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Binomials Arising from Buchberger Algorithm on Polyomino Ideals

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Abstract. A polyomino is a finite set of unit squares joined side by side on the Cartesian plane. Qureshi introduced an ideal constructed from a polyomino which is called "polyomino ideal". In this paper, we study the binomials arising from Buchberger Algorithm on polyomino ideals. We also introduce socket wrench polyominoes and study the Gröbner bases of the ideal $I_{\mathcal{P}}$ and some algebraic properties of $K[\mathcal{P}]$.

2020 Mathematics Subject Classifications: 05B50, 05E40, 13C05

Key Words and Phrases: Buchberger Algorithm, Gröbner Bases, Polyomino, Radical Ideal

1. Introduction

A polyomino is a finite set of unit squares joined side by side on the Cartesian plane. They are discussed in a lot of papers. Look at: [2, 3] for Combinatorics; [17–19] for its relation to the tiling problem on the plane; [12] for the relation between polyominoes and Dyck Words and Motzkin Word; and [41] for statistical physics.

The relation between polyominoes and commutative algebra was introduced by Qureshi, introducing an ideal constructed from a polyomino which is called polyomino ideal [34]. The polyomino ideal is a generalization of ideals generated by the set of 2-minor of a matrix. Generally, the ideal of t-minors is a central topic in Commutative Algebra and has some applications in algebraic statistics [33, 40]. There were a lot of research related to the ideal generated by the set of $t-$minor of a matrix [22, 27].

Since it was introduced by Qureshi in 2012, many interesting question have arisen about polyomino ideal. Here are some recent works and related results:

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- The primality of the polyomino ideal is studied in many articles [5, 7, 24–26, 31, 32, 35, 36, 38]. In [24, 25, 36], it is proved that $K[\mathcal{P}]$ is a domain if $\mathcal{P}$ is simple. In [31], it is proved that if $K[\mathcal{P}]$ is a domain then the polyomino have no zig-zag walks. They also conjectured that the converse direction is true. Later, it was proved in [5] and [7] that the conjecture holds for two special classes of non-simple polyominoes, called closed path and weakly closed paths. Still, a complete classification of polyominoes with prime polyomino ideal is not known.
- The algebraic properties, like when $K[\mathcal{P}]$ is Cohen-Macaulay or Gorenstein, are known only for some specific polyominoes. In [36] the authors show that if $\mathcal{P}$ is simple then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain, by identifying their quotient ring with the toric ring of a weakly chordal graph. In [6], the authors show that if $\mathcal{P}$ is a closed path polyomino having no zig-zag walks then $K[\mathcal{P}]$ is a Cohen-Macaulay domain. In [34], Qureshi established that the Cohen-Macaulay property holds for convex polyominoes and characterized all stack polyominoes $\mathcal{P}$ for which $K[\mathcal{P}]$ is Gorenstein. The Gorensteiness is also studied in [1, 8, 10, 16, 35, 37].
- Gröbner basis of polyomino ideals are studied in [6, 20, 25, 26, 32, 34].
- The König type property is studied for simple thin polyominoes in [21], for closed path polyominoes in [13], and for grid polyominoes in [14].
- The linearly related polyominoes are studied in [15].
- The Charney-Davis conjecture for simple thin-polyominoes are studied in [29].
- The primary decomposition of polyomino ideals, like closed paths, and more in general for polyocollections is studied in [9].
- Another challenging problem is to compute the h-polynomial of $K[\mathcal{P}]$ in terms of the rook polynomial of $\mathcal{P}$ [8, 16, 28, 30, 35, 37].

An important class of ideals other than the prime ideal is the radical ideal. Radical ideal plays an important role in Algebraic Geometry, for example the Strong Nullstellensatz Theorem [11]. Qureshi gave an example of a non-simple polyomino with non-prime polyomino ideal [36] that is radical.

The radicality of an ideal can be studied from the Gröbner bases of the ideal. If we can define a monomial order such that every element in the Gröbner bases has square-free initial monomial then the ideal is radical [23, Problem 1.8(b)]. The Gröbner bases of an ideal can be computed by using Buchberger Algorithm [23, Section 1.3].

In this paper, we study some elements arising from Buchberger Algorithm to polyomino ideals. In the second section, polyominoes and some terminologies related to our study will be defined. In the third section, we will perform the Buchberger Algorithm in polyomino ideals. In the fourth sections, we will apply the results from previous sections to a class of polyominoes that we call *socket wrench polyominoes*. We prove that for the socket wrench polyomino $\mathcal{P}$, the ideal I_P has square-free quadratic Gröbner bases for a suitable monomial order. We also study some algebraic properties of the K-algebra $K[\mathcal{P}] = S/I_{\mathcal{P}}$.

2. Preliminaries

In this section, we will recall the definitions and terminologies about polyomino and polyomino ideal from [5] and [34]. Consider the set $\mathbb{Z}^2$ and define the partial order: $(i,j) \leq (k,\ell)$ if and only if $i \leq k$ and $j \leq \ell$.

(i) Let $a, b \in \mathbb{Z}^2$ with $a \leq b$. The set $[a,b] = \{c \in \mathbb{Z}^2 \mid a \leq c \leq b\}$ is called an *interval*.

(ii) Let $a = (i,j)$ and $b = (k,\ell)$. The elements a and b are called the *diagonal corners* of the interval $[a,b]$, and the elements (i,ℓ) and (k,j) are called the *antidiagonal corners* of the interval $[a,b]$. Particularly, the elements (i,j) and (k,ℓ) are called *left lower corner* and *the right upper corner*, respectively, of the interval $[a,b]$. Similarly, the elements (i,ℓ) and (k,j) are called *the left upper corner* and *the right lower corner*, respectively, of the interval $[a,b]$.

(iii) If $b = a + (1,1)$ then the interval $[a,b]$ is called a *cell*.

(iv) The *edges* of a cell $[a, a+(1,1)]$ are the intervals: $[a, a+(1,0)]$, $[a, a+(0,1)]$, $[a+(0,1), a+(1,1)]$, and $[a+(1,0), a+(1,1)]$.

(v) Let $\mathcal{P}$ be a finite collection of cells in $\mathbb{Z}^2$. The collection of all *vertices* of $\mathcal{P}$, denoted by $V(\mathcal{P})$ is the union of all corners from each cells in $\mathcal{P}$.

(vi) Let $a = (i,j), b = (k,\ell) \in \mathbb{Z}^2$. The vertices a and b are called in *horizontal position* if $j = \ell$ and in *vertical position* if $i = k$.

(vii) Let $\mathcal{P}$ be a finite collection of cells in $\mathbb{Z}^2$. Let C and D be two cells in $\mathcal{P}$. The cells C and D are called *connected* if there exists a sequence of cells $C = C_1, \ldots, C_m = D$ in $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for all $i = 1, 2, \ldots, m-1$.

(viii) A finite collection of cells $\mathcal{P}$ in $\mathbb{Z}^2$ is called a *polyomino* if any two cells in $\mathcal{P}$ are connected.

(ix) A *walk* from cell C to cell D in $\mathbb{Z}^2$ is a sequence of cells $\mathcal{C} : C = C_1, \ldots, C_m = D$ in $\mathbb{Z}_2$ such that $C_i \cap C_{i+1}$ is an edge of C_i and C_{i+1} for all $i = 1, 2, \ldots, m-1$. If $C_i \neq C_j$ for all $i \neq j$ then $\mathcal{C}$ is called a *path*. A polyomino $\mathcal{P}$ is called *simple* if for any two cells C and D not belonging to $\mathcal{P}$, there exist a path $\mathcal{C} : C = C_1, \ldots, C_m = D$ such that $C_i \notin \mathcal{P}$ for all $i = 1, \ldots, m$.

(x) Let $\mathcal{P}$ be a polyomino and $(i,j), (k,\ell) \in V(\mathcal{P})$ such that $i < k$ and $j < \ell$. The interval $[(i,j),(k,\ell)]$ is called an *inner interval* of $\mathcal{P}$ if any cell $[(r,s),(r+1,s+1)]$ is an element in $\mathcal{P}$ for all $i \leq r \leq k-1$ and $j \leq s \leq \ell - 1$.

(xi) Let $\mathcal{P}$ be a polyomino. The interval $[(i,j),(k,j)]$ with $i < k$ is called in a *horizontal edge interval* of $\mathcal{P}$ if the interval $[(\ell,j),(\ell+1,j)]$ are edges of cells of $\mathcal{P}$ for all $\ell = i, \ldots, k-1$. If $[(i-1,j),(i,j)]$ and $[(k,j),(k,j)]$ are not edges af cells if $\mathcal{P}$ then the interval $[(i,j),(k,j)]$ is called a *maximal horizontal edge interval* of $\mathcal{P}$. We define the *vertical edge interval* and the *maximal vertical edge interval* similarly.

(xii) Let $\mathcal{P}$ be a polyomino and K be a field. Define the polynomial ring S over K with variables x_{ij} for all $(i,j) \in V(\mathcal{P})$. Each inner interval $[(i,j),(k,\ell)]$ in $\mathcal{P}$ is associated to $x_{ij}x_{k\ell} - x_{i\ell}x_{kj} \in S$, that is called the *inner 2-minor* of $\mathcal{P}$. The set of all inner 2-minors of $\mathcal{P}$ is denoted by S_2.

(xiii) Let $\mathcal{P}$ be a polyomino. The ideal $I_{\mathcal{P}} \subseteq S$ generated by S_2 is called the *polyomino ideal* of $\mathcal{P}$ and $K[\mathcal{P}] = S/I_{\mathcal{P}}$ the coordinate ring of $\mathcal{P}$.

(xiv) Let $J \subseteq S$ be a binomial ideal and $f = f^+ - f^-$ be a binomial in J. The binomial f is called *redundant* if it can be expressed as a linear combination of binomials in J od lower degree. The binomal f is called *irredundant* if it is not redundant. We also denote by V_f^+ the set of vertices v such that x_v divides f^+ and by V_f^- the set of vertices v such that x_v divides f^-.

