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Tensor Product Semi-Groups

Roshdi Khalil[1,*], R. Al-Mirbati[2], and D. Drissi[3]

[1] *Department of Mathematics, University of Jordan, Amman-Jordan*
[2] *Gulf University for Science and Technology, Kuwait*
[3] *Department of Mathematics, Kuwait University, Kuwait*

Abstract. Let X and Y be Banach spaces and $L(X,Y)$ be the space of all bounded linear operators from X to Y.If $X=Y$ we write $L(X)$ for$L(X,Y)$. Let $X\otimes Y$ be the tensor product of X and Y, and $X\overset{\alpha}{\otimes}Y$ be the completion of $X\otimes Y$ with respect to a uniform cross norm α. In this paper, we present an extension of the Hille-Yosida Theorem to tensor product semigroups.

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1. Introduction

One parameter semigroups of operators have been a useful tool in the study of the so-called abstract Cauchy problem. Such a problem states as follows: Let A be a linear operator on a Banach space X, find a continuously differentiable function $T(.,x)$ from $[0,\infty)$ into the domain of A such that T satisfies the differential equation $\frac{d}{dt}T(t,x)=AT(t,x)$, $(t\geq 0)$, $T(0,x)=x$, for all $x\in Dom(A)$. So much work has been done on one parameter semigroups of operators as well as its relation to the abstract Cauchy problem. For more on such topics we refer to [3, 4, 9].

We begin recalling some standard definitions. Let X be a Banach space and $L(X)$ be the space of bounded linear operators on X. By a one parameter semigroup of operators on X we mean a map $T:[0,\infty)\to L(X)$ such that

(i) $T(0)=I$, the identity operator on X.

(ii) $T(s+t)=T(s)T(t)$ for all $s,t\geq 0$, the semigroup property.

*Corresponding author.

Email address: `roshdi@ju.edu.jo` (R. Khalil)

The linear operator A whose domain $\mathfrak{D}(A)$ is given by

$$\mathfrak{D}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

such that

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d^+}{dt}\left(T(t)x\right)|_{t=0} \text{ for } x \in \mathfrak{D}(A)$$

is called the infinitesimal generator of the semigroup $(T(t))_{t \geq 0}$. The generator A is always a closed, densely defined operator. It is well known that, when A is a densely defined linear operator with non empty resolvent set, then the abstract Cauchy problem has a unique solution, for all x in the domain of A, if and only if A generates a strongly continuous semigroup. (Pazy, [9, 10], Goldstein, [3]).

There are many important results on one parameter semigroups of operators. We mention two of such results:

I. Characterization of the infinitesimal generator of a semigroup.

II. "Hille-Yosida Theorem": The norm of the resolvent operator $R_\lambda(A)$ of the infinitesimal generator of a C_0 semigroup tends to zero at infinity. More precisely, $\left\| R_\lambda(A) \right\| \leq \frac{M}{\lambda - \omega}$ for large λ, which is known as the Hille-Yosida Inequality.

In this paper, we introduce what we call a tensor product semigroup. We show that every tensor product semigroup is a two parameter semigroup. We study the relation between a tensor product semigroup and its components. As not every two parameter semigroup on $X \overset{\alpha}{\otimes} Y$ defines a T.P.S., we present a condition under which a two parameter semigroup be a T.P.S. We show that the operator $\overline{A_1 \otimes I + I \otimes A_2}$, is the infinitesimal generator of a C_0 T.P.S., where A_1,A_2 generate the semigroup components of the T.P.S. . Equality of $\overline{A_1 \otimes I + I \otimes A_2}$ and $\overline{A_1 \otimes I} + \overline{I \otimes A_2}$ is proved as well.

Throughout this paper, $X \overset{\vee}{\otimes} Y$ ($X \overset{\wedge}{\otimes} Y$) denote the completion of the injective (the projective) tensor products of X and Y . If P and Q are elements in $L(X)$ and $L(Y)$ respectively, then $P \otimes Q$ denotes the tensor product operator on $X \otimes Y$. Further, we write $X \overset{\alpha}{\otimes} Y$ to denote either one of the tensor products (the projective or the injective). For more details on tensor product spaces and tensor product of operators, we refer the reader to [8].

