beta_0 d\beta_1 d\sigma^2} \\ &\times\left[\exp\left(-\frac{1}{2b^2}(\beta_0-a)^2\right)\right. \\ &\left.\times\exp\left(-\frac{1}{2g^2}(\beta_1-\delta)^2\right)\times\exp\left(-\frac{1}{\sigma^2\beta}\right)\right]d\beta_0 d\beta_1 d\sigma^2\end{aligned} \tag{41}$$

All the above three equations involve a triple integral over all possible values of β_0, β_1, and σ_b^2. In the Bayesian context, the integral goes from $(-\infty, +\infty)$ because we're integrating over all possible values that the parameters could take. In practice, the values are constrained to make sense in the context of the problem in hand.
The equations are written in such a way to clearly distinguish three common terms involved; in the numerator, the terms for each equation $\frac{\beta_0}{\sigma^r}$, $\frac{\beta_1}{\sigma^r}$ and $\frac{\sigma^2}{\sigma^r}$ are part of the function we want to find the expected value of, where r is the number of data points. The product $\prod_{j=1}^{r}E_m$ represents the likelihood function for the model. The terms involving exponential functions are the prior distributions of β_0, respectively. We multiply these three parts together and integrate over the parameters to get the expected value of β_0, β_1, and σ_b^2 under the posterior distribution. In the denominator, the integral of the likelihood over all possible parameter values, which serves as a normalization factor to ensure that the result is a valid probability distribution. This is often called the evidence or marginal likelihood in Bayesian statistics. The ratio of the numerator to the denominator gives the expected value of β_0, β_1, and σ_b^2 which is an estimate for this parameter under the posterior distribution. The integrals above are typically not solvable analytically, especially for complex models. Thus, numerical methods like MCMC are often used to approximate these integrals.

6. Simulation Study

In this study, the MCMC numerical method is used to approximate the calculation of the posterior expectation of the parameters β_0, β_1, and σ_b^2, derived in section 5. This serves as the Bayesian point estimate for this parameters. We run the MCMC simulation for 200,000 iterations. An iteration here corresponds to one cycle through the algorithm – proposing new values for the parameters, checking whether these new values are likely

given the data (using the likelihood function), and deciding whether to accept these new values or keep the old ones [15]. The first 5,000 iterations are discarded. This is known as the "burn-in" period. The idea is that early iterations are based on our initial (potentially poor) guess, and so we have to give the algorithm some time to converge towards the true values before we start collecting data. After the burn-in period, we start collecting data, but not at every iteration. Instead, we only keep the parameter values every 50th iteration. This is known as "thinning". The purpose is to reduce the correlation between successive samples, which can bias the results. After the burn-in and thinning, you are left with 5,000 samples. These are the values of the parameters at different points in the MCMC simulation. Each sample represents a possible set of parameters that could have generated the data. Finally, we need to check whether the algorithm has converged – that is, whether it has found the true posterior distribution. This is typically done using various statistical tests, such as the Geweke diagnostic, which compares the mean of the early and late portions of the chain, and the Heidelberger-Welch diagnostics, which involve a stationarity test (to see if the chain has reached equilibrium) and a half-width test (to see if the chain has run long enough to achieve a desired precision) [17, 24].
Through this process, we have generated many different samples of parameter values. The collection of these samples approximates the posterior distribution of the parameters and the triple integral in the formula for the posterior mean of β_0 can then be approximated by taking the average of β_0 over these samples.
In order to assess the performance of the simulation and the convergence of the chain, in Figure 1 we show the trace plots for the three parameters of interest β_0,β_1 and σ_b^2. The trace plot of each parameter displays the sequence of sampled values (on the y-axis) at each step in the MCMC chain (on the x-axis). It enables us to visualize the path taken by the Markov Chain over the iterations. By looking at the trace plot, we can assess whether the chain has converged to the target distribution. Ideally, the plot should look like a 'hairy caterpillar' – it should oscillate around a constant mean without any trend, and it should cover the entire range of plausible values for the parameter, meaning it mixes well. Figure 1 show the values of β_0, β_1 and σ_b^2 within a stable range after a certain number of iterations. This indicates that the MCMC simulation has "converged" on an estimate for β_0, β_1 and σ_b^2. It also show that these three parameters values are "mixing" well, meaning they move freely and explore the entire range of plausible values rather than getting stuck in particular regions [18]. This indicates that the MCMC simulation is effectively exploring the full range of possible values for β_0, β_1 and σ_b^2. We conclude from Figure 1 that the MCMC simulation is likely to provide a very good estimate for this three parameter.

In this study, the benefits of Bayesian estimates based on the Median Ranked Set Samples (MRSS) have been explored. This exploration is carried out via MCMC numerical simulations following the steps outlined below:

- Generate Ranked Set Samples (RSS) of size 'n' from the full posterior distribution for the case when m = 1. Here, 'm' refers to the number of cycles used in the sampling process. In many practical applications, one cycle (m = 1) is commonly used.