3. Buchberger Algorithm in Polyomino Ideal

Let $\mathcal{P}$ be a polyomino. We define an ordering in the set $V(\mathcal{P})$ in the following way: $(i,j) <_{\mathcal{P}} (k,\ell)$ if and only if

- $j < \ell$ or
- $j = \ell$ and $i < k$.

By this ordering, all the vertices in a polyomino with n vertices can be labelled with positive integer $1, 2, \ldots, n$ from left to right, starting from the vertices with the lowest ordinate to the vertices with the highest ordinate. Below is an example of such labelling.

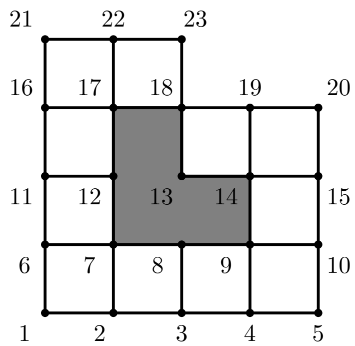

Figure 1: Labelling the Set $V(\mathcal{P})$.

By this labelling, the polynomial ring associated to the polyomino ideal is $S = K[x_1, \ldots, x_n]$. We use the lexicographic monomial order with $x_1 > x_2 > \cdots > x_n$. For the sake of simplicity, the elements $x_a \in R$ will be written with a. We define the degree of monomials $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ with $\sum_{i=1}^{n} a_i$. We also define the *interval determined by* $\{a,b\}$ as the interval with diagonal corners $\{a,b\}$ or antidiagonal corners $\{a,b\}$. Now, we are ready to perform the Buchberger Algorithm. Since S_2 consists of inner 2-minors then the polynomial obtained by the Buchberger Algorithm in each step is again a binomial consisting of two monomials of the same degree. The *degree* of this binomial is defined as the degree of both monomials.

3.1. Binomials of Degree Three

The result in this subsection can also be derived from [34, Theorem 4.1] and [32, Proposition 3.2]. We start the Buchberger Algorithm by computing the S-polynomial $\mathcal{S}(F,G)$ for every $F,G \in S_2$. The S-polynomial $\mathcal{S}(F,G)$ is defined by

$$\mathcal{S}(F,G) = \frac{\operatorname{lcm}(\operatorname{in}_<(F),\operatorname{in}_<(G))}{c_F \cdot \operatorname{in}_<(F)} - \frac{\operatorname{lcm}(\operatorname{in}_<(F),\operatorname{in}_<(G))}{c_G \cdot \operatorname{in}_<(G)}$$

where $\operatorname{in}_<(F)$ (resp. $\operatorname{in}_<(F)$) denotes the *initial monomial* of F (resp. G) with respect to $<$ and c_F (resp. c_G) denotes the coefficient of $\operatorname{in}_<(F)$ (resp. $\operatorname{in}_<(G)$) in F (resp. G).

- If the initial monomial of F and G are relatively prime then $\mathcal{S}(F,G)$ is reduced to zero.
- If the greatest common divisor of their initial monomials is a monomial of degree two then $F = G$ and $\mathcal{S}(F,G) = 0$.
- If the greatest common divisor of their initial monomials is a monomial of degree one then $\mathcal{S}(F,G)$ is a binomial of degree three.

We will compute $\mathcal{S}(F,G)$ in the last possibility and find the condition for the S-polynomial to be not reduced to zero. Consider the case when F and G have common factor in their non-initial monomials (reader may see [6, Remark 1] for more general result). So, let $F = ab - pq$ and $G = ac - pr$ with initial monomials ab and ac, respectively. Note that $\mathcal{S}(F,G) = p(br - cq)$ and the interval determined by $\{b,r\}$ is an inner interval. We conclude that $\mathcal{S}(F,G)$ is reduced to zero.

Now we assume that F and G have no common factor in their non-initial monomial. Let F and G be the inner 2-minors associated to inner intervals $[a,b]$ and $[c,d]$, respectively. Without losing of generality, assume that $a \leq_{\mathcal{P}} c$. Write $F = ab - pq$ and $G = cd - rs$ with $p <_{\mathcal{P}} q$ and $r <_{\mathcal{P}} s$. We conclude that $a <_{\mathcal{P}} d$. We consider three cases.

(i) If $a = c$ then $\mathcal{S}(F,G) = brs - dpq$.

Consider the location of vertex d, there are 4 cases to discuss (see the figure below).

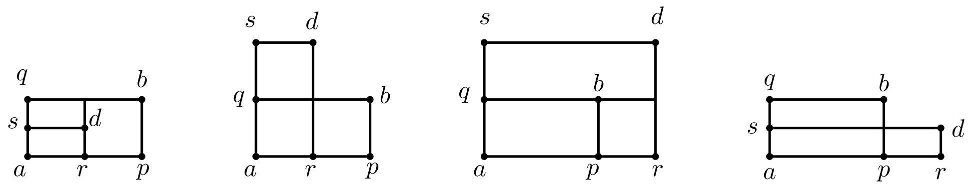

Figure 2: Case $a = c$.

For the first three cases, the interval determined by $\{d, q\}$ is an inner interval. Since s, d are in horizontal position and s, q are in vertical position, then there is a vertex y such that the inner interval determined by $\{q, d\}$ is the inner interval determined by $\{s, y\}$. By checking all possible configurations (see the figure below), we see that the inner interval determined by $\{b, r\}$ is the inner interval determined by $\{p, y\}$.

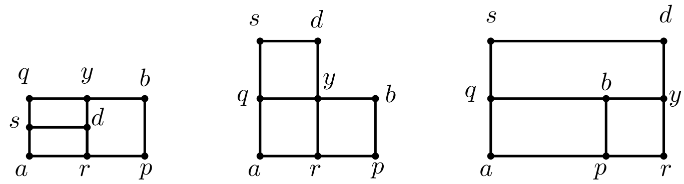

Figure 3: The first three cases.

Note that

$$\mathcal{S}(F, G) = brs - dpq = (br - py)s + p(ys - dq).$$

Therefore, $\mathcal{S}(F, G)$ is reduced to zero.

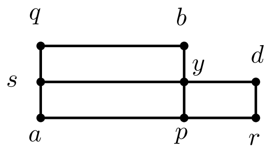

Figure 4: The fourth case.

For the fourth case, the interval determined by $\{d, p\}$ is an inner interval (see the figure above). Similarly, there is a vertex y such that the inner interval determined by $\{p, d\}$ is the inner interval determine by $\{r, y\}$. Note that

$$\mathcal{S}(F, G) = brs - dpq = (bs - qy)r + q(ry - dp).$$

Therefore, $S(F, G)$ is reduced to zero.

(ii) If $a \neq c$ and $b = d$ then $\mathcal{S}(F, G) = ars - cpq$

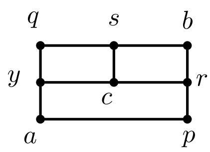

Figure 5: Case $a \neq c$ and $b = d$.

Note that there exists a vertex y such that q, c are the antidiagonal corners of the inner interval determined by $\{q, c\}$. Since

$$\mathcal{S}(F, G) = (ar - py)s + p(ys - cq)$$

we conclude that $\mathcal{S}(F, G)$ is reduced to zero.

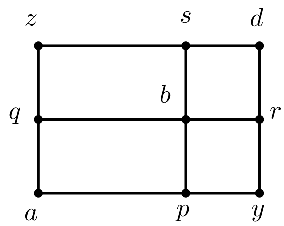

Figure 6: Case $a \neq c$ and $b \neq d$.

(iii) If $a \neq c$ and $b \neq d$ then $b = c$ and $\mathcal{S}(F, G) = ars - dpq$.

Note that ars is the initial monomial of $S(F, G)$ and this monomial can only be divided by initial monomials ar or as from an inner 2-minor. This is the only case when $[a, r]$ or $[a, s]$ is an inner interval. If $[a, r]$ is an inner interval then

$$\mathcal{S}(F, G) = (ar - qy)s + q(ys - pd)$$

with y is the antidiagonal corner other than q from the inner interval $[a, r]$. If $[a, s]$ is an inner interval then

$$\mathcal{S}(F, G) = (as - pz)r + p(zr - qd)$$

with z is the antidiagonal corner other than p from the inner interval $[a, s]$.

From the observation above, we conclude the following theorem.

Theorem 1. *Let F and G be the inner 2-minors associated to inner interval $[a, b]$ and $[c, d]$, respectively, with $a \leq_{\mathcal{P}} c$. The binomial $\mathcal{S}(F, G)$ is not reduced to zero by all inner 2-minors if and only if*

- *$b = c$ and*
- *the interval determined by $\{a, r\}$ for all r that is an antidiagonal corner of $[c, d]$ is not an inner interval.*

Definition 1. *Let $\mathcal{P}$ be a polyomino. Define S_3 to be the set of all binomials $a_1a_3a_5 - a_2a_4a_6$ such that*

- *a_i, a_{i+1} are in vertical position for $i = 1, 3, 5$*
- *a_i, a_{i+1} are in horizontal position for $i = 2, 4, 6$ (with $a_7 = a_1$)*
- *$a_1 <_{\mathcal{P}} a_2 <_{\mathcal{P}} a_3 <_{\mathcal{P}} a_4$ and $a_1 <_{\mathcal{P}} a_6 <_{\mathcal{P}} a_5 <_{\mathcal{P}} a_4$*
- *both intervals determined by $\{a_1, a_3\}$ and $\{a_1, a_5\}$ are not inner intervals*
- *the interval determined by $\{a_1, b\}$ and $\{b, a_4\}$ are inner interval with b is the intersection of the segments a_2a_3 and a_5a_6.*

Note that there are no elements in S_3 whose initial monomial is divisible by the initial monomial of an inner 2-minor. Therefore, we conclude that S_3 is the set of all binomials of degree three arising from Buchberger Algorithm in the polyomino ideal with respect to the given monomial order $<_{\mathcal{P}}$.