2. Tensor Product Semigroups

Definition 1. *Let X, Y be Banach spaces, and $(T(s))_{s \geq 0}, (S(t))_{t \geq 0}$ be one parameter families of operators in $L(X), L(Y)$ respectively. The family $(T(s) \otimes S(t))_{s,t \geq 0}$ is called a* ***tensor product semigroup****, (abbreviated* ***T.P.S.****) on the Banach space $X \overset{\alpha}{\otimes} Y$ if*

1. $T(0) \otimes S(0) = I_{X \otimes Y}$,
2. $T(s_1 + s_2) \otimes S(t_1 + t_2) = \left(T(s_1) \otimes S(t_1)\right)\left(T(s_2) \otimes S(t_2)\right)$,

This is equivalent to

$$T(0)\overset{\alpha}{\otimes}S(0)=I_{X\overset{\alpha}{\otimes}Y} \text{ and } T(s_1+s_2)\overset{\alpha}{\otimes}S(t_1+t_2)=\left(T(s_1)\overset{\alpha}{\otimes}S(t_1)\right)\left(T(s_2)\overset{\alpha}{\otimes}S(t_2)\right).$$

Thus the family $\left(T(s)\overset{\alpha}{\otimes}S(t)\right)_{s,t\geq 0}$ is a T.P.S. defined on the complete space $X\overset{\alpha}{\otimes}Y$. For short, we will write $(T(s)\otimes S(t))_{s,t\geq 0}$ for $\left(T(s)\overset{\alpha}{\otimes}S(t)\right)_{s,t\geq 0}$, and I for each of $I_{X\otimes Y}$, and $I_{X\overset{\alpha}{\otimes}Y}$. It should be remarked that if we know $T(s)\otimes S(t)$ on $X\otimes Y$, then we know $T(s)\overset{\alpha}{\otimes}S(t)$ on $X\overset{\alpha}{\otimes}Y$.

One can define a **T.P.S.** $(T(s)\otimes S(t))_{s,t\geq 0}$, to be **uniformly continuous** on $X\overset{\alpha}{\otimes}Y$ if $\underset{(s,t)\to(0^+,0^+)}{Lim}\|T(s)\otimes S(t)-I\otimes I\|=0$, and to be **strongly continuous** on $X\overset{\alpha}{\otimes}Y(C_0)$ if

$$\underset{(s,t)\to(0^+,0^+)}{Lim}\left\|T(s)\overset{\alpha}{\otimes}S(t)z-z\right\|=0$$

for all $z\in X\overset{\alpha}{\otimes}Y$.

One can easily see that the limit in (2) can be replaced by

$$\underset{(s,t)\to(0^+,0^+)}{Lim}\left\|(T(s)\otimes S(t))\left(x\otimes y\right)-x\otimes y\right\|=0,$$

for all $x\in X, y\in Y$. The proof is a consequence of the following

Lemma 1. *Let X, Y be Banach spaces and α be a uniform crossnorm on $X\otimes Y$. If $\left\|\left(A_i\otimes B_i\right)z-(A\otimes B)z\right\|\to 0$ as $i\to\infty$, for all $z\in X\otimes Y$ and none of the sequences $\left(A_i\right),\left(B_i\right)$ has a subsequence that converges to zero pointwise, then $\left(A_i\otimes B_i\right)_i$ is uniformly bonded. Moreover, each of $\left(A_i\right)_i,\left(B_i\right)_i$ is uniformly bounded.*

