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Universal Distance Spectra of Join of Graphs

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Abstract. Let G be a simple undirected graph of order n. In this paper, we introduce a new distance matrix called the universal distance matrix of G, denoted as $U^D(G)$ and it is defined as

$$U^D(G) = \alpha Tr(G) + \beta D(G) + \gamma J + \delta I,$$

where $Tr(G)$ is the diagonal matrix whose elements are the vertex transmissions, and $D(G)$ is the distance matrix of G. Here J is the all-ones matrix, and I is the identity matrix and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\beta \neq 0$. This unified definition enables us to derive the spectra of different matrices associated with the distance matrix of graphs. The set of eigenvalues of the universal distance matrix namely, $\{\rho_1, \rho_2, \ldots, \rho_n\}$ is known as the universal distance spectrum of G. As a consequence, by taking appropriate values for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\beta \neq 0$, we obtain the eigenvalues of distance matrix, distance Laplacian matrix, distance signless Laplacian matrix, generalized distance matrix, distance Seidal matrix and distance matrices of graph complements. In this paper, we obtain the universal distance spectra of regular graph, join of two regular graphs, joined union of three regular graphs, generalized joined union of n disjoint graphs with one arbitrary graph H using the Schur complement of a block matrix.

2020 Mathematics Subject Classifications: 05C50

Key Words and Phrases: Universal distance spectrum, Seidal matrix, Joined Union, Complete split graph.

1. Introduction

Consider a graph G consisting of the vertex set $V(G)$ and the edge set $E(G)$ on n vertices. *Degree* of a vertex is the number of edges incident on that vertex. A graph G is *regular* if every vertex has the same degree. The *adjacency matrix* $A(G) = (a_{ij})$ of G, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ symmetric matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } d(v_i, v_j) = 1 \\ 0, & \textit{otherwise}. \end{cases}$$

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Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of G. The *diameter* is the maximum distance between all pairs of vertices of a graph G. The *complement* of G is denoted by $\overline{G}$ and is the graph whose vertex set is the same as that of G and two vertices are adjacent in G if and only if they are not adjacent in $\overline{G}$. The *join* of two graphs G_1 and G_2, denoted by $G_1 \nabla G_2$ is the graph obtained by joining every vertex of G_1 with every vertex of G_2. The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and edge set is $E(G_1) \cup E(G_2)$. As usual, we denote by C_n, the cycle graph and K_n, the complete graph, on n vertices.

The *distance matrix* of a connected graph G of order n, denoted by $D(G)$, is the symmetric $n \times n$ matrix (b_{ij}), where $b_{i,j} = d(v_i, v_j)$ (the length of a shortest path connecting vertices v_i and v_j). The *transmission of a vertex* v, denoted by $Tr_G(v)$ is defined as the sum of the distances from v to all other vertices in G, i.e.,

$$Tr_G(v) = \sum_{u \in V} d(u, v).$$

The matrix $Tr(G)$ is a diagonal matrix whose entries are the transmissions of vertices of G.

For a connected graph G, the *distance Laplacian matrix* of G is the matrix $D^{\mathbb{L}}(G) = Tr(G) - D(G)$ and the *distance signless Laplacian matrix* of G is the matrix $D^{\mathbb{Q}}(G) = Tr(G) + D(G)$. These two matrices have been introduced by M. Aouchiche and P. Hansen [2]. In [13], Haritha and Chithra defined a matrix called distance Seidal matrix. The *distance Seidal matrix* of G is the matrix $D^{\mathbb{S}}(G) = J - I - 2D(G)$. For a connected graph G, Cui et al.[8] have introduced the *generalized distance matrix*, and it is denoted by $D_\alpha(G)$. It is defined as the convex combination of $Tr(G)$ and $D(G)$. It is of the form $D_\alpha(G) = \alpha Tr(G) + (1-\alpha) D(G)$, $\alpha \in [0, 1]$.

In [12], Haemers et al. have derived the characteristic polynomials of various universal adjacency matrices in terms of the characteristic polynomials of the adjacency matrices of the components of G. In [6, 7], Cardoso et al. obtained the generalization of Fiedler's lemma which can be applied to the $H-$join of regular graphs. In [18], Saravanan et al. have determined the universal adjacency spectra of $H-$ join of graphs using another generalization of Fiedler's lemma. Several authors [1, 4, 8, 10, 11, 14, 15, 20] have determined the distance spectra of graphs that are obtained by applying different graph operations, as well as the distance spectra that characterize the graphs from an application perspective. Recently, in [3, 16, 17], the authors have determined the upper bounds for the extremal graphs related to reciprocal distance Laplacian spectral radius. The books [5, 9] are excellent resources on spectra of graphs for interested readers.

Motivated by these, we define a new distance matrix is called the *universal distance matrix* of G. For $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\beta \neq 0$, the universal distance matrix $U^D(G)$ is defined as $U^D(G) = \alpha Tr(G) + \beta D(G) + \gamma J + \delta I$, where $Tr(G)$ is the diagonal matrix whose elements are the vertex transmissions, and $D(G)$ is the distance matrix of G. Here J is the all-ones matrix, and I is the identity matrix. The set of eigenvalues of the universal distance matrix namely, $\{\rho_1, \rho_2, \ldots, \rho_n\}$ is known as the universal distance spectrum of G. By taking appropriate values for $\alpha, \beta, \gamma,$ and δ, we obtain the eigenvalues for the universal distance matrix and various matrices related to distance. Consequently, we also determine the spectrum of universal distance matrix of the graph complement of G. Here we determine the universal distance spectra of regular graphs and graphs obtained using graph operations such as join, joined union, generalized joined union of regular graphs of diameter two.