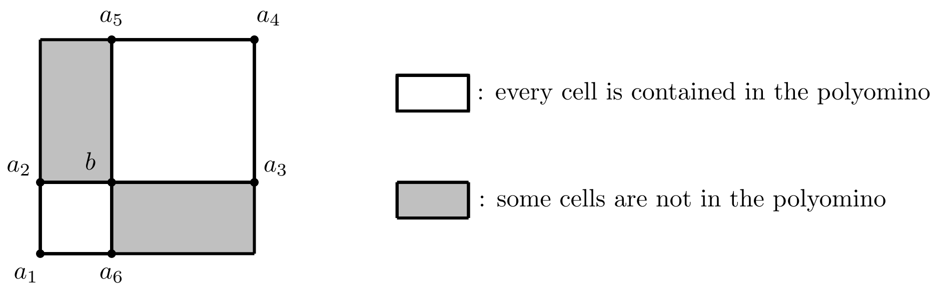

Figure 7: Binomials in S_3.

3.2. Binomials of Degree Four

Now, we have the set $S_2 \cup S_3$. We want to compute $\mathcal{S}(F, G)$ for $F \in S_3, G \in S_2$ or $F, G \in S_3$. For the case $F \in S_3$ and $G \in S_2$, note that if $F = a_1a_3a_5 - a_2a_4a_5 \in S_3$ then the pairs (a_1, a_3), (a_1, a_5), and (a_3, a_5) are not pair of diagonal corners of an inner interval, thus we only need to consider the case when the greatest common divisor of their initial monomials has degree one. For the case both $F, G \in S_3$, we will consider the case when the greatest common divisor of their initial monomials is a monomial of degree two.

3.2.1. The Case $F \in S_3$ and $G \in S_2$

Let $F = abc - def$ and $G = pq - rs$, with $|\{a, b, c\} \cap \{p, q\}| = 1$ as illustrated below.

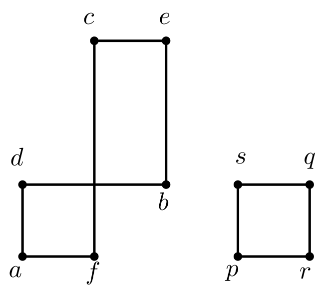

Figure 8: The Binomials F and G.

Recall that the intervals determined by $\{a, b\}$ and $\{c, d\}$ are not inner intervals. First, we observe the case if F and G also have common monomial divisor on their non-initial monomials. We show that $\mathcal{S}(F, G)$ can be reduced to zero. From the structure of F and G, the possible cases are:

(i) $a = p$ and $d = s$;

(ii) $a = p$ and $f = r$;

(iii) $b = p$ and $e = s$;

(iv) $b = q$ and $d = s$;

(v) $c = p$ and $e = r$;

(vi) $c = q$ and $r = f$.

The first two cases can occur simultaneously. In that case, we notice that q is the intersection of the segments cf and bd. Notice that $\mathcal{S}(F,G) = df(bc - qe)$ is reduced to zero since $bc - qe \in S_2$. Now, we examine all the cases separately.

(i) If $a = p$ and $d = s$ then $\mathcal{S}(F,G) = d(bcr - efq)$.

- If $a < r < f$ then consider the interval determined by $\{r, c\}$. If it is an inner interval then $bcr - efq$ is reduced to zero by similar argument in section 3.1 (case (iii)). If it is not an inner interval, since the interval determined by $\{r, b\}$ is not an inner interval, then $bcr - efq \in S_3$ and it is reduced to zero.

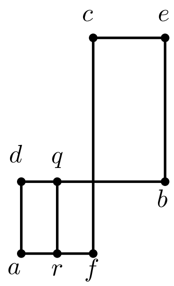

Figure 9: Case $a = p$, $d = s$ and $a < r < f$.

- If $a < f < r$, let z be the intersections of segments cf and bd, then

$$bcr - efq = (-e)(qf - zr) + (-r)(ez - bc)$$

is reduced to zero.

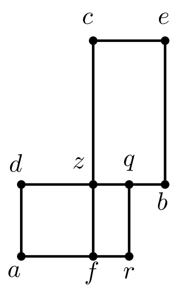

Figure 10: Case $a = p$, $d = s$ and $a < f < r$.

(ii) The case $a = p$ and $f = r$ is similar with the first case.

(iii) If $b = p$ and $e = s$ then $\mathcal{S}(F,G) = e(acr - dfq)$. Note the the interval determined by $\{a, r\}$ and the interval determined by $\{a, c\}$ are not inner intervals. Therefore $acr - dfq \in S_3$ and $\mathcal{S}(F,G)$ is reduced to zero.

(iv) If $b = q$ and $d = s$ then $\mathcal{S}(F, G) = d(acr - pef)$. Since the interval determined by $\{a, b\}$ is not inner interval then $a < p < d$ and the interval determined by $\{a, r\}$ is not an inner interval. Therefore $acr - pef \in S_3$ and $\mathcal{S}(F, G)$ is reduced to zero.

(v) The case $c = p$ and $e = r$ is similar with the case (iii).

(vi) The case $c = q$ and $r = f$ is similar with the case (iv).

Now, we can assume that F and G have no common monomial divisor in their non-initial monomials. We consider every possibility of $\{a, b, c\} \cap \{p, q\}$.

(i) If $q = c$ then $\mathcal{S}(F, G) = abrs - pdef$. Note that r, c, f are in vertical position. Consider the vertex r.

If $b <_{\mathcal{P}} r$ then the interval determined by $\{p, e\}$ is an inner interval with s as one of the antidiagonal corner. Let y be the other antidiagonal corner. It is clear that y, b, e are in vertical position and $b <_{\mathcal{P}} y <_{\mathcal{P}} e$. Therefore,

$$\mathcal{S}(F, G) = (abr - dfy)s + df(ys - pe).$$

The binomial $abr - dfy$ is reduced to zero by S_2 or an element of S_3. Hence, $\mathcal{S}(F, G)$ is reduced to zero.

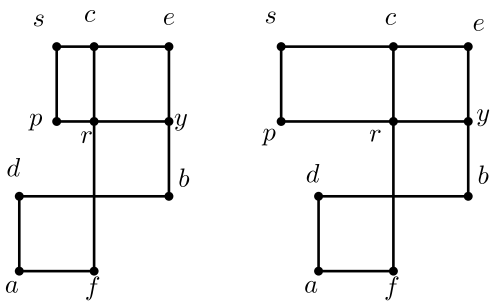

Figure 11: Case $q = c$ and $b <_{\mathcal{P}} r$.

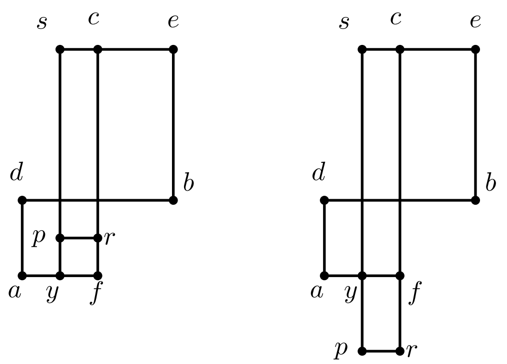

Figure 12: Case $q = c$ and $r <_{\mathcal{P}} b$.

If $r <_{\mathcal{P}} b$ then the interval determined by $\{p, f\}$ is an inner interval. Let y be the vertex such that the interval determined by $\{y, r\}$ is the inner interval determined by $\{p, f\}$. It is clear that a, y, f are in horizontal position and $a <_{\mathcal{P}} y <_{\mathcal{P}} f$ since the interval determined by $\{a, c\}$ is not an inner interval. Therefore

$$\mathcal{S}(F, G) = (abs - dey)r + de(yr - pf)$$

and is reduced to zero of the previous arguments.

(ii) If $q = b$ then $\mathcal{S}(F, G) = acrs - pdef$. Let u be the intersection of the segments cf and db. Consider the vertex s.

If $u <_{\mathcal{P}} s$ then $[p, e]$ is an inner interval with r as one of the antidiagonal corner. Let y be the other antidiagonal corner. Therefore,

$$\mathcal{S}(F, G) = (acs - dyf)r + df(yr - pe)$$

and is reduced to zero.

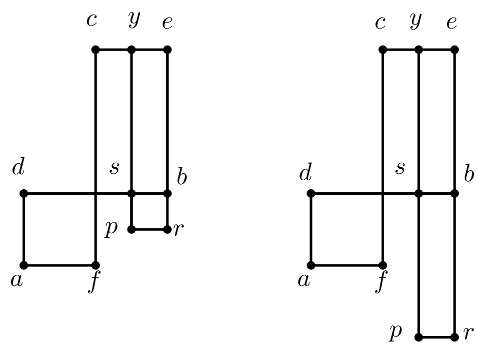

Figure 13: Case $q = b$ and $u <_{\mathcal{P}} s$.

If $s \leq_{\mathcal{P}} u$ then the interval determined by $\{p, d\}$ is an inner interval.

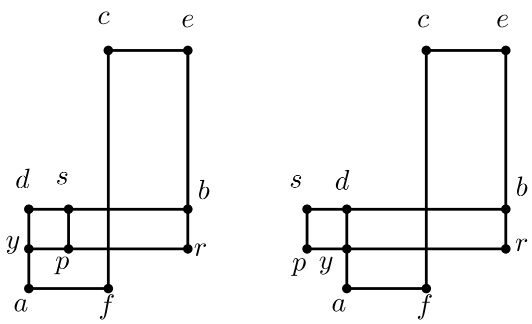

Figure 14: Case $q = b$ and $s \leq_{\mathcal{P}} u$.

Let y be a vertex such that the interval determined by $\{p, d\}$ is the interval determined by $\{y, s\}$. Note that a, y, d are in vertical position and $a <_{\mathcal{P}} y <_{\mathcal{P}} d$ since the interval determined by $\{a, b\}$ is not an inner interval. Therefore

$$\mathcal{S}(F,G) = (arc - yef)s + ef(ys - pd)$$

and is reduced to zero.

(iii) If $q = a$ then $\mathcal{S}(F,G) = bcrs - pdef$. Let u be the intersection of the segments cf and bd. Note that

$$\mathcal{S}(F,G) = (-rs)(ue - bc) + (-e)(pdf - usr)$$

and therefore is reduced to zero.