Proof. For a fixed $0\neq x_0\in X$ one can see that each vector in the space $\left[x_0\right]\otimes Y$ is of the form $x_0\otimes y$ for some $y\in Y$. Therefore, $\left[x_0\right]\otimes Y$ is a Banach space. Since $\left\|\left(A_i\otimes B_i\right)z-(A\otimes B)z\right\|\overset{i\to\infty}{\to}0$ for every $z\in\left[x_0\right]\otimes Y$, then $\left(A_i\otimes B_i\right)$ is a pointwise bounded sequence of bounded operators on the Banach space $\left[x_0\right]\otimes Y$. Which implies by the Uniform Boundedness Principle that $\left(\left(A_i\otimes B_i\right)_{|_{[x_0]\otimes Y}}\right)_i$ is uniformly bounded. That is $\left\|\left(A_i\otimes B_i\right)_{|_{[x_0]\otimes Y}}\right\|\leq c$ for all i. But

$$\begin{aligned}\left\|\left(A_i\otimes B_i\right)_{|_{[x_0]\otimes Y}}\right\| &= \sup_{\substack{y\in Y\\ \|x_0\otimes y\|=1}}\left\|\left(A_i\otimes B_i\right)\left(x_0\otimes y\right)\right\| \\ &= \sup_{\substack{y\in Y\\ \|y\|=\frac{1}{\|x_0\|}}}\left\|\left(A_i\otimes B_i\right)\left(x_0\otimes y\right)\right\|\end{aligned}$$

$$= \sup_{\substack{y\in Y \\ \|y\|=1}} \left\|A_i x_0 \otimes B_i y\right\| = \sup_{\substack{y\in Y \\ \|y\|=1}} \left\|A_i x_0\right\| \left\|B_i y\right\|.$$

Thus $\sup_i \left(\sup_{\substack{y\in Y \\ \|y\|=1}} \left\|A_i x_0\right\| \left\|B_i y\right\| \right) \le c$. In other words $\left\|A_i x_0\right\| \left\|B_i y\right\| \le c$ for all i for all $y \in Y$. Under the assumption that $A_i x_0$ does not converge to zero, we obtain that $\left(B_i\right)$ is uniformly bounded on Y. Repeating the same approach and choosing $0 \neq y_0 \in Y$, one can show that $\left(A_i\right)$ is uniformly bounded on X. The lemma is then completely proved.

One can easily prove the following result.

Lemma 2. *Let X, Y be Banach spaces and $(T(s))_{s\ge 0}, (S(t))_{t\ge 0}$, be one parameter families of operators in $L(X), L(Y)$ respectively. Then the following are equivalent:*

a. $T(s)$ is a one parameter semigroup on X.

b. $T(s) \otimes I$ is a one parameter semigroup on $X \overset{\alpha}{\otimes} Y$.

c. $I \otimes T(s)$ is a one parameter semigroup on $Y \overset{\alpha}{\otimes} X$.

The following Lemma is essential for Theorem 1. Its proof is different from the proof in [7].

Lemma 3. *Let X, Y be Banach spaces, α any crossnorm on $X \otimes Y$. Let $a, c \in X, b, d \in Y$ be nonzero vectors. If $a \otimes b = c \otimes d$, then there exists a nonzero scalar β such $a = \beta c, b = \frac{1}{\beta} d$.*

Proof. Let $x^* \in X^*$. Then $x^*(a)b = x^*(c)d$. In particular, this holds for an x^* satisfying that $x^*(c) = \|c\|$. That is, $\frac{x^*(a)}{\|c\|} b = d$. It is clear that $x^*(a)$ is not zero. Choose $\frac{x^*(a)}{\|c\|} = \beta$. Then $\left(a - \beta c\right) \otimes b = 0$. Thus, $x^*\left(a - \beta c\right) b = 0$ for all $x^* \in X^*$. Choosing $x^* \in X^*$, such that $x^*\left(a - \beta c\right) = \left\|a - \beta c\right\|$ completes the proof.