2. Main Results

In this section, we discuss the universal distance spectra of $r-$regular graph, join of two regular graphs and joined union of graphs. Also, we obtain the universal distance spectrum of Petersen graph, complete bipartite graph, wheel graph, complete split graph and joined union of graphs related to complete graph.

2.1. Universal Distance Spectrum of $r-$ Regular Graph

In this subsection, we describe the universal distance spectrum of $r-$ regular graph and obtain the universal distance spectrum of $r-$ regular graph. In particular, we obtain the universal distance spectrum of Petersen graph.

Theorem 1. *Let G be a $r-$regular graph of order n with diameter at most two. The adjacency eigenvalues of G are denoted by $r = \lambda_1, \lambda_2, \ldots, \lambda_n$. The eigenvalues of the universal distance matrix of G are*

$$\{(\alpha+\beta)(2n-r-2)+\gamma n+\delta, \quad \alpha(2n-r-2)+(-2-\lambda_i)\beta+\delta, \quad i=2,3,\ldots,n\}$$

Proof. Let G represent a $r-$ regular graph of order n with diameter at most two. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of the graph G.

In G, for all $v \in V(G)$, we have $Tr(v) = r + 2(n-r-1) = 2n-r-2$.

The universal distance matrix of G can be written as

$$\begin{aligned} U^D(G) &= \alpha Tr(G) + \beta D(G) + \gamma J_n + \delta I_n, \quad for\ \alpha, \beta, \gamma, \delta \in \mathbb{R},\ \beta \neq 0. \\ &= \alpha(2n-r-2)I_n + \beta\big[A(G) + 2A(\overline{G})\big] + \gamma J_n + \delta I_n \\ &= \alpha(2n-r-2)I_n + \beta(2J_n - 2I_n - A(G)) + \gamma J_n + \delta I_n \end{aligned}$$

where J_n is an all ones matrix of order n and I_n is the identity matrix of order n.

Let $X = (1\ 1\ 1 \ldots\ 1)^T$ be the all ones vector of order n. Since G is a $r-$regular graph, it follows that X is the Perron vector corresponding to $\rho_1(G) = (\alpha+\beta)(2n-r-2)+\gamma n+\delta$. Note that since G is $r-$regular, $A\left(\overline{G}\right)$ is $(n-1-r)-$ regular and $X = (1\ 1\ 1 \ldots\ 1)^T$ is also an eigenvector corresponding to $\lambda_1\left(A\left(\overline{G}\right)\right)$. For each $i \in \{2, 3, \ldots, n\}$, let λ_i and X_i be an eigenvalue and the corresponding eigenvector of λ_i, respectively, of $A(G)$. Then $X^T X_i = 0$ and

$$\begin{aligned} U^D(G)X_i &= \left[\alpha(2n-r-2)I_n + \beta(2J_n - 2I_n - A(G)) + \gamma J_n + \delta I_n\right]X_i \\ &= \left[\alpha(2n-r-2) + (-2-\lambda_i)\beta + \delta\right]X_i, \quad i = 2, 3, \ldots, n. \end{aligned}$$

This completes the proof.

Corollary 1. *The universal distance spectrum of Petersen graph consists precisely of* $15(\alpha+\beta)+10\gamma+\delta$, $15\alpha-3\beta+\delta$ *with algebraic multiplicity 5 and* $15\alpha+\delta$ *with algebraic multiplicity 4.*

2.2. Eigenvalues of Universal Distance Matrix of Join of Graphs

In this subsection, we describe the universal distance spectrum of join of two regular graphs and obtain the universal distance spectrum of this graph. Also, we obtain the universal distance spectra of complete bipartite graph, wheel graph and complete split graph.

Theorem 2. *For* $i \in \{1, 2\}$, *let* G_i *be an* r_i-*regular graph of order* n_i *and let* $r_i = \lambda_1^i, \lambda_2^i, \ldots, \lambda_{n_i}^i$ *be the eigenvalues of* $A(G_i)$. *The characteristic polynomial of* $G = G_1 \nabla G_2$, *denoted by* $P(G:x)$, *is given by*

$$P(G:x) = \left[x^2 - (s_1+s_2)x + \left[s_1 s_2 - (\beta+\gamma)^2 n_1 n_2\right]\right] \prod_{s=2}^{n_1} \left[x - \left[\alpha(2n_1 - r_1 + n_2 - 2)\right] + \beta\left(-\lambda_s^1\right) + \delta\right] \prod_{j=2}^{n_2} \left[x - \left[\alpha(2n_2 - r_2 + n_1 - 2)\right] + \beta\left(-\lambda_j^2\right) + \delta\right];$$

where
$s_1 = \alpha(2n_1 - r_1 + n_2 - 2) + \beta(2 - r_1) + \gamma n_1 + \delta,$
$s_2 = \alpha(2n_2 - r_2 + n_1 - 2) + \beta(2 - r_2) + \gamma n_2 + \delta.$

Proof. Let G_1 and G_2 be r_1- and r_2- regular graphs of orders n_1 and n_2, respectively. Consider the vertex sets of G_1 and G_2 with, $V(G_1)$ and $V(G_2)$, respectively. Clearly, the graph G has diameter at most two with the vertex set $V(G_1) \cup V(G_2)$.