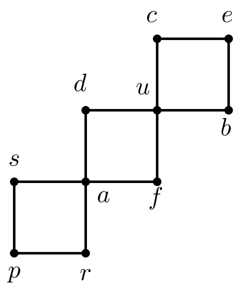

Figure 15: Case $q = a$.

(iv) If $p = a$ then $\mathcal{S}(F,G) = bcrs - qdef$. Consider the vertex r which is in horizontal position with a and f.

If $a <_{\mathcal{P}} r <_{\mathcal{P}} f$ then the interval determined by $\{q, d\}$ is an inner interval and it is an interval determined by $\{s, y\}$ for some vertex y. Therefore,

$$S(F,G) = (bcr - efy)s + ef(ys - qd)$$

is reduced to zero.

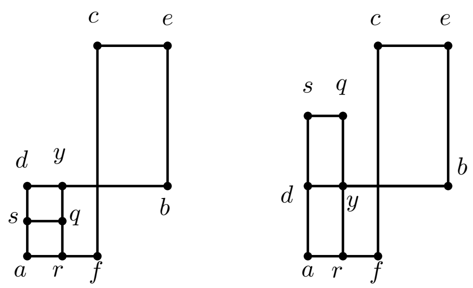

Figure 16: Case $a <_{\mathcal{P}} r <_{\mathcal{P}} f$.

If $a <_{\mathcal{P}} f <_{\mathcal{P}} r$ then the interval determined by $\{q, f\}$ is an inner interval and it is an interval determined by $\{r, y\}$ for some vertex y. Note that

$$\mathcal{S}(F,G) = (bcs - dey)r + ed(ry - fq)$$

is reduced to zero if $a <_{\mathcal{P}} s <_{\mathcal{P}} d$. If $a <_{\mathcal{P}} d <_{\mathcal{P}} s$, note that $y <_{\mathcal{P}} c$ since $[a, c]$ is not an inner interval. The binomial $bcs - dey$ is not in S_3. But

$$bcs - dey = (bc - ez)s + (-e)(dy - sz)$$

with z is the intersections of the segments cf and bd. So, $\mathcal{S}(F, G)$ is reduced to zero.

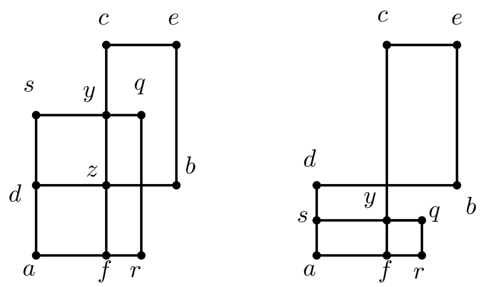

Figure 17: Case $a <_{\mathcal{P}} f <_{\mathcal{P}} r$.

(v) If $p = b$ then $\mathcal{S}(F, G) = acrs - qdef$. Note that the vertices b, e, s are in vertical position. Consider the vertex s.

If $e <_{\mathcal{P}} s$ then there is a vertex y such that the inner interval $[e, q]$ has s, y as the antidiagonal corners. Therefore,

$$\mathcal{S}(F, G) = (arc - dyf)s + (-df)(qe - ys)$$

is reduced to zero.

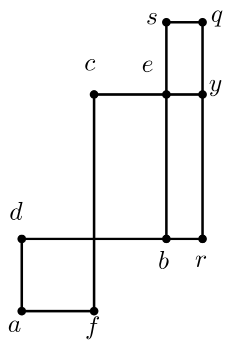

Figure 18: Case $p = b$ and $e <_{\mathcal{P}} s$.

If $s <_{\mathcal{P}} e$, note that there is no inner 2-minor whose initial monomial divides the initial monomial of $\mathcal{S}(F, G)$, which is $acrs$. Hence the binomials whose initial monomial may divide $acrs$ are the elements of S_3. From Theorem 1, the only initial monomials of the element in S_3 that may divide $acrs$ are ars, acr, or asc. Let x, y, z be the vertices such that $ars - xdq$, $acr - dyf$, $asc - efz$ are the corresponding binomials, respectively.

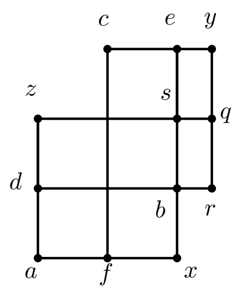

Figure 19: Case $p = b$ and $s <_{\mathcal{P}} e$.

- The binomial $ars - xdq$ is not in S_3 since $[a, b]$ is not an inner interval.
- The binomial $acr - dyf$ is contained in S_3 if and only if the interval determined by $\{q, e\}$ is an inner interval. If the interval is an inner interval then

$$\mathcal{S}(F, G) = (acr - dyf)s + df(ys - qe)$$

is reduced to zero.

- The binomial $asc - efz$ is contained in S_3 if and only if the interval determined by $\{q, d\}$ is an inner interval. If the interval is an inner interval then

$$\mathcal{S}(F, G) = (asc - efz)r + ef(zr - qd).$$

is reduced to zero.

(vi) If $p = c$ then $\mathcal{S}(F, G) = abrs - qdef$. Note that the vertices e, c, r are in horizontal position. Consider the vertex r.

If $e <_{\mathcal{P}} r$ then there exists a vertex y such that the inner interval $[e, q]$ has r, y as the antidiagonal corners. Therefore,

$$\mathcal{S}(F, G) = (abs - dyf)r + (-df)(qe - yr)$$

is reduced to zero.

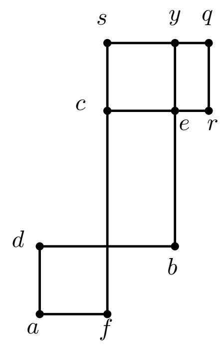

Figure 20: Case $p = c$ and $e <_{\mathcal{P}} r$.

If $r <_{\mathcal{P}} e$, note that there is no inner 2-minor whose initial monomial divides the initial monomial of $\mathcal{S}(F,G)$, which is $abrs$. Hence the binomials whose initial monomial may divide $abrs$ are the elements of S_3. From Theorem 1, the only initial monomials of the element in S_3 that may divide $abrs$ are ars, abr, or abs. Let x, y, z be the vertices such that $ars - qxf$, $abr - dey$, $abs - dfz$ be the corresponding binomials, respectively.

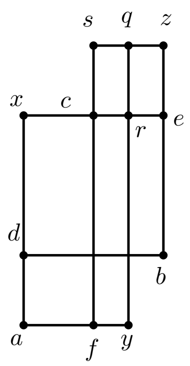

Figure 21: Case $p = c$ and $r <_{\mathcal{P}} e$.

- The binomial $ars - qxf$ is not in S_3 since $[a, c]$ is not an inner interval.
- The binomial $abr - dey$ is contained in S_3 if and only if the interval determined by $\{q, f\}$ is an inner interval. If the interval is an inner interval then

$$\mathcal{S}(F,G) = (abr - dey)s + de(ys - fq)$$

  is reduced to zero.
- The binomial $abs - dfz$ is contained in S_3 if and only if the interval determined by $\{q, e\}$ is an inner interval. If the interval is an inner interval then

$$\mathcal{S}(F,G) = (abs - dfz)r + df(zr - eq)$$

  is reduced to zero.

We summarize our discussion above to the following theorem.

Theorem 2. *Let $F = a_1a_3a_5 - a_2a_4a_6$ be the element in S_3 as in Definition 1 and $G = pq - rs$ be the element in S_2 associated to the inner interval $[p, q]$ with lower-right and upper-left corners r and s, respectively, then $\mathcal{S}(F,G)$ is not reduced to zero by $S_2 \cup S_3$ if and only if one of the following statement holds:*

- *$p = a_3$, $a_4 > s$ and both intervals determined by $\{q, a_2\}$ and $\{q, a_4\}$ are not inner intervals*
- *$p = a_5$, $a_4 > r$ and both intervals determined by $\{q, a_4\}$ and $\{q, a_6\}$ are not inner intervals.*

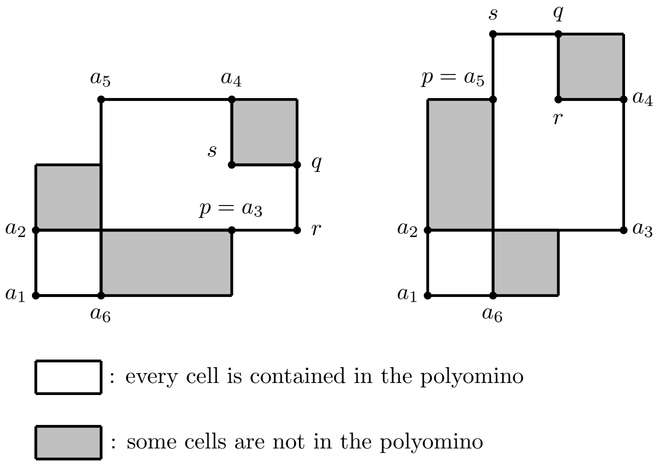

Figure 22: Binomial in Theorem 2.

3.2.2. The Case $F \in S_3$ and $G \in S_3$

Let $F = A_1A_2A_3 - B_1B_2B_3$ and $G = a_1a_2a_3 - b_1b_2b_3$ with initial monomials $A_1A_2A_3$ and $a_1a_2a_3$, respectively. We assume that $A_1 <_{\mathcal{P}} A_2 <_{\mathcal{P}} A_3$, $B_1 <_{\mathcal{P}} B_2 <_{\mathcal{P}} B_3$, $a_1 <_{\mathcal{P}} a_2 <_{\mathcal{P}} a_3$, $b_1 <_{\mathcal{P}} b_2 <_{\mathcal{P}} b_3$, and $|\{A_1, A_2, A_3\} \cap \{a_1, a_2, a_3\}| = 2$. We consider every possibility of $\{A_1, A_2, A_3\} \cap \{a_1, a_2, a_3\}$

(i) If $\{A_1, A_2, A_3\} \cap \{a_1, a_2, a_3\} = \{A_1, A_2\}$.

This case is only possible if $A_1 = a_1$ and $A_2 \in \{a_2, a_3\}$.