Theorem 1. *Let X, Y be Banach spaces, $(T(s))_{s\ge 0}, (S(t))_{t\ge 0}$ one parameter families of operators in $L(X), L(Y)$ respectively. Then the family $T(s) \otimes S(t)$ is a T.P.S. on $X \overset{\alpha}{\otimes} Y$ if and only if there is a unique $0 \neq \beta \in \Re$, and unique one parameter semigroups $\left(\widehat{T}(s)\right)_{s\ge 0}$, $\left(\widehat{S}(t)\right)_{t\ge 0}$ on X, Y respectively, such that*

$$\beta T(s) = \widehat{T}(s) \text{ and } \frac{1}{\beta} S(t) = \widehat{S}(t) \text{ for all } s, t \ge 0.$$

Proof. If $\beta = 1$ then $(T(s))_{s\ge 0}, (S(t))_{t\ge 0}$ define one parameter semigroups. Therefore, from Lemma 3, each of $T(s) \otimes I$ and $I \otimes S(t)$ is a one parameter semigroup on $X \overset{\alpha}{\otimes} Y$. Consequently,

$$(T(s) \otimes I)(I \otimes S(t)) = T(s) \otimes S(t) = (I \otimes S(t))(T(s) \otimes I)$$

is a T.P.S. on $X \overset{\alpha}{\otimes} Y$.
If $\beta \neq 1$, then $T(s), S(t)$ are not semigroups of operators since $T(0) = \frac{1}{\beta} I \neq I$ even though, $T(s) \otimes S(t)$ is a T.P.S.

To show necessity, let $T(s) \otimes S(t)$ be a T.P.S. on $X \overset{\alpha}{\otimes} Y$. then $T(0) \otimes S(0) = I \otimes I$, and by Lemma 3, there exists $0 \neq \gamma \in \mathfrak{R}$ such that $T(0) = \gamma I$, and $S(0) = \frac{1}{\gamma} I$. Define the families $\widehat{T}(s)$ and $\widehat{S}(t)$ from $\mathfrak{R}^{+^2}$ into $L\left(X \overset{\alpha}{\otimes} Y\right)$ so that $\widehat{T}(s) = \frac{1}{\gamma} T(s)$ and $\widehat{S}(t) = \gamma S(t), s, t \geq 0$. Clearly, $\widehat{T}(s) \otimes \widehat{S}(t)$ is the T.P.S. $T(s) \otimes S(t)$. Moreover, $\widehat{T}(s), \widehat{S}(t)$ are one parameter semigroups on X, Y respectively. Indeed, $\widehat{T}(0) = \frac{1}{\gamma} T(0) = I$ and $\widehat{S}(0) = \gamma S(0) = I$. To show the semigroup property for $\widehat{T}(s)$, let $s_1, s_2 \in \mathfrak{R}^{+^2}$ and let $x \in X$. Then for any $0 \neq y \in Y$ we have

$$
\begin{aligned}
& \left\| \widehat{T}(s_1 + s_2)x - \widehat{T}(s_1)\widehat{T}(s_2)x \right\| \\
= \; & \frac{1}{\|y\|} \left\| \left(\widehat{T}(s_1 + s_2)x - \widehat{T}(s_1)\widehat{T}(s_2)x \right) \otimes y \right\| \\
= \; & \frac{1}{\|y\|} \left\| \left(\left(\widehat{T}(s_1 + s_2) \otimes I \right) - \left(\widehat{T}(s_1)\widehat{T}(s_2) \otimes I \right) \right) (x \otimes y) \right\| \\
= \; & \frac{1}{\|y\|} \left\| \left(\widehat{T}(s_1 + s_2) \otimes \widehat{S}(0+0) \right) - \left(\widehat{T}(s_1)\widehat{T}(s_2) \otimes \widehat{S}(0)\widehat{S}(0) \right) (x \otimes y) \right\| \\
= \; & \frac{\left\| \left[\left(\widehat{T}(s_1 + s_2) \otimes \widehat{S}(0+0) \right) - \left(\widehat{T}(s_1) \otimes \widehat{S}(0) \right) \left(\widehat{T}(s_2) \otimes \widehat{S}(0) \right) \right] (x \otimes y) \right\|}{\|y\|} \\
= \; & \frac{1}{\|y\|} \left\| \left(\left(T(s_1 + s_2) \otimes S(0+0) \right) - \left(T(s_1) \otimes S(0) \right) \left(T(s_2) \otimes S(0) \right) \right) (x \otimes y) \right\|.
\end{aligned}
$$

Therefore $\widehat{T}(s_1 + s_2) = \widehat{T}(s_1)\widehat{T}(s_2)$. Similarly, $\left(\widehat{S}(t)\right)_{t \geq 0}$ satisfies the semigroup property. Hence $\widehat{T}(s)$ and $\widehat{S}(t)$ are one parameter semigroups on X, Y respectively.