In G_1, we have $Tr_{G_1}(v) = 2(n_1 - r_1 - 1) + r_1 + n_2$, for all $v \in V(G_1)$.
In G_2, we have $Tr_{G_2}(v) = 2(n_2 - r_2 - 1) + r_2 + n_1$, for all $v \in V(G_2)$.

Label the vertices of the graph G such that the first n_1 vertices are from G_1 and the next n_2 vertices are from G_2.

The universal distance matrix of G can be written as

$$U^D(G) = \begin{pmatrix} U(G_1) & (\beta+\gamma) J_{n_1\times n_2} \\ (\beta+\gamma) J_{n_2\times n_1} & U(G_2) \end{pmatrix}$$

where

$$U^D(G_1) = \alpha(2n_1 - r_1 + n_2 - 2) I_{n_1} + \beta(2I_{n_1} - A(G_1)) + \gamma J_{n_1} + \delta I_{n_1}$$

$$U^D(G_2) = \alpha(2n_2 - r_2 + n_1 - 2) I_{n_2} + \beta(2I_{n_2} - A(G_2)) + \gamma J_{n_2} + \delta I_{n_2}$$

Let $\mathbf{1_n} = (1\ 1\ 1\ldots\ 1)^T$ be an all ones vector of order n. Since G_1 is a r_1- regular graph, $\mathbf{1_{n_1}}$ is the eigenvector corresponding to the eigenvalue r_1 of $A(G_1)$. Similarly, G_2 is a r_2- regular graph, $\mathbf{1_{n_2}}$ is the eigenvector corresponding to the eigenvalue r_2 of $A(G_2)$.

Let $\mathbf{w}$ be an orthogonal vector to $\mathbf{1_{n_1}}$, and $A(G_1)\mathbf{1_{n_1}} = \lambda_1^1\mathbf{w}$. We take $W = \begin{pmatrix}\mathbf{w} & 0\end{pmatrix}^T$ and since $J^T_{n_1\times n_2}W = 0$, we get

$$U^D(G)W = \Big[(2n_1 - r_1 + n_2 - 2)\alpha + \left(-\lambda_s^1\right)\beta + \delta\Big]W; \quad s = 2,3,\ldots,n_1.$$

This shows that $(2n_1 - r_1 + n_2 - 2)\alpha + \left(-\lambda_s^1\right)\beta + \delta$ is an eigenvalue of $U^D(G)$ and $W = \begin{pmatrix}\mathbf{w} & 0\end{pmatrix}^T$ is the corresponding eigenvector.

Similarly, let $\mathbf{x}$ be an orthogonal vector to $\mathbf{1_{n_2}}$, and $A(G_2)\mathbf{1_{n_2}} = \lambda_1^2\mathbf{x}$. We take $X = \begin{pmatrix}0 & \mathbf{x}\end{pmatrix}^T$ and since $J^T_{n_2\times n_1}\mathbf{x} = 0$, we get

$$U^D(G)X = \Big[(2n_2 - r_2 + n_1 - 2)\alpha + \left(-\lambda_j^2\right)\beta + \delta\Big]X; \quad j = 2,3,\ldots,n_2.$$

This shows that $(2n_2 - r_2 + n_1 - 2)\alpha + \left(-\lambda_j^2\right)\beta + \delta$ is an eigenvalue of $U^D(G)$ and $X = \begin{pmatrix}0 & \mathbf{x}\end{pmatrix}^T$ is the corresponding eigenvector.

Totally, we have $n_1 + n_2 - 2$ eigenvalues of $U^D(G)$. Then the other two eigenvalues of $U^D(G)$ are derived from the quotient matrix

$$S = \begin{pmatrix} s_1 & (\beta+\gamma)n_2 \\ (\beta+\gamma)n_1 & s_2 \end{pmatrix}$$

where

$$s_1 = \alpha(2n_1 - r_1 + n_2 - 2) + \beta(2 - r_1) + \gamma n_1 + \delta$$

$s_2 = \alpha(2n_2 - r_2 + n_1 - 2) + \beta(2 - r_2) + \gamma n_2 + \delta$

The characteristic equation of S is $x^2 - (s_1 + s_2)x + \left[s_1 + s_2 - (\beta+\gamma)^2 n_1 n_2\right] = 0$ and its roots are the eigenvalues of $U^D(G)$.

This completes the proof.

Corollary 2. *The universal distance spectrum of complete bipartite graph* $K_{p,\,q} = \overline{K_p}\nabla\overline{K_q}$ *consists of the eigenvalues* $\{\alpha(2p+q-2)+\delta\}^{p-1}$, $\{\alpha(2q+p-2)+\delta\}^{q-1}$ *and* $\frac{1}{2}\Big[(t_1+t_2) \pm \sqrt{(t_1-t_2)^2 + 4(\beta+\gamma)^2 pq}$, *where* $t_1 = \alpha(2p+q-2)+2\beta+p\gamma+\delta$, $t_2 = \alpha(2q+p-2)+2\beta+q\gamma+\delta$.

Proof. By substituting $n_1 = p, r_1 = 0, n_2 = q, r_2 = 0, \lambda_2^1 = \lambda_3^1 = \cdots = \lambda_p^1 = 0$, and $\lambda_2^2 = \lambda_3^2 = \cdots = \lambda_q^2 = 0$, in Theorem 2, the universal spectrum of $K_{p,\,q}$ graph is obtained. Hence the result.