- If $A_2 = a_2$ then $B_2 = b_2$ and

$$\mathcal{S}(F, G) = (-B_2)(a_3B_1B_3 - A_3b_1b_3).$$

Without loss of generality, we may assume $B_1 <_{\mathcal{P}} b_1$ (for the possibility $B_1 = b_1$, we have the interval determined by $\{A_3, b_3\}$ is an inner interval and hence $\mathcal{S}(F, G) = B_2B_1(A_3b_3 - a_3B_3)$ is reduced to zero).

  - If $b_3 = B_3$ then
$$\mathcal{S}(F, G) = (-B_2B_3)(a_3B_1 - A_3b_1)$$
is reduced to zero since $A_3b_3 - a_3B_3$ is an inner 2-minor.
  - If $B_3 < b_3$ (see Figure 23 in the left side) then $a_3B_1B_3 - A_3b_1b_3 \in S_3$ since the interval determined by $\{b_1, B_3\}$ and the interval determined by $\{A_3, a_3\}$, both are not inner intervals. Therefore $\mathcal{S}(F, G)$ is reduced to zero.
  - If $B_3 < b_3$ (see Figure 23 in the right side), note that the interval determined by $\{a_3, B_3\}$ is an inner interval and is the same with the interval determined by $\{b_3, y\}$ for some vertex y. Therefore,
$$\mathcal{S}(F, G) = (-B_2b_3)(B_1y - A_3b_1) + (-B_2B_1)(a_3B_3 - b_3y)$$
is reduced to zero.

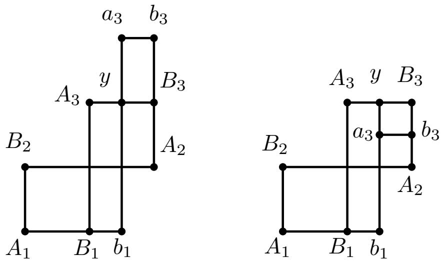

Figure 23: Case $A_1 = a_1$ and $A_2 = a_2$.

- If $A_2 = a_3$ then

$$\mathcal{S}(F,G) = A_3b_1b_2b_3 - a_2B_1B_2B_3$$

has initial monomial $a_2B_1B_2B_3$.

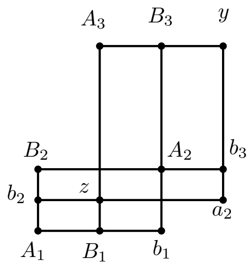

Figure 24: Case $A_1 = a_1$ and $A_2 = a_3$.

Note that there is no inner 2-minor whose initial monomial divides $a_2B_1B_2B_3$. Moreover, the binomials in S_3 whose initial monomial divides $a_2B_1B_2B_3$ are the binomial with initial monomial $a_2B_1B_3$. This can only happen when the interval determined by $\{B_3, b_3\}$ is an inner interval and the binomial in S_3 that satisfies the property is $a_2B_1B_3 - b_1yz$ where z is a vertex such that $[A_1, z]$ is the inner interval determined by $\{b_2, B_1\}$. Therefore, $\mathcal{S}(F,G)$ is reduced to zero since

$$\mathcal{S}(F,G) = -B_2(a_2B_1B_3 - b_1yz) + b_1(A_3b_2b_3 - B_2yz)$$

and $A_3b_2b_3 - B_2yz$ is an element in S_3.

(ii) If $\{A_1, A_2, A_3\} \cap \{a_1, a_2, a_3\} = \{A_1, A_3\}$. This case is only possible if $A_1 = a_1$ and $A_3 \in \{a_2, a_3\}$.

For the case $A_3 = a_2$, since $\mathcal{S}(F,G) = -\mathcal{S}(G,F)$ then it is similar with the case $A_1 = a_1$ and $A_2 = a_3$. We conclude that $\mathcal{S}(F,G)$ is reduced to zero if and only if the interval determined by $\{B_3, b_3\}$ is an inner interval.

For the case $A_3 = a_3$, we have $B_1 = b_1$ and

$$\mathcal{S}(F,G) = B_1(A_2b_2b_3 - a_2B_2B_3).$$

By similar argument with the case $A_1 = a_1$ and $A_2 = a_2$, we may assume $B_3 <_{\mathcal{P}} b_3$ and we have three subcases

- If $b_2 = B_2$ then $\mathcal{S}(F, G)$ is reduced to zero since $A_2b_3 - a_2B_3$ is an inner 2-minor.
- If $B_2 < b_2$ then $\mathcal{S}(F, G)$ is reduced to zero since $a_2B_2B_3 - A_2b_2b_3 \in S_3$.
- If $b_2 < B_2$, notice that the interval determined by $\{B_3, a_2\}$ is an inner interval that is the same with $[y, b_3]$ for some vertex y. Therefore, $S(F, G)$ is reduced to zero since we can write

$$\mathcal{S}(F, G) = B_1b_3(A_2b_2 - B_2y) + B_1B_2(b_3y - a_2B_3).$$

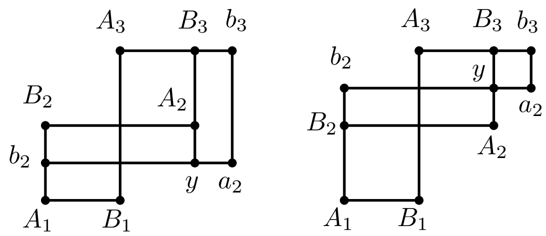

Figure 25: Case $A_1 = a_1$ and $A_3 = a_3$.

(iii) If $\{A_1, A_2, A_3\} \cap \{a_1, a_2, a_3\} = \{A_2, A_3\}$ then $A_2 = a_2$ and $A_3 = a_3$. Thus, $B_3 = b_3$ and

$$S(F, G) = B_3(A_1b_1b_2 - a_1B_1B_2).$$

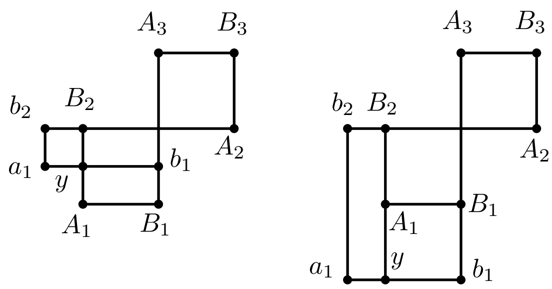

Figure 26: Case $A_2 = a_2$ and $A_3 = a_3$.

Similarly, we may assume $b_2 <_{\mathcal{P}} B_2$, and this implies that $[a_1, B_2]$ is an inner interval having $\{b_2, y\}$ as the antidiagonal corners for some vertex y. Therefore, $S(F, G)$ is reduced to zero since

$$S(F, G) = -B_1B_3(a_1B_2 - b_2y) + B_3b_2(A_1b_1 - B_1y).$$

We summarize this discussion with the following theorem.

Theorem 3. *Let $F = A_1A_2A_3 - B_1B_2B_3$ and $G = a_1a_2a_3 - b_1b_2b_3$ with initial monomials $A_1A_2A_3$ and $a_1a_2a_3$, respectively, in S_3 with $A_1 <_{\mathcal{P}} A_2 <_{\mathcal{P}} A_3$, $B_1 <_{\mathcal{P}} B_2 <_{\mathcal{P}} B_3$, $a_1 <_{\mathcal{P}} a_2 <_{\mathcal{P}} a_3$, $b_1 <_{\mathcal{P}} b_2 <_{\mathcal{P}} b_3$, and $|\{A_1, A_2, A_3\} \cap \{a_1, a_2, a_3\}| = 2$. The binomial $S(F, G)$ is not reduced to zero by $S_2 \cup S_3$ if and only if*

- *$A_1 = a_1$,*
- *the interval determined by $\{B_3, b_3\}$ is not inner interval, and*
- *$A_3 = a_2$ or $A_2 = a_3$ holds.*

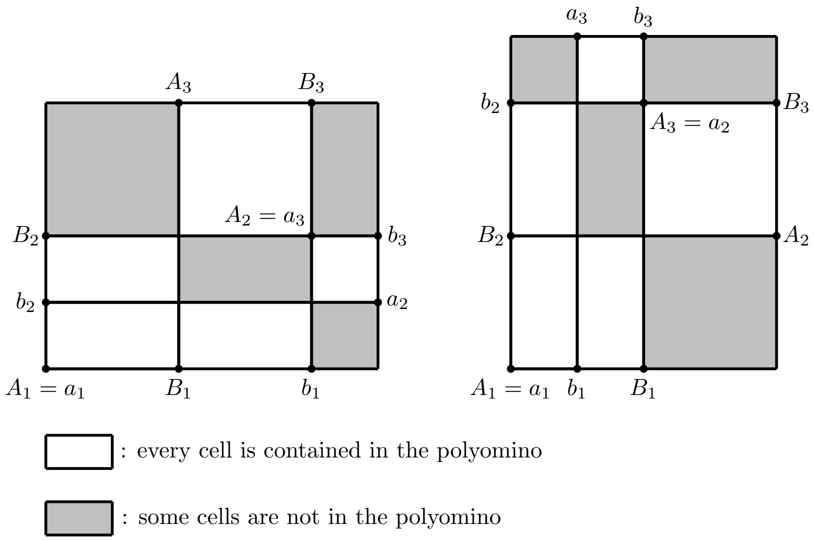

Figure 27: Binomials in Theorem 3.

Note that there are no elements in $S_2 \cup S_3$ whose initial monomial divides the binomials $\mathcal{S}(F, G)$ discussed in Theorem 2 and 3. We conclude that all binomials of degree four from the Buchberger Algorithm is of the form $a_1a_2a_3a_4 - b_1b_2b_3b_4$ with $a_1 <_{\mathcal{P}} a_2 <_{\mathcal{P}} a_3 <_{\mathcal{P}} a_4$, $b_1 <_{\mathcal{P}} b_2 <_{\mathcal{P}} b_3 <_{\mathcal{P}} b_4$, and initial monomial $a_1a_2a_3a_4$, and can be illustrated in the following figure.