The proof of Theorem 1 shows that if $(T(s))_{s \geq 0}, (S(t))_{t \geq 0}$, are one parameter semigroups on X, Y respectively, then the family $(T(s) \otimes S(t))_{s,t \geq 0}$ is a T.P.S. on $X \overset{\alpha}{\otimes} Y$.

As for the continuity of tensor product semigroups it is not difficult to see

Lemma 4. *Let X, Y be Banach spaces, $(T(s))_{s \geq 0}, (S(t))_{t \geq 0}$, one parameter families of operators in $L(X), L(Y)$ respectively. If $T(s) \otimes S(t)$ is a T.P.S. and $\widehat{T}(s)$, $\widehat{S}(t)$ are as in Theorem 1, then the following are equivalent*

a. $T(s) \otimes S(t)$ *is uniformly (strongly) continuous.*

b. $\widehat{T}(s) \otimes I$ *and* $I \otimes \widehat{S}(t)$ *are uniformly (strongly) continuous.*

c. $\widehat{T}(s)$ *and* $\widehat{S}(t)$ *are uniformly (strongly) continuous.*

Now, if $(T(s))_{s\geq 0}, (S(t))_{t\geq 0}$, are one parameter families of operators in $L(X)$, $L(Y)$ respectively and $T(s)\otimes S(t)$ is a T.P.S., then:

If $T(s)\otimes S(t)$ is uniformly (strongly) continuous, then the map $F(s,t) : \mathfrak{R}^{+^2} = [0,\infty)\times[0,\infty) \to L\left(X \overset{\alpha}{\otimes} Y\right)$ defined by $F(s,t) \to T(s)\otimes S(t)$ is continuous in the uniform (strong) operator topology. Further, $F(s,t)$ is uniformly (strongly) continuous if and only if it is separately uniformly (strongly) continuous.

The proof of the following proposition is straight forward, and will be omitted.

Proposition 1. *Let $L(s,t)$ be a 2-parameter semigroup on the Banach space $X \overset{\alpha}{\otimes} Y$, such that*

$$\begin{aligned} L(s,0)\left(x\otimes y\right) &= \left(f(s)x\right)\otimes y \text{ for all } x\in X, y\in Y,\\ L(0,t)\left(x\otimes y\right) &= x\otimes\left(g(t)y\right) \text{ for all } x\in X, y\in Y, \end{aligned}$$

where f,g are any functions on X,Y respectively. Then

1. *$\left(f(s)\right)_{s\geq 0}$, and $\left(g(t)\right)_{t\geq 0}$ are one parameter semigroups on X,Y respectively.*
2. *$L(s,t)$ is uniformly (strongly) continuous if and only if each of the one parameter semigroups $L(s,0)$ and $L(0,t)$ is uniformly (strongly) continuous.*

It follows from the definition of a two-parameter semigroup [7], we observe that a T.P.S. $(T(s)\otimes S(t))_{s,t\geq 0}$ defines a two-parameter semigroup $(L(s,t))_{s,t\geq 0} = L(s,t) = T(s)\otimes S(t)$. Note that $L(s,t) = L(s,0)L(0,t)$, where $L(s,0) = T(s)\otimes S(0) = T(s)\otimes I$ and $L(0,t) = T(0)\otimes S(t) = I\otimes S(t)$.