Corollary 3. *The universal distance spectrum of wheel graph* $W_n = C_n\nabla K_1$ *consists of the eigenvalues* $n\alpha+\delta$, $\alpha(2n-3) - \beta cos\left(\frac{2(i-1)\pi}{n}\right) + \delta$; $i = 2,3,\ldots,n$ *and* $\frac{1}{2}\Big[(l_1+l_2) \pm \sqrt{(l_1-l_2)^2 + 4(\beta+\gamma)^2 n}$, *where* $l_1 = \alpha(2n-3)+\gamma n+\delta$, $l_2 = n\alpha + 2\beta+\gamma+\delta$

Proof. By substituting $n_1 = n,\ r_1 = 2,\ n_2 = 1, r_2 = 0$, and $\lambda_i^1 = 2cos\left(\frac{2(i-1)\pi}{n}\right)+\delta;\ i = 2,3,\ldots,n$, in Theorem 2, we obtain the universal distance spectrum of W_n graph. Hence the result.

Corollary 4. *The universal distance spectrum of complete split graph* $CS_{m,\,n-m} = K_m\nabla\overline{K_{n-m}}$ *consists of the eigenvalues* $\{\alpha(n-1)-\beta+\delta\}^{m-1}$, $\{\alpha(2n-m-2)+\delta\}^{n-m-1}$ *and* $\frac{1}{2}\Big[(g_1+g_2) \pm \sqrt{(g_1-g_2)^2 + 4(\beta+\gamma)^2 m(n-m)}$, *where* $g_1 = \alpha(n-1)+\beta(3-m)+\gamma m+\delta$, $g_2 = \alpha(3n-3m-2)+2\beta+\gamma(n-m)+\delta$.

Proof. In Theorem 2, by substituting $n_1 = m,\ r_1 = m-1,\ \lambda_2^1 = \lambda_3^1 = \cdots = \lambda_m^1 = -1$ and $\lambda_2^2 = \lambda_3^2 = \cdots = \lambda_{n-m}^2 = 0,\ n_2 = n-m,\ r_2 = 0$, we obtain the result.

2.3. Eigenvalues of Universal Distance Matrix of Joined Union of Graphs

In this subsection, we describe the universal distance spectrum of joined union of three regular graphs and obtain the universal distance spectrum of this graph. In particular, we obtain the universal distance spectrum of joined union of graphs related to complete graph.

Theorem 3. *Let G_i be r_i- regular graph of order n_i, for $i = 1, 2, 3$. Let $A(G_i)$ denote the adjacency matrix of G_i and the eigenvalues be $r_i = \lambda_1^i, \lambda_2^i, \ldots, \lambda_{n_i}^i$, respectively. Let $G = G_1 \nabla (G_2 \cup G_3)$. The graph G is the join of G_1 and union of two graphs $G_2 \cup G_3$. The universal distance spectrum of G consists of the eigenvalues*

(i) $\left[\alpha (N + n_1 - r_1 - 2) - 2\beta + \delta\right] - \beta\lambda_l^1; \quad l = 2, 3, \ldots, n_1,$

(ii) $\left[\alpha (2N + n_1 - r_2 - 2) - 2\beta + \delta\right] - \beta\lambda_m^2; \quad m = 2, 3, \ldots, n_2,$

(iii) $\left[\alpha (2N + n_1 - r_3 - 2) - 2\beta + \delta\right] - \beta\lambda_s^3; \quad s = 2, 3, \ldots, n_3,$

and the eigenvalues of the matrix

(iv)
$$\begin{bmatrix} \alpha (N + n_1 - r_1 - 2) + (2n_1 - r_1 - 2)\beta + \gamma n_1 + \delta & (\beta + \gamma) n_2 & (\beta + \gamma) n_3 \\ (\beta + \gamma) n_1 & \alpha (2N + n_1 - r_2 - 2) + (2n_2 - r_2 - 2)\beta + \gamma n_2 + \delta & (2\beta + \gamma) n_3 \\ (\beta + \gamma) n_1 & (2\beta + \gamma) n_2 & \alpha (2N + n_1 - r_2 - 2) + (2n_3 - r_3 - 2)\beta + \gamma n_3 + \delta \end{bmatrix}$$

where $N = \sum_{i=1}^{3} n_i$.

Proof. Let G_i be r_i-regular graph of order n_i, for $i = 1, 2, 3$. Let $V(G_i) = \left\{v_1^i, v_2^i, \ldots, v_{n_i}^i\right\}$ be the vertex set of the graph G_i. Consider the adjacency spectrum of G_i, $r_i = \lambda_1^i, \lambda_2^i, \ldots, \lambda_{n_i}^i$.

Let $G = G_1 \nabla (G_2 \cup G_3)$. The vertex set $V(G) = V(G_1) \cup V(G_2) \cup V(G_3)$ and $N = \sum_{i=1}^{3} n_i$. Obviously, G is of diameter two.

For all $v_j^1 \in V(G_1)$, we have $Tr_{G_1}\left(v_j^1\right) = N + n_1 - r_1 - 2;\ j = 1, 2, \ldots, n_1$.
For all $v_k^2 \in V(G_2)$, we have $Tr_{G_2}\left(v_k^2\right) = 2N + n_1 - r_2 - 2;\ k = 1, 2, \ldots, n_2$.
For all $v_l^3 \in V(G_3)$ we have, $Tr_{G_3}\left(v_l^3\right) = 2N + n_1 - r_3 - 2;\ l = 1, 2, \ldots, n_3$.