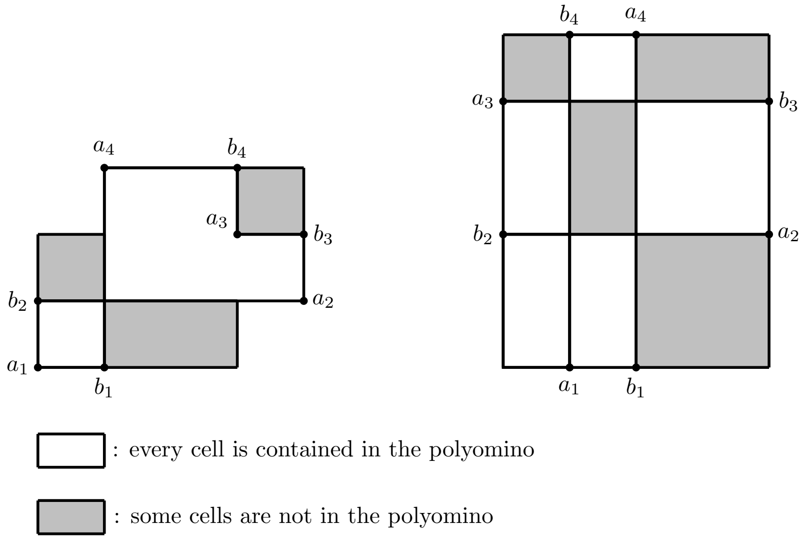

Figure 28: Binomials in Theorem 2 and Theorem 3.

4. The Socket Wrench Polyominoes

Consider the following polyomino constructed from 8 unit squares forming a 3×3 square without the unit square in the center and continued by adding n unit squares to the left of the unit square on the leftmost cell on the middle row. We call this polyomino *a socket wrench polyomino*, because it looks like the socket wrenches used by mechanics to tighten or loosen nuts and bolts. We use for this polyomino the same labelling with reference to Section 3.

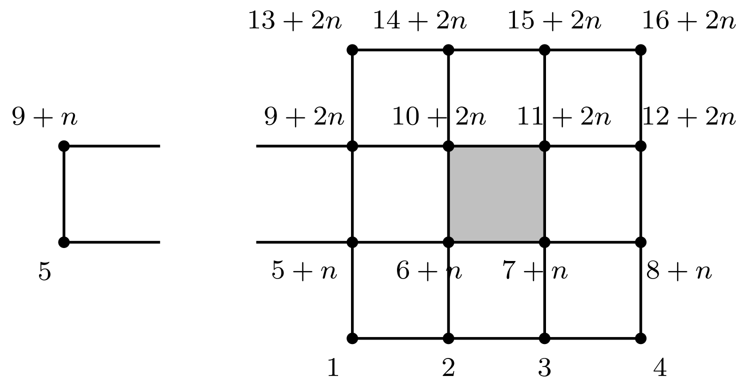

Figure 29: A Socket Wrench Polyomino with n Additional Unit Squares.

Theorem 4. *Let $\mathcal{P}$ be a socket wrench polyomino then the polyomino ideal $I_{\mathcal{P}}$ is a radical.*

Proof. We claim that according to this labelling and lexicographic order, the ideal $I_{\mathcal{P}}$ has $S_2 \cup S_3$ as the Gröbner bases. Since the initial monomial of every element in $S_2 \cup S_3$

is square-free then we have the result.

By Theorem 1, every element of S_3 is one of the following forms:

- $x_i x_{11+2n} x_{14+2n} - x_{i+4+n} x_{15+2n} x_{6+n}$, $5 \le i \le 4+n$
- $x_i x_{12+2n} x_{14+2n} - x_{i+4+n} x_{16+2n} x_{6+n}$, $5 \le i \le 4+n$
- $x_i x_{11+2n} x_{13+2n} - x_{i+4+n} x_{15+2n} x_{5+n}$, $5 \le i \le 4+n$
- $x_i x_{12+2n} x_{13+2n} - x_{i+4+n} x_{16+2n} x_{5+n}$, $5 \le i \le 4+n$

To complete our claim, we observe the following.

(i) $\mathcal{S}(F,G)$ with $F,G \in S_2$ is either reduced to zero or contained in S_3.

(ii) Suppose there are $F \in S_3$ and $G \in S_2$ such that $\mathcal{S}(F,G)$ is not reduced to zero. Write $F = a_1a_3a_5 - a_2a_4a_6$ as in Definition 1 and $G = pq - rs$. Here, $a_1 <_{\mathcal{P}} a_3 <_{\mathcal{P}} a_5$, $a_6 <_{\mathcal{P}} a_2 <_{\mathcal{P}} a_4$, $p <_{\mathcal{P}} q$, and $r <_{\mathcal{P}} s$. From the structure of elements in S_3, we have $a_1 \in [5, 4+n]$, $a_3 \in \{11+2n, 12+2n\}$, and $a_5 \in \{13+2n, 14+2n\}$. By Theorem 2, we have $p = a_3$ or $p = a_5$. Suppose $p = a_5$, then $[p,q]$ can not be an inner interval. Therefore $p = a_3$. Since $[p,q]$ is an inner interval, then $p = a_3 = 11+2n$ and $q = 16+2n$. But then $s = a_4 = 15+2n$, contradiction with $a_4 > s$.

(iii) Suppose there are $F,G \in S_3$ such that $\mathcal{S}(F,G)$ is not reduced to zero. By the definition of S_3, since the non-initial monomial of a binomial in S_3 is completely determined by its initial monomial then we can eliminate the cases when the initial monomials of F,G are relatively prime or equal.

  (a) If the greatest common divisor of their initial monomial is a monomial of degree two, we will have a similar argument as in the previous case but using Theorem 3 that it will come to a contradiction.

  (b) If the greatest common divisor of their initial monomial is a monomial of degree one, by our classifications above, we need to consider several cases of $\mathcal{S}(F,G)$.

  - $\mathcal{S}(x_i x_{11+2n} x_{14+2n} - x_{i+4+n} x_{15+2n} x_{6+n}, x_j x_{12+2n} x_{14+2n} - x_{j+4+n} x_{16+2n} x_{6+n})$, $5 \le i < j \le 4+n$.

    The above expression is equal to

$$
\begin{aligned}
& x_i x_{11+2n} x_{j+4+n} x_{16+2n} x_{6+n} - x_{i+4+n} x_{15+2n} x_{6+n} x_j x_{12+2n} \\
= \; & x_{6+n} x_{11+2n} x_{16+2n} (x_i x_{j+4+n} - x_{i+4+n} x_j) \\
& + x_{6+n} x_{i+4+n} x_j (x_{11+2n} x_{16+2n} - x_{15+2n} x_{12+2n})
\end{aligned}
$$

    and is reduced to zero.

  - $\mathcal{S}(x_i x_{11+2n} x_{14+2n} - x_{i+4+n} x_{15+2n} x_{6+n}, x_j x_{11+2n} x_{13+2n} - x_{j+4+n} x_{15+2n} x_{5+n})$, $5 \le i < j \le 4+n$.

    The above expression is equal to

$$
x_i x_{14+2n} x_{j+4+n} x_{15+2n} x_{5+n} - x_j x_{13+2n} x_{i+4+n} x_{15+2n} x_{6+n}
$$

$$
\begin{aligned}
&= x_{15+2n}x_{14+2n}x_{5+n}(x_ix_{j+4+n} - x_jx_{i+4+n}) \\
&\quad +x_{15+2n}x_jx_{i+4+n}(x_{14+2n}x_{5+n} - x_{6+n}x_{13+2n})
\end{aligned}
$$

and is reduced to zero.

- $\mathcal{S}(x_ix_{11+2n}x_{14+2n} - x_{i+4+n}x_{15+2n}x_{6+n}, x_ix_{12+2n}x_{13+2n} - x_{i+4+n}x_{16+2n}x_{5+n})$, $5 \leq i \leq 4+n$.
  The above expression is equal to

$$
\begin{aligned}
& x_{11+2n}x_{14+2n}x_{i+4+n}x_{16+2n}x_{5+n} - x_{12+2n}x_{13+2n}x_{i+4+n}x_{15+2n}x_{6+n} \\
= {}& x_{i+4+n}x_{11+2n}x_{16+2n}(x_{5+n}x_{14+2n} - x_{6+n}x_{13+2n}) \\
& +x_{i+4+n}x_{6+n}x_{13+2n}(x_{11+2n}x_{16+2n} - x_{12+2n}x_{15+2n})
\end{aligned}
$$

  and is reduced to zero.

- $\mathcal{S}(x_ix_{12+2n}x_{14+2n} - x_{i+4+n}x_{16+2n}x_{6+n}, x_ix_{11+2n}x_{13+2n} - x_{i+4+n}x_{15+2n}x_{5+n})$, $5 \leq i \leq 4+n$.
  The above expression is equal to

$$
\begin{aligned}
& x_{12+2n}x_{14+2n}x_{i+4+n}x_{15+2n}x_{5+n} - x_{i+4+n}x_{16+2n}x_{6+n}x_{11+2n}x_{13+2n} \\
= {}& x_{i+4+n}x_{12+2n}x_{15+2n}(x_{5+n}x_{14+2n} - x_{6+n}x_{13+2n}) \\
& +x_{i+4+n}x_{6+n}x_{13+2n}(x_{12+2n}x_{15+2n} - x_{11+2n}x_{16+2n})
\end{aligned}
$$

  and is reduced to zero.

- $\mathcal{S}(x_ix_{12+2n}x_{14+2n} - x_{i+4+n}x_{16+2n}x_{6+n}, x_jx_{12+2n}x_{13+2n} - x_{j+4+n}x_{16+2n}x_{5+n})$, $5 \leq i < j \leq 4+n$.
  The above expression is equal to

$$
\begin{aligned}
& x_ix_{14+2n}x_{j+4+n}x_{16+2n}x_{5+n} - x_{i+4+n}x_{16+2n}x_{6+n}x_jx_{13+2n} \\
= {}& x_{16+2n}x_{14+2n}x_{5+n}(x_ix_{j+4+n} - x_jx_{i+4+n}) \\
& +x_{16+2n}x_jx_{i+4+n}(x_{14+2n}x_{5+n} - x_{13+2n}x_{6+n})
\end{aligned}
$$

  and is reduced to zero.