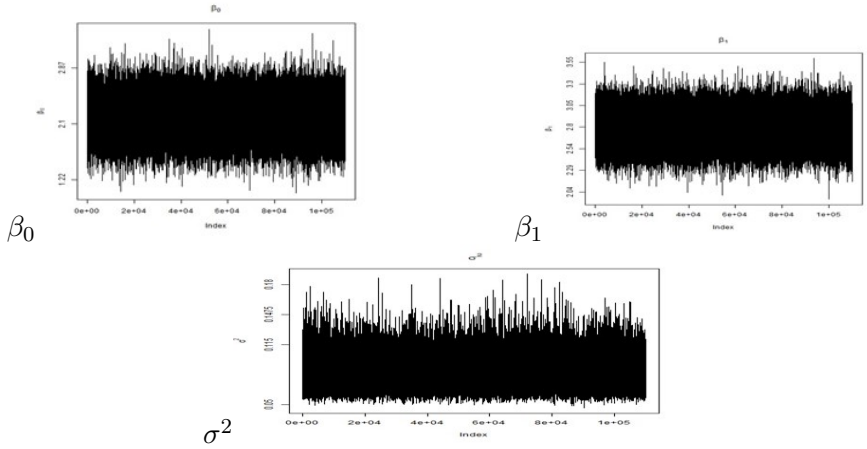

Figure 1: Trace plot of the parameters β_0, β_1, and σ^2

- Use these RSS samples to calculate the Bayesian estimates as described in Section 5. These estimates rely on the MRSS samples.

The estimates are then analyzed in two ways:

- Point Estimation: The posterior mean and standard error (SE) for each parameter is computed for different values of 'm'. These values are presented in Table 1. This table includes the mean and SE for parameters β_0, β_1 and σ_b^2, under different 'm' values. The 'm' values are increased from 3 to 7. A key observation from this table is that as the value of 'm' increases, the Bayesian standard error for all parameters decreases. This indicates that as the cycle of sampling increases, the accuracy of the estimates also increases.

- Interval Estimation: 95% credible intervals for the mean posterior of the parameters are also calculated. These intervals, representing a range of values within which the true parameter value lies with 95% certainty, are presented in Table 2. Here, for each parameter and for different 'm' values, the 2.5th percentile and the 97.5th percentile values are reported. The range between these two percentile values forms the 95% credible interval.

The tables illustrate that as the number of cycles 'm' increases, the posterior means become more accurate (SE decreases) and the credible intervals become narrower. These outcomes suggest that by using more cycles in the MRSS sampling, we can achieve more precise estimates and more confidence about where the true parameter values lie.
From the tables provided, we can also infer that as the number of cycles (m) increases, not only does the precision of our estimates improve (seen by decreasing standard error), but our credible intervals become narrower. This indicates that our certainty about the true values of the parameters increases. The findings support the use of larger 'm' values in applications where precision is critical. However, larger 'm' values may involve more

complex computations or more extensive data collection. The fact that standard errors decrease as 'm' increases suggests that the estimator becomes more efficient with more cycles, which is advantageous in providing more reliable estimates.

Table 1: The Posterior mean and the standard error (SE) for the parameters at different **m**.

	$\hat{\beta}_0$		$\hat{\beta}_1$		$\hat{\sigma^2}$	
	Mean	SE	Mean	SE	Mean	SE
m=3	1.5	0.65	2.24	0.54	0.17	0.057
m=5	1.24	0.55	2.13	0.41	0.09	0.0096
m=7	1.12	0.34	2.03	0.30	0.07	0.0067

The 95% credible intervals for the parameters at different **m**.

	$\hat{\beta}_0$		$\hat{\beta}_1$		$\hat{\sigma^2}$	
	2.5%	97.5%	2.5%	97.5%	2.5%	97.5%
m=3	1.22	2.87	2.04	3.55	0.05	0.18
m=5	1.12	2.06	1.85	2.34	0.06	0.11
m=7	0.97	1.76	1.67	2.65	0.05	0.10

Table 3 compare the efficiency of Bayesian and frequentist Mean Rank Set Sampling (MRSS) methods for estimating the parameters of a simple regression model. The efficiency is calculated as the ratio of the Mean Squared Errors (MSEs) of the parameter estimates from the two methods. Specifically, the MSE of a parameter estimate using the frequentist MRSS method (denoted by $\hat{\theta_2}$) is divided by the MSE of the same parameter estimate using the Bayesian MRSS method (denoted by $\hat{\theta_1}$).
The table lists the efficiency ratios for the three parameters of interest: β_0, β_1 and σ_b^2. Each row in the table corresponds to a different value of 'm', the size of the cycle in MRSS. Each efficiency value in the table (like 1.29, 1.76, etc.) is calculated as the MSE of the frequentist MRSS estimate divided by the MSE of the Bayesian MRSS estimate for a given parameter and 'm' value. This efficiency ratio measures how well the Bayesian method performs in comparison to the frequentist method. The MSE is a common measure of an estimator's quality: the smaller the MSE, the better the estimator. The efficiencies are all greater than one, indicating that the Bayesian MRSS method is more efficient than the frequentist MRSS method for all parameter estimates and all values of 'm'. This is because a smaller MSE (which implies better performance) for the Bayesian method results in a larger efficiency ratio. As 'm' increases from 3 to 7, the efficiencies also increase. This shows that the advantage of the Bayesian MRSS method over the frequentist MRSS method becomes more pronounced as 'm' increases. In summary from Table 3, we conclude that in this particular application, the Bayesian MRSS method provides more efficient (lower MSE) estimates of the regression parameters than the frequentist MRSS method, and this advantage increases as the size of the MRSS cycle ('m') increases.