Label the vertices of graph G such that the first n_1 vertices are from G_1, the next n_2 vertices are from G_2 and the next n_3 vertices are from G_3.

The universal distance matrix of G can be expressed as

$U^D(G) =$

$$\begin{bmatrix} \begin{matrix}\left[\alpha\left(N+n_1-r_1-2\right)\\ -2\beta+\delta\right]I_{n_1}+\\ \left(2\beta+\gamma\right)J_{n_1}-\beta A\left(G_1\right)\end{matrix} & \left(\beta+\gamma\right)J_{n_1\times n_2} & \left(\beta+\gamma\right)J_{n_1\times n_3} \\ \left(\beta+\gamma\right)J_{n_2\times n_1} & \begin{matrix}\left[\alpha\left(2N+n_1-r_2-2\right)\\ -2\beta+\delta\right]I_{n_2}+\left(2\beta+\gamma\right)J_{n_2}\\ -\beta A\left(G_2\right)\end{matrix} & \left(2\beta+\gamma\right)J_{n_2\times n_3} \\ \left(\beta+\gamma\right)J_{n_3\times n_1} & \left(2\beta+\gamma\right)J_{n_3\times n_2} & \begin{matrix}\left[\alpha\left(2N+n_1-r_3-2\right)\\ -2\beta+\delta\right]I_{n_3}+\\ \left(2\beta+\gamma\right)J_{n_3}-\beta A\left(G_3\right)\end{matrix} \end{bmatrix}$$

Let $\mathbf{1}_n = (1\ 1\ 1\ \ldots 1)^T$ be all ones vector of order n. Since $G_i;\ i = 1, 2, 3$, is a r_i- regular graph, it follows that r_i is the largest eigenvalue and the corresponding eigenvector is $\mathbf{1}_{n_i}$. The remaining eigenvectors are orthogonal to $\mathbf{1}_{n_i}$.

Consider λ, μ, ζ as the eigenvalues of the adjacency matrices of G_1, G_2, G_3 with corresponding eigenvectors as $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively. Also, they satisfy $\mathbf{1}_{n_1}^T\mathbf{u} = 0,\ \mathbf{1}_{n_2}^T\mathbf{v} = 0,\ \mathbf{1}_{n_3}^T\mathbf{w} = 0$, respectively. Then, $\begin{pmatrix}\mathbf{u}^T & 0_{1\times n_2} & 0_{1\times n_3}\end{pmatrix}^T, \begin{pmatrix}0_{1\times n_1} & \mathbf{v}^T & 0_{1\times n_3}\end{pmatrix}^T$ and $\begin{pmatrix}0_{1\times n_1} & 0_{1\times n_2} & \mathbf{w}^T\end{pmatrix}^T$ are the eigenvectors of $U^D(G)$ with corresponding eigenvalues $\left[\alpha\left(N+n_1-r_1-2\right)-2\beta+\delta\right]-\beta\lambda_l^1;\quad l = 2, 3, \ldots, n_1$, $\left[\alpha\left(2N+n_1-r_2-2\right)-2\beta+\delta\right]-\beta\lambda_m^2;\quad m = 2, 3, \ldots, n_2$, and $\left[\alpha\left(2N+n_1-r_3-2\right)-2\beta+\delta\right]-\beta\lambda_s^3;\quad s = 2, 3, \ldots, n_3$, respectively.

Totally, we have $N-3$ eigenvectors and they are orthogonal to $\begin{pmatrix}\mathbf{u}^T & 0_{1\times n_2} & 0_{1\times n_3}\end{pmatrix}^T$, $\begin{pmatrix}0_{1\times n_1} & \mathbf{v}^T & 0_{1\times n_3}\end{pmatrix}^T$ and $\begin{pmatrix}0_{1\times n_1} & 0_{1\times n_2} & \mathbf{w}^T\end{pmatrix}^T$. For a suitable choice of $a \neq 0, b \neq 0, c \neq 0$, the other three eigenvectors of $U^D(G)$ can be represented by $\begin{pmatrix}a\mathbf{1}_{n_1}^T & b\mathbf{1}_{n_2}^T & c\mathbf{1}_{n_3}^T\end{pmatrix}^T$.

Consider ρ as an eigenvalue of the matrix $U^D(G)$ with the corresponding eigenvector $\mathbf{Z} = \begin{pmatrix}a\mathbf{1}_{n_1}^T & b\mathbf{1}_{n_2}^T & c\mathbf{1}_{n_3}^T\end{pmatrix}^T$. We know that $U^D(G)\mathbf{Z} = \rho\mathbf{Z}$ and $A(G_i) = r_i\mathbf{1}_{\mathbf{n_i}};\ i = 1, 2, 3$. Hence we have the system of linear equations as follows:

$$\left[\alpha\left(N+n_1-r_1-2\right)+\left(2n_1-r_1-2\right)\beta+\gamma n_1+\delta\right]a+\left[\left(\beta+\gamma\right)n_2\right]b+\left[\left(\beta+\gamma\right)n_3\right]c = \rho a,$$

$$\left[\left(\beta+\gamma\right)n_1\right]a+\left[\alpha\left(2N+n_1-r_2-2\right)+\left(2n_2-r_2-2\right)\beta+\gamma n_2+\delta\right]b+\left[\left(2\beta+\gamma\right)n_3\right]c = \rho b,$$

$$\left[\left(\beta+\gamma\right)n_1\right]a+\left[\left(2\beta+\gamma\right)n_2\right]b+\left[\alpha\left(2N+n_1-r_2-2\right)+\left(2n_3-r_3-2\right)\beta+\gamma n_3+\delta\right]c = \rho c.$$

Eliminating $a, b,$ and c, we obtain the nontrivial solution for the system of equations. This nontrivial solution yields the eigenvalues of $U^D(G)$ corresponding to ρ.