- $\mathcal{S}(x_ix_{11+2n}x_{13+2n} - x_{i+4+n}x_{15+2n}x_{5+n}, x_jx_{12+2n}x_{13+2n} - x_{j+4+n}x_{16+2n}x_{5+n})$, $5 \leq i < j \leq 4+n$.
  The above expression is equal to

$$
\begin{aligned}
& x_ix_{11+2n}x_{j+4+n}x_{16+2n}x_{5+n} - x_{i+4+n}x_{15+2n}x_{5+n}x_jx_{12+2n} \\
= {}& x_{5+n}x_{11+2n}x_{16+2n}(x_ix_{j+4+n} - x_jx_{i+4+n}) \\
& +x_{5+n}x_jx_{i+4+n}(x_{11+2n}x_{16+2n} - x_{12+2n}x_{15+2n})
\end{aligned}
$$

  and is reduced to zero.

And we are done with the proof.

Remark 1. *Note that we can rotate the socket wrench polyominoes* 90°, 180°, *and* 270° *and get the same conclusion since we also can rotate the labelling and using the same monomial order.*

We also can prove a stronger result by using the similar argument with [5, Section 4].

Theorem 5. *Let $\mathcal{P}$ be a socket wrench polyomino then the ideal $I_{\mathcal{P}}$ is prime.*

Proof. We label $\mathcal{P}$ according to Figure 29. Let $\{V_i\}_{i\in I}$ be the set of maximal vertical edge intervals of $\mathcal{P}$ and $\{H_j\}_{j\in J}$ be the set of maximal horizontal edge intervals of $\mathcal{P}$, where $I = \{1, 2, \ldots, n+4\}$ and $J = \{1, 2, 3, 4\}$. Let $\{v_i\}_{i\in I}$ and $\{h_j\}_{j\in J}$ be the set of variables associated respectively to $\{V_i\}_{i\in I}$ and $\{H_j\}_{j\in J}$, respectively. Let w be another variable different from v_i and h_j. Let $A = \{3, 4, 7+n, 8+n\}$. Define

$$\alpha : V(\mathcal{P}) \to K[\{v_i, h_j, w\} : i \in I, j \in J]$$
$$r \mapsto v_i h_j w^k$$

with $r \in V_i \cap H_j$, $k = 0$ if $r \notin V(A)$, and $k = 1$ if $r \in V(A)$.

Consider the following surjective ring homomorphism

$$\phi : K[x_r : r \in V(\mathcal{P})] \to K[\alpha(v) : v \in V(\mathcal{P})]$$
$$x_r \mapsto \alpha(r)$$

The toric ideal $J_{\mathcal{P}}$ is the kernel of ϕ. We will prove that $I_{\mathcal{P}} = J_{\mathcal{P}}$.

We start by proving $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. Let $f = x_p x_q - x_r x_s$ be a generator of $I_{\mathcal{P}}$ that associated to the inner interval $[p, q]$. We may assume that p, r and q, s, respectively, are on the same maximal vertical edge interval. Then, p, s and q, r, respectively, are on the same maximal horizontal edge interval. If $[p, q] \cap A = \emptyset$ then $f \in J_{\mathcal{P}}$. Consider the case $[p, q] \cap A \neq \emptyset$. If $[p, q] = A$ then $f \in J_{\mathcal{P}}$. If $[p, q] \neq A$, by the construction of $\mathcal{P}$, then either s, q or p, s must be two vertices of A. In the first case, p, r are not the vertices of A. In the second case, r, q are not the vertices of A. In both cases, we conclude that $f \in J_{\mathcal{P}}$.

Now, it remains to prove that $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$. We will prove this by showing that every binomial of degree two in $J_{\mathcal{P}}$ belongs to $I_{\mathcal{P}}$ and every irredundant binomial in $J_{\mathcal{P}}$ is of degree two (or for some cases, it is in $I_{\mathcal{P}}$).

For the first part, let $f = x_p x_q - x_r x_s$ be a binomial in $J_{\mathcal{P}}$. If p, q are in horizontal or vertical position, since $\phi(f) = 0$ then we can easily argue that $\{p, q\} = \{r, s\}$ and $f = 0 \in I_{\mathcal{P}}$. We consider the case p, q are the diagonal corners of an interval (the case p, q are the antidiagonal corners can be done similarly). Let v_p and h_p be the variables associated to the maximal vertical and horizontal edge intervals that contain p, respectively. We define $v_q, v_r, v_s, h_q, h_r, h_s$ similarly. We will prove that r, s are the antidiagonal corners of $[p, q]$ and argue that $f \in I_{\mathcal{P}}$. We divide into three cases:

- If $p, q \in A$. Since $\phi(x_p x_q) = v_p v_q h_p h_q w^2$ then w^2 divides $\phi(x_r x_s)$. Thus, $r, s \in A$. If $r = p$ or $r = q$ then $\{r, s\} = \{p, q\}$ and $f = 0 \in I_{\mathcal{P}}$. Therefore r is an antidiagonal corner of $[p, q]$. If $\phi(x_r) = v_p h_q w$ then $\phi(x_s) = v_q h_p w$ and s is also an antidiagoal corner of $[p, q]$. The same conclusion for $\phi(x_r) = v_q h_p w$. Clearly, $[p, q] = [3, n+8]$ is an inner interval and thus $f \in I_{\mathcal{P}}$.

- If exactly one of p, q belongs to A. We consider the case $p \in A$. By the construction of $\mathcal{P}$, we have $p \in \{3, 7+n\}$ and $q \in \{12+2n, 16+2n\}$. Since w divides $\phi(x_p)$ then $r \in A$ or $s \in A$. We may assume that $r \in A$. If $r = p$ then $s = q$ and $f = 0 \in I_{\mathcal{P}}$. If p, r are not in horizontal position then H_r contains an edge of A but $H_p \neq H_r$ and H_q does not contain any edge of A. Therefore h_r divides $\phi(x_r x_s)$ but does not divide $\phi(x_p x_q)$, a contradiction. Now, p, r are in horizontal position. By the construction of $\mathcal{P}$, we conclude that r, q are in vertical position. Thus, $\phi(x_s) = v_p h_q$ and therefore r, s are the antidiagonal corners of $[p, q]$. Since $p \in \{3, 7+n\}$ and $q \in \{12+2n, 16+2n\}$ then $[p, q]$ is an inner interval and thus $f \in I_{\mathcal{P}}$. The case $q \in A$ is also true by symmetry.

- If both p, q do not belong to A. Similarly, we have that $f = 0 \in I_{\mathcal{P}}$ or r, s are the antidiagonal corners of $[p, q]$. Let $\mathcal{P}'$ be a polyomino obtained by removing the cells that has common vertices with A. Note that $\mathcal{P}'$ is a simple polyomino. Let ϕ' be the restriction of ϕ on $K[x_a : a \in V(\mathcal{P})\backslash A]$ and $J_{\mathcal{P}'}$ be the kernel of ϕ'. Note that $f \in J_{\mathcal{P}'}$ By [36, Theorem 2.2], we have that $I_{\mathcal{P}'} = J_{\mathcal{P}'}$. Therefore $f \in J_{\mathcal{P}'} = I_{\mathcal{P}'} \subset I_{\mathcal{P}}$.

For the second part, let f be an irredundant binomial in $J_{\mathcal{P}}$. Clearly, f has degree at least two. Suppose that f has degree at least three and choose f with the least degree. Suppose that every variable of f is in $K[x_a : a \in V(\mathcal{P})\backslash A]$. Define $\mathcal{P}'$ as the previous case then f is a binomial in $J_{\mathcal{P}'}$ and f is irredundant in $J_{\mathcal{P}'}$. Since $I_{\mathcal{P}'} = J_{\mathcal{P}'}$ then f is an irredundant binomial in $I_{\mathcal{P}'}$ which means that f must be a binomial of degree two, a contradiction. Now, suppose that x_{v_1} is a variable of f with $v_1 \in A$. Write $f = f^+ - f^-$. We may assume that x_{v_1} divides f^+. If x_{v_1} divides f^- then $f = x_{v_1}(g^+ - g^-)$. Since $J_{\mathcal{P}}$ is prime then $g = g^+ - g^- \in J_{\mathcal{P}}$. If the degree of g is at least three then g must be irredundant. But, this contradict the choice of f. If the degree of g is two then by the previous part, we conclude that $g \in I_{\mathcal{P}}$ and $f = x_{v_1} g \in I_{\mathcal{P}}$. Now, suppose that x_{v_1} does not divide f^-. We may assume that no x_v divides both f^+ and f^- for $v \in A$. Since w divides $\phi(f^+)$ and $\phi(f^+) = \phi(f^-)$ then there exists $v_1' \in A$ such that $x_{v_1'}$ divides f^-. Let V_{v_1} and H_{v_1} be the maximal vertical and horizontal edge intervals, respectively, that contain v_1. Since v_{v_1} divides $\phi(f^+)$ and $\phi(f^+) = \phi(f^-)$ then there exists $v_2' \in V_{v_1}$ such that $x_{v_2'}$ divides f^-. Similarly, there exists $v_3' \in H_{v_1}$ such that $x_{v_3'}$ divides f^-. Define $V_{v_1'}$ and $H_{v_1'}$ similarly. We also get that there exists $v_2 \in V_{v_1'}$ and $v_3 \in H_{v_1'}$ such that both x_{v_2} and x_{v_3} divide f^+. Consider the following cases:

- If v_1 and v_1' are on the same horizontal edge interval of $\mathcal{P}$. By the construction of $\mathcal{P}$ then the interval determined by v_1, v_2 is an inner interval. By [5, Lemma 2.2] with three vertices $v_1, v_2 \in V_f^+$ dan $v_1' \in V_f^-$, we get a contradiction.

- If v_1 and v_1' are on the same vertical edge interval of $\mathcal{P}$. Similarly we get a contradiction by [5, Lemma 2.2] and three vertices $v_1, v_3 \in V_f^+$ dan $v_1' \in V_f^-$.