Table 3: The efficiency of Bayesian regression parameters using MRSS concerning MRSS with different **m**. efficiency $(\hat{\theta_1}, \hat{\theta_2})$= MSE$(\hat{\theta_1})$/MSE$(\hat{\theta_2})$

	efficiency $(\hat{\beta}_0, \hat{\beta_{0M}})$	efficiency $(\hat{\beta}_1, \hat{\beta_{1M}})$	efficiency $(\hat{\sigma^2}, \hat{\sigma^2_M})$
m=3	1.29	1.76	1.01
m=5	1.78	2.88	1.98
m=7	2.21	3.07	2.34

7. Discussion

In the present study, we developed a Bayesian model for estimating the parameters of a simple linear regression model through median ranked set sampling (MRSS). Our results demonstrate that Bayesian estimators derived from this approach are more efficient than those from the frequentist MRSS method detailed in [6].

This study marks the first exploration of a Bayesian model for estimating coefficients of a simple linear regression model via MRSS. The primary goal was to investigate the potential of Bayesian statistical analysis for a straightforward and efficient estimation of the simple linear regression model via MRSS, especially when compared with the traditional frequentist MRSS approach.

The paper aligns with the results of Alodat et al., [6] and AlSaleh and AlSharafat [3], emphasizing that Bayesian estimators prove to be more efficient than those derived from simple random sampling. We also noticed a decrease in the standard error (SE) of Bayesian estimates based on MRSS as the sample size m increases, reinforcing findings by De Iorio et al. [10] and Helo et al., [14], both of whom demonstrated a similar trend in ranked set sampling and the Bayesian estimation of Weibull parameters respectively.

This study assumes a symmetric distribution for random errors. When compared to the frequentist model in Alodat et al. [6], our Bayesian model's estimates exhibit a noticeable advantage: they are simpler both in mathematical formulation and calculation. This ease and efficiency in handling underline the potential advantages of Bayesian estimation when dealing with MRSS in simple linear regression models.

The findings of this study bear significant implications for statistical modeling and inferential analysis. They establish the value of Bayesian approaches in providing robust and efficient estimates in the context of a simple linear regression model, using median ranked set sampling (MRSS).

One key implication of the study is that it demonstrates the efficiency of Bayesian estimators in comparison to their frequentist MRSS counterparts. This points to the potential of Bayesian approaches to be more widely applied in statistical modeling and inferential analysis, providing researchers with more robust and precise tools to analyze their data.

In addition, this study suggests that as the sample size (m) increases, the standard error (SE) of Bayesian estimates based on MRSS decreases. This trend could inform future data collection strategies, by encouraging the collection of larger sample sizes to improve the precision of Bayesian estimates.

8. Conclusion

Related to our knowledge, in the literature, there are no previous studies on Bayesian estimating of the linear regression parameters using MRSS. Thus, the emphasis of this paper is to estimate the simple linear regression parameters via MRSS in Bayesian approach.

This study investigated the Bayesian estimation of parameters in a simple linear regression model, employing MRSS. Using Markov Chain Monte Carlo (MCMC) numerical simulations, Bayesian estimates were obtained and analyzed. The findings underscored the efficiency of Bayesian estimators obtained through MRSS, surpassing their frequentist counterparts using the same MRSS methodology, at least for one MRSS design. Additionally, it was observed that the Bayesian standard error for all parameters diminished as the values of m increased.

However, this study is not without its limitations. The efficiency of the Bayesian estimators was established in the context of one MRSS specific design and this may not hold true for all MRSS design. MRSS is a specific type of RSS where the median item from each subset is chosen. The way these subsets are chosen and ranked can vary, and these specifics constitute the MRSS design. Further research should investigate the performance of Bayesian estimators in a more diverse range of MRSS design. Additionally, this study has taken a symmetric distribution of random errors as an assumption. Future investigations could benefit from considering asymmetric or heavy-tailed error distributions to expand the scope of this research.

Moreover, while this study highlights the increased efficiency and simplicity of the Bayesian approach, it is crucial to further explore the computational aspects and feasibility of these methods, especially in the context of large-scale data. Future research should also delve into the practical implications and applications of the methodology proposed in this study, in various fields and for a variety of research questions.

In conclusion, this study has paved the way for further research in the field of Bayesian estimation using MRSS, providing a springboard for more extensive studies, wider applications, and refined methodologies.

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