This completes the proof.

Corollary 5. *The universal distance spectrum of* $G = K_{n_1}\nabla\left(K_{n_2}\cup K_{n_3}\right)$ *consists of the eigenvalues*

(i) $(\alpha N - \beta + \delta - \alpha)$ *with algebraic multiplicity* $n_1 - 1$*,*

(ii) $\alpha\left(2N - n_1 - n_2 - 1\right) - \beta + \delta$ *with algebraic multiplicity* $n_2 - 1$*,*

(iii) $\alpha\left(2N - n_1 - n_3 - 1\right) - \beta + \delta$ *with algebraic multiplicity* $n_3 - 1$*,*

and the eigenvalues of the matrix

(iv)
$$\begin{pmatrix} \alpha\left(N-1\right)+\left(n_1-1\right)\beta+\gamma n_1+\delta & \left(\beta+\gamma\right)n_2 & \left(\beta+\gamma\right)n_3 \\ \left(\beta+\gamma\right)n_1 & \alpha\left(2N-n_1-n_2-1\right)+\left(n_2-1\right)\beta+\gamma n_2+\delta & \left(2\beta+\gamma\right)n_3 \\ \left(\beta+\gamma\right)n_1 & \left(2\beta+\gamma\right)n_2 & \alpha\left(2N-n_1-n_3-1\right)+\left(n_3-1\right)\beta+\gamma n_3+\delta \end{pmatrix}$$

Proof. In Theorem 3, by substituting $r_i = n_i - 1,\ \lambda_2^i, \lambda_3^i, \ldots \lambda_{n_i}^i = -1,$ for all $i = 1, 2, 3,$ we obtain the universal distance spectrum of G.

This completes the proof.

3. Eigenvalues of Universal Distance Matrix of Generalized Joined Union of Graphs

The generalized joined union is a nice graph operation. It is also called H-join [6] or generalized composition [19].

Let $H = (V, E)$ be any arbitrary graph of order n and $G_i = (V_i, E_i)$ be regular graphs of order n_i; $i = 1, 2, \ldots n$. The generalized joined union graph is denoted by $G\left(P, Q\right) = H\left(G_1, G_2, \ldots, G_n\right)$ with vertex set

$$P\left(G\right) = \bigcup_{i=1}^{n} V\left(G_i\right)$$

and edge set

$$Q\left(G\right) = \left(\bigcup_{i=1}^{n} E\left(G_i\right)\right) \cup \left(\bigcup_{v_i, v_j \in E(H)} \left\{\mathcal{E}\left(G_i \nabla G_j\right)\right\}\right).$$

where $\mathcal{E}\left(G_i \nabla G_j\right) = \left\{xy : x \in V\left(G_i\right), y \in V\left(G_j\right)\right\}$.

This graph G can be constructed by taking the union of $G_1, G_2, \ldots, G_n$ and joining every pair of vertices between G_i and G_j whenever v_i and v_j are adjacent in H.

Theorem 4. *Suppose H is a graph and its vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ with diameter at most 2. Let G_i be a r_i-regular graph of order n_i. Denote the adjacency eigenvalues of G_i as $r_i = \lambda_1^i, \lambda_2^i, \ldots, \lambda_{n_i}^i;\ i = 1, 2, \ldots, n$, respectively. The universal distance spectrum of the generalized joined union $G = H(G_1, G_2, \ldots, G_n)$ consists of the eigenvalues $\alpha(2N - r_i - m_i - 2) - (\lambda_k^i + 2)\beta + \delta;\ i = 1, 2, \ldots, n,\ k = 2, 3, \ldots, n_i$, where $N = \sum_{i=1}^{n} n_i$ and $m_i = \sum_{\mathcal{E}(G_i \nabla G_j)} n_j$ and the other n eigenvalues of the quotient matrix*

$$\begin{bmatrix} R_{11} & [\beta\ d_H(v_1, v_2) + \gamma]n_2 & \ldots & [\beta\ d_H(v_1, v_n) + \gamma]n_n \\ [\beta\ d_H(v_2, v_1) + \gamma]n_1 & R_{22} & \ldots & [\beta\ d_H(v_2, v_n) + \gamma]n_n \\ \vdots & \vdots & \vdots & \vdots \\ [\beta\ d_H(v_n, v_1) + \gamma]n_1 & [\beta\ d_H(v_n, v_2) + \gamma]n_2 & \ldots & R_{nn} \end{bmatrix},$$

where $R_{ii} = \alpha(2N - r_i - m_i - 2) - (r_i - 2n_i + 2)\beta + \gamma n_i + \delta,;\ i = 1, 2, \ldots, n$, and $d_H(v_i, v_j)$ is the length of the shortest path between v_i and v_j in H.