3. The Infinitesimal Generator of a T.P.S.

Let $(T(s)\otimes S(t))_{s,t\geq 0}$ be a C_0 T.P.S. on $X \overset{\alpha}{\otimes} Y$ and A_1, A_2 be the infinitesimal generators of the one parameter C_0 semigroups $\left(\widehat{T}(s)\right)_{s\geq 0}, \left(\widehat{S}(t)\right)_{t\geq 0}$ on X,Y respectively, where $\left(\widehat{T}(s)\right)_{s\geq 0}, \left(\widehat{S}(t)\right)_{t\geq 0}$ are as in Theorem 1.

Remark 1. *Let us recall the followings.*

1. *Let X be a normed space and A be a linear operator, $A : \mathfrak{D}(T) \subseteq X \to X$. A subspace Z of the domain $\mathfrak{D}(A)$ is called a **core** for A if Z is dense in $\mathfrak{D}(A)$ for the graph norm $\|A\|_A := \|x\| + \|Ax\|$. [2]*
2. *A function $G : \mathfrak{R}^{+^2} \to X \overset{\alpha}{\otimes} Y$, is said to be differentiable at $(0,0)$ if there exists a linear transformation $\mathfrak{L} : \mathfrak{R}^{+^2} \to X \overset{\alpha}{\otimes} Y$ such that*

$$\lim_{(s,t)\to(0^+,0^+)} \frac{\|G(s,t) - G(0,0) - \mathfrak{L}\left((s,t)-(0,0)\right)\|}{\|(s,t)\|} = 0.$$

In other words,

$$G(s,t) - G(0,0) = \mathfrak{L}\ (s,t) + R(s,t),$$

where

$$\lim_{(s,t)\to(0^+,0^+)} \frac{\|R(s,t)\|}{\|(s,t)\|} = 0.$$

3. *The transformation* $\mathfrak{L}$ *above, if it exists, is unique, and it is called the derivative of* G *at* $(0,0)$.

4. *For a fixed* $z \in X \overset{\alpha}{\otimes} Y$*, if* $G(s,t)z = (T(\cdot) \otimes S(\cdot\cdot))z$*, then (2) becomes*

$$\left(T(s) \overset{\alpha}{\otimes} S(t)\right) z - z = \mathfrak{L}\ (s,t)z + R(s,t)z,$$

and (2) comes to

$$\lim_{(s,t)\to(0^+,0^+)} \frac{\|R(s,t)z\|}{\|(s,t)\|} = 0,$$

for all z *where (2) holds.*

5. *If it is shown that for any* z *satisfying (4), one has the same* $\mathfrak{L}(\cdot,\cdot\cdot)$ *in (4), then one can consider the derivative as the linear transformation from* $\mathfrak{R}^{+^2} \to \mathscr{L}\left(X \overset{\alpha}{\otimes} Y\right)$*, where* $\mathscr{L}\left(X \overset{\alpha}{\otimes} Y\right)$ *is the space of linear (not necessarily bounded) operators on* $X \overset{\alpha}{\otimes} Y$*, in the following sense:*

   For all z *such that (4) holds, there is a linear transformation* $\widehat{\mathfrak{L}} : \mathfrak{R}^{+^2} \to \mathscr{L}\left(X \overset{\alpha}{\otimes} Y\right)$*, where* $(s,t) \mapsto \widehat{\mathfrak{L}}(s,t)$ *such that* $\widehat{\mathfrak{L}}(s,t)z = \mathfrak{L}(s,t)z$.

6. *In case of item 5 holds, if moreover,* $\mathfrak{L}$ *is of the form* $(\mathfrak{L}_1, \mathfrak{L}_2)$ *then for any* $(s,t) \in \mathfrak{R}^{+^2}$ *the domain of* $\widehat{\mathfrak{L}}(s,t)$ *is* $\mathfrak{D}(s\mathfrak{L}_1 + t\mathfrak{L}_2)$*, the domain of* $s\mathfrak{L}_1 + t\mathfrak{L}_2$ *which is* $\mathfrak{D}(\mathfrak{L}_1) \cap \mathfrak{D}(\mathfrak{L}_2)$.