- If v_1 and v_1' are the diagonal corners of $[3, 8+n]$. We may assume that $v_1 = 3$ and $v_1' = 8+n$. Consider v_3'. If $v_3' = 4$ then $v_2 \in \{12+2n, 16+2n\}$ and we get a contradiction by [5, Lemma 2.2] and three vertices $v_1, v_2 \in V_f^+$ dan $v_3' \in V_f^-$.

Therefore $v_3' \neq 4$. In particular, v_3' is not the antidiagonal corner of $[3, 8+n]$. With the similar arguments, we conclude that v_2' is not the antidiagonal corner of $[3, 8+n]$. Looking at the construction of $\mathcal{P}$, we see that the vertices v_1, v_1', v_2', v_3' lie on $\mathcal{P}$ as the following figure:

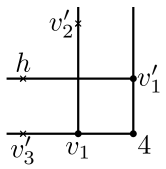

Figure 30: Illustration for v_1, v_1', v_2', v_3' on $\mathcal{P}$.

Note that $[v_3', v_1']$ is an inner interval with 4 as one of the antidiagonal. Let h be the other antidiagonal. Notice that

$$f = \left(f^+ - \frac{f^-}{x_{v_1'}x_{v_3'}}x_h x_4\right) - \frac{f^-}{x_{v_1'}x_{v_3'}}(x_{v_1'}x_{v_3'} - x_h x_4).$$

Since $x_{v_1'}x_{v_3'} - x_h x_4 \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$ then $f^+ - \frac{f^-}{x_{v_1'}x_{v_3'}}x_h x_4 \in J_{\mathcal{P}}$. But, both x_4 and $x_{v_2'}$ divide $\frac{f^-}{x_{v_1'}x_{v_3'}}x_h x_4$ and x_{v_1} divide f^+. By [5, Lemma 2.2] and three vertices $4, v_2', v_1$ we get $f^+ - \frac{f^-}{x_{v_1'}x_{v_3'}}x_h x_4$ is redundant and f is also redundant, a contradiction.

- We argue similarly for the case v_1 and v_1' are the antidiagonal corners of $[3, 8+n]$.

Corollary 1. *Let $\mathcal{P}$ be a socket wrench polyomino then $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain.*

Proof. By the previous theorem, we have $I_{\mathcal{P}}$ is a toric ideal and has square-free quadratic Gröbner bases for the suitable monomial order. By a theorem of Sturmfels [23, Corollary 4.26] we conclude that $K[\mathcal{P}]$ is normal and by a theorem of Hochster [4, Theorem 6.3.5] we have that $K[\mathcal{P}]$ is Cohen-Macaulay. Therefore $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain.

Next we compute the h-polynomial of socket wrench polyominoes and prove that $K[\mathcal{P}]$ is Gorenstein if and only if there is no unit square that we add in the definition of the socket wrench polyominoes. We refer the definition of $(\mathcal{L}, \mathcal{C})$-polyomino in [8]. The socket wrench polyominoes are $(\mathcal{L}, \mathcal{C})$-polyominoes by the following figure

Here, we take symmetry to the definition of $(\mathcal{L}, \mathcal{C})$-polyomino so it is suitable to the socket wrench. The results in [8] do not change. We also can rotate the socket wrench polyominoes by 180° to see that the socket wrench polyominoes are $(\mathcal{L}, \mathcal{C})$-polyominoes.

We recall some terminologies from [8] and [37].

(i) For a polyimino $\mathcal{P}$, *the rook number* $r(\mathcal{P})$ is the maximum number of non-attacking rooks that can be placed in $\mathcal{P}$.

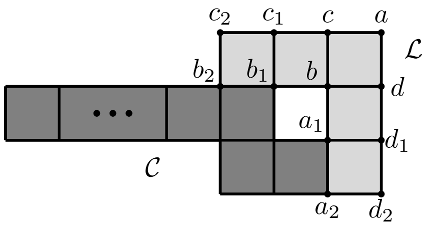

Figure 31: Socket wrench polyominoes are $(\mathcal{L}, \mathcal{C})$-polyominoes.

(ii) For a polyomino $\mathcal{P}$, denote by r_k the number of ways to placed k rook in $\mathcal{P}$ in non-attacking position, conventionally $r_0 = 1$.

(iii) The polyomino $\mathcal{P}$ is *thin* if it does not contain square tetromino.

(iv) Let $\mathcal{P}$ be a simple thin polyomino. A cell C of P is *single* if there exists a unique maximal inner interval of $\mathcal{P}$ containing C. If any maximal inner interval of $\mathcal{P}$ has exactly one single cell, we say that P has the *S-property.*

We also use some terminologies from [5] and [31].

(i) Let $\mathcal{P}$ be a polyomino. A sequence of distinct inner interval $\mathcal{W} : I_1, \ldots, I_\ell$ of $\mathcal{P}$ such that v_i, z_i are diagonal (resp. antidiagonal) corners and u_i, v_{i+1} are antidiagonal (resp. diagonal) corners of I_i, for $i = 1, \ldots, \ell$, is a *zig-zag walk* of $\mathcal{P}$, if

   (a) $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$ for $i = 1, 2, \ldots, \ell - 1$;
   (b) v_i and v_{i+1} are on the same edge interval of $\mathcal{P}$, for $i = 1, \ldots, \ell$.
   (c) for any $i, j \in \{1, \ldots, \ell\}$, with $i \neq j$, does not exist an inner interval J of $\mathcal{P}$ such that $z_i, z_j \in J$.

(ii) A polyomino is called *closed path* if $\mathcal{P}$ is a sequence of cells $A_1, A_2, \ldots, A_n, A_{n+1}$, $n > 5$ such that

   (a) $A_1 = A_{n+1}$;
   (b) $A_i \cap A_{i+1}$ is a common edge, for $i = 1, 2, \ldots, n$;
   (c) $A_i \neq A_j$ for all $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$;
   (d) For all $i \in \{1, 2, \ldots, n\}$ and for all $j \notin \{i-2, i-1, i, i+1, i+2\}$ then $V(A_i) \cap V(A_j) = \emptyset$, where $A_{-1} = A_{n-1}$, $A_0 = A_n$, $A_{n+1} = A_1$, $A_{n+2} = A_2$.

In [31, Corollary 3.6], the authors prove that if there exists a zig-zag walk in $\mathcal{P}$ then the ideal $I_{\mathcal{P}}$ is not prime. For socket wrench polyominoes, since $I_{\mathcal{P}}$ is prime then $\mathcal{P}$ has no zig-zag walk. Now, we are ready to prove the next theorem.

Theorem 6. *Let $\mathcal{P}$ be a socket wrench polyomino with n additional unit squares. Then:*

(i) the h-polynomial of $K[\mathcal{P}]$ is

$$h_{K[\mathcal{P}]}(t) = 1 + (n+8)t + (7n+16)t^2 + (11n+8)t^3 + (3n+1)t^4;$$

(ii) $reg(K[\mathcal{P}]) = 4$*;*

(iii) $K[\mathcal{P}]$ *is Gorenstein if and only if* $n = 0$*.*

Proof. Since $\mathcal{P}$ is a $(\mathcal{L}, \mathcal{C})$-polyomino and $\mathcal{C}$ is a simple and thin polyomino then by [8, Theorem 5.2], we obtain that

$$h_{K[\mathcal{P}]}(t) = \sum_{k=0}^{r(\mathcal{P})} r_k t^k$$

and $\mathrm{reg}(K[\mathcal{P}]) = r(\mathcal{P})$. Note that $r(\mathcal{P}) = 4$ since the first, the second and the third row can not contain more than one rook, two rooks, one rook, respectively, and we can place four rooks like illustrated in the figure below

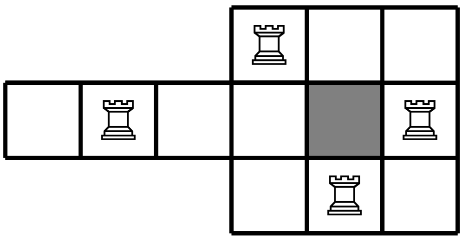

Figure 32: $r(\mathcal{P}) = 4$.

We also can easily get $r_1 = n+8$, $r_2 = 7n+16$, $r_3 = 11n+8$, and $r_4 = 3n+1$ by some counting arguments. Then

$$h_{K[\mathcal{P}]}(t) = 1 + (n+8)t + (7n+16)t^2 + (11n+8)t^3 + (3n+1)t^4$$

and $\mathrm{reg}(K[\mathcal{P}]) = 4$, *(i)* and *(ii)* are proven.

For *(iii)*, by [39, Theorem 4.2] since $1 \neq 3n+1$ for $n > 0$ then $K[\mathcal{P}]$ is not Gorenstein. If $n = 0$, then $\mathcal{P}$ is a closed path having no zig-zag walks. We notice that the single cells of $\mathcal{P}$ are the four cells in the middle of the first row, the third row, the first column, and the third column. We also notice that the maximal intervals of $\mathcal{P}$ are the four intervals containing three cells in the first row, the third row, the first column, and the third column. Each of them only has one single cell, and thus $\mathcal{P}$ has the S-property. By [8, theorem 5.7], we conclude that $K[\mathcal{P}]$ is Gorenstein.

5. Conclusion

In this paper, we classify some few-degree binomials that arise from the Buchberger Algorithm on polyomino ideal. Based on the labelling and the monomial order that were explained at the beginning of the third section, we obtain that the Buchberger Algorithm produces binomials of degree three (Theorem 1) and binomials of degree four (Theorem 2 and 3). We also give a class of polyominoes (the socket wrench polyominoes) that has Gröbner bases of degree at most three with respect to the previous labelling and monomial order, and hence the polyomino ideal is radical. We also study some properties of the polyomino ideal $I_{\mathcal{P}}$ of the socket wrench polyominoes. The ideal $I_{\mathcal{P}}$ is prime (Theorem 4). The quotient ring $K\mathcal{P}$ is a normal Cohen-Macaulay domain (Corollary 1). The h-polynomial, regularity, and Goreinsteness are given in Theorem 6. The problem about Gröbner bases and radicality of polyomino ideal are still open.

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