Proof. Using the appropriate labelling of the vertices of the graph G, the universal distance spectrum of the generalized distance matrix can be expressed in the following form

$$U^D(G) = \alpha Tr(G) + \beta D(G) + \gamma J + \delta$$

$$= \begin{bmatrix} S_{11} & [\beta\ d_H(v_1, v_2) + \gamma]J_{n_1 \times n_2} & \ldots & [\beta\ d_H(v_1, v_n) + \gamma]J_{n_1 \times n_n} \\ [\beta\ d_H(v_2, v_1) + \gamma]J_{n_2 \times n_1} & S_{22} & \ldots & [\beta\ d_H(v_2, v_n) + \gamma]J_{n_2 \times n_n} \\ \vdots & \vdots & \vdots & \vdots \\ [\beta\ d_H(v_n, v_1) + \gamma]J_{n_n \times n_1} & [\beta\ d_H(v_n, v_2) + \gamma]J_{n_2 \times n_n} & \ldots & S_{nn}, \end{bmatrix}$$

where $S_{ii} = \Big[\alpha(2N - r_i - m_i - 2) - 2\beta + \delta\Big]I_{n_i} + (2\beta + \gamma)J_{n_i} - A(G_i)\beta;\ i = 1, 2, \ldots, n$, I_{n_i} is the identity matrix of order n_i, and J_{n_i} is the all-ones matrix of order n_i.

Since G_i is r_i-regular, $\mathbf{1_{n_i \times 1}}$ the all-ones vector is an eigenvector of $A(G_i)$ corresponding to eigenvalue r_i. The remaining eigenvectors are orthogonal to $\mathbf{1_{n_i \times 1}}$. Consider λ the eigenvalue of $A(G_i)$ corresponding to the eigenvector $X_i = \left(x_1^i\ x_2^i\ \ldots\ x_{n_i}^i\right)^T$, satisfying $\mathbf{1^T_{n_i \times 1}} X_i = 0;\ i = 2, 3, \ldots, n$. Consider the vector y_i^T, where

$$y_1 = \begin{cases} x_j^1, & v_j^1 \in V(G_1);\ j = 2, 3, \ldots, n_1. \\ 0, & otherwise \end{cases}$$

$$y_2 = \begin{cases} x_j^2, & v_j^2 \in V(G_2);\ j = 2,3,\ldots,n_2. \\ 0, & otherwise \end{cases}$$

$\ldots$

$$y_n = \begin{cases} x_j^n, & v_j^n \in V(G_n);\ j = 2,3,\ldots,n_n. \\ 0, & otherwise \end{cases}$$

Clearly, the vector y_i^T is an eigenvector of $U^D(G)$ corresponding to the eigenvalue $\alpha(2N - r_i - m_i - 2) - \left(\lambda_k^i + 2\right)\beta + \delta;\ i = 1,2,\ldots,n,\ k = 2,3,\ldots,n_i$. Totally, we have $N - n$ mutually orthogonal eigenvectors of $U^D(G)$. These vectors are orthogonal to the vector

$$\mathbf{1^i} = \begin{cases} \mathbf{1_{n_i \times 1}}, & v_j^i \in V(G_i); \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n_i. \\ 0, & otherwise \end{cases}$$

For suitable choice of arbitrary values $\alpha_1, \alpha_2, \ldots, \alpha_n$ we have $\mathbf{1} = \left(\alpha_1 \mathbf{1}^1\ \alpha_2 \mathbf{1}^2\ \ldots\ \alpha_n \mathbf{1}^n\right)$ as the eigenvector corresponding to the eigenvalues of the $n \times n$ quotient matrix of $U^D(G)$ of the form

$$\begin{bmatrix} R_{11} & \left[\beta\ d_H(v_1, v_2) + \gamma\right] n_2 & \ldots & \left[\beta\ d_H(v_1, v_n) + \gamma\right] n_n \\ \left[\beta\ d_H(v_2, v_1) + \gamma\right] n_1 & R_{22} & \ldots & \left[\beta\ d_H(v_2, v_n) + \gamma\right] n_n \\ \vdots & \vdots & \vdots & \vdots \\ \left[\beta\ d_H(v_n, v_1) + \gamma\right] n_1 & \left[\beta\ d_H(v_n, v_2) + \gamma\right] n_2 & \ldots & R_{nn} \end{bmatrix},$$

where $R_{ii} = \alpha(2N - r_i - m_i - 2) - (r_i - 2n_i + 2)\beta + \gamma n_i + \delta,;\ i = 1,2,\ldots,n$.

This completes the proof.

Corollary 6. *The universal distance spectrum of complete $t-$partite graph $G = K_{n_1,n_2,\ldots,n_t}$ with $N = \sum_{i=1}^{t} n_i$ consists of the eigenvalues $\alpha(N + n_i - 2) - 2\beta + \delta;\ i = 1,2,\ldots,t$ with algebraic multiplicity n_i and t eigenvalues of the matrix*

$$\begin{pmatrix} M_{11} & (\beta+\gamma) n_2 & \ldots & (\beta+\gamma) n_t \\ (\beta+\gamma) n_1 & M_{22} & \ldots & (\beta+\gamma) n_t \\ \vdots & \vdots & \ldots & \vdots \\ (\beta+\gamma) n_1 & (\beta+\gamma) n_2 & \ldots & M_{tt}, \end{pmatrix}$$

where $M_{ii} = \alpha(N + n_i - 2) + (2n_i - 2)\beta + \gamma n_i + \delta;\ i = 1,2,\ldots,t$.

Proof. In Theorem 4, by substituting $r_i = 0, m_i = N - n_i;\ i = 1,2,\ldots,t$, we obtain the universal distance spectrum of G.

Example 1. *Consider the graph $G = H(G_1, G_2, G_3)$ as depicted in Figure 1, where $H = P_3$ the path graph of order 3, $G_1 = C_4$ the cycle graph of order 4, $G_2 = K_2$ and $G_3 = K_3$ the complete graphs of order 2 and 3, respectively. The universal distance matrix $U^D(G)$ of the generalized joined union $G = H(G_1, G_2, G_3)$ is a block matrix of the form*

$$\begin{pmatrix} W_{11} & (\beta+\gamma) J_{n_1\times n_2} & (2\beta+\gamma) J_{n_1\times n_3} \\ (\beta+\gamma) J_{n_2\times n_1} & W_{22} & (\beta+\gamma) J_{n_2\times n_3} \\ (2\beta+\gamma) J_{n_3\times n_1} & (\beta+\gamma) J_{n_3\times n_2} & W_{33} \end{pmatrix},$$

where $W_{ii} = \alpha(2N - r_i - m_i - 2) - (r_i - 2n_i + 2)\beta + \gamma n_i + \delta,;\ i = 1, 2, 3.$

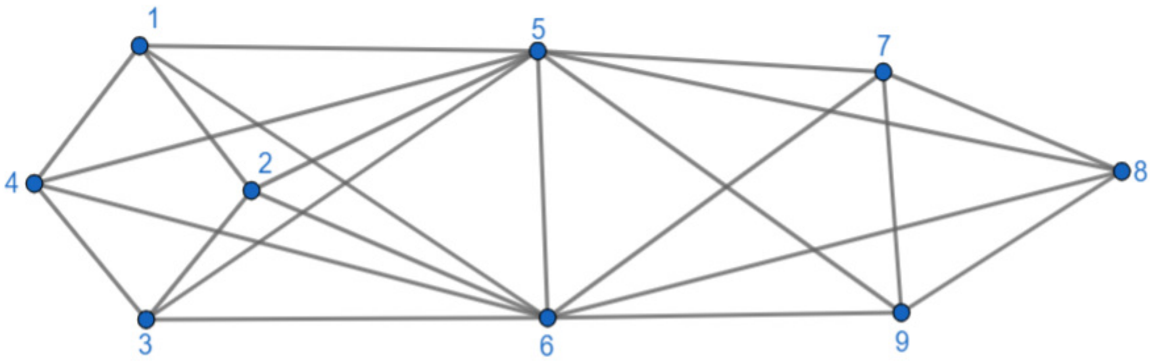

Figure 1: $P_3(C_4, K_2, K_3)$

The adjacency spectra of G_1, G_2 and G_3 are $spec_A(G_1) = \{2, 0, 0, -2\}$, $spec_A(G_2) = \{1, -1\}$, and $spec_A(G_3) = \{2, -1, -1\}$, respectively. Then from Theorem 4, the universal distance spectrum of $G = H(G_1, G_2, G_3)$ consists of the eigenvalues

(i) $12\alpha - 2\beta + \delta$ *with algebraic multiplicity 2,*

(ii) $12\alpha + \delta$,

(iii) $8\alpha - \beta + \delta$,

(iv) $12\alpha - \beta + \delta$ *with algebraic multiplicity 2,*

and the eigenvalues of the matrix

(v) $$\begin{pmatrix} 12\alpha + 4\beta + 4\gamma + \delta & 2(\beta+\gamma) & 3(2\beta+\gamma) \\ 4(\beta+\gamma) & 8\alpha + \beta + 2\gamma + \delta & 3(2\beta+\gamma) \\ 4(2\beta+\gamma) & 2(\beta+\gamma) & 12\alpha + 2\beta + 3\gamma + \delta \end{pmatrix}$$

Note that, when $\alpha = 0, \beta = 1, \gamma = 0, \delta = 0$, $U^D(G) = D(G)$ and we obtain the distance spectrum of G as

$$spec_D(G) = \{11.3523, 0, -0.3523, -1, -1, -1, -2, -2, -4\}.$$

Also, when $\alpha = 0, \beta = -2, \gamma = 1, \delta = -1$, $U^D(G) = D_S(G) = J - I - 2D(G)$ and we obtain the eigenvalues of the distance Seidal matrix of G.

4. Conclusion

In this paper, we have introduced a new unified matrix called the Universal Distance matrix. As a consequence, we can obtain the eigenvalues of distance matrix, distance Laplacian matrix, distance signless Laplacian matrix, generalized distance matrix, distance Seidal matrix and distance matrices of graph complements. We have derived the universal distance spectra of $r-$ regular graphs, join of two regulars, joined union of three regular graphs and generalized joined union of $G_1, G_2, \dots, G_n$ regular graphs with an arbitrary graph of order n. Also, we obtained the universal distance spectra of Petersen graph, complete bipartite graph, wheel graph, complete split graph. We have also illustrated our results through an example for H-join of graphs. Our current study pertains only to regular graphs. This study can be extended to general graphs. We conclude with the following open problems:

Problem 1. Characterize graphs with minimal universal distance spectral radius.

Problem 2. Find $k-$ transmission regular graphs for particular values of $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Problem 3. Find the upper bound for the largest universal distance eigenvalue and universal distance energy.

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