Definition 2. *Let* $(T(s) \otimes S(t))_{s,t\geq 0}$ *be a T.P.S. on* $X \overset{\alpha}{\otimes} Y$*. The* ***infinitesimal generator*** A *of* $(T(s) \otimes S(t))_{s,t\geq 0}$ *is defined as follows*

$$\begin{aligned}
\mathfrak{D}(A) &= \left\{ z \in X \overset{\alpha}{\otimes} Y : \left(T(s) \overset{\alpha}{\otimes} S(t)\right) z \text{ is differentiable at } (0,0) \right\}, \\
Az &= D\left(T(s) \overset{\alpha}{\otimes} S(t)\right) z_{|_{(s,t)=(0,0)}} \text{ for } z \in \mathfrak{D}(A),
\end{aligned}$$

where $\mathfrak{D}(A)$ *is the domain of* A*, and* $D\left(T(s) \overset{\alpha}{\otimes} S(t)\right) z_{|_{(s,t)=(0,0)}}$ *is the derivative of* $T(s) \overset{\alpha}{\otimes} S(t)z$ *as a function of two variables at* $(s,t) = (0,0)$.

Lemma 5. $(A_1 \otimes I)(x \otimes y) = \frac{\partial}{\partial s}\left((T(s) \otimes S(t))(x \otimes y)\right)_{|_{(s,t)=(0,0)}}$*, and* $(I \otimes A_2)(x \otimes y) = \frac{\partial}{\partial t}\left((T(s) \otimes S(t))(x \otimes y)\right)_{|_{(s,t)=(0,0)}}$ *for all* $x \in \mathfrak{D}(A_1)$*,* $y \in \mathfrak{D}(A_2)$*, where* A_1 *and* A_2 *are the infinitesimal generators of the coordinate semigroups respectively.*

Proof. Let $x \in \mathfrak{D}(A_1)$, $y \in \mathfrak{D}(A_2)$. Then

$$\begin{aligned}\frac{\partial}{\partial s}\left(T(s)\otimes S(t)(x\otimes y)\right)_{|(s,t)=(0,0)} &= \underset{h\to 0^+}{Lim}\frac{(T(h)\otimes S(0))(x\otimes y)-(T(0)\otimes S(0))(x\otimes y)}{h}\\ &= \underset{h\to 0^+}{Lim}\left(\left(\frac{T(h)\ -I}{h}x\right)\otimes y\right)\\ &= \left[\underset{h\to 0^+}{Lim}\left(\frac{T(h)-I}{h}x\right)\right]\otimes y=(A_1x)\otimes y.\end{aligned}$$

Similarly for $I\otimes A_2$.

Now, Let $\left(T(s)\overset{\alpha}{\otimes} I\right)_{s\geq 0}$, $(T(s))_{s\geq 0}$ be one parameter C_0 semigroups on the Banach spaces $X\overset{\alpha}{\otimes} Y$, X with infinitesimal generators A,A_1 respectively. Then, one can easily see:

a. $\mathfrak{D}(A_1)\otimes Y$ is a subspace of $\mathfrak{D}(A)$.

b. $\mathfrak{D}(A_1)\otimes Y$ is dense in $X\overset{\alpha}{\otimes} Y$.

c. $\mathfrak{D}(A_1)\otimes Y$ is invariant under $T(s)\overset{\alpha}{\otimes} I$.

d. $\mathfrak{D}(A_1)\otimes Y$ is a core for A.

Lemma 6. *Suppose that $(T(s))_{s\geq 0}$, $(S(t))_{t\geq 0}$ are one parameter C_0 semigroups on the Banach spaces X,Y with infinitesimal generators A_1, A_2 respectively. Then $\overline{A_1\otimes I}$ and $\overline{I\otimes A_2}$ are the infinitesimal generators of the one parameter C_0 semigroups $\left(T(s)\overset{\alpha}{\otimes} I\right)_{s\geq 0}$, $\left(I\overset{\alpha}{\otimes} S(t)\right)_{t\geq 0}$ respectively on $X\overset{\alpha}{\otimes} Y$.*

Proof. First let $z=x\otimes y$, for some $x\otimes y\in\mathfrak{D}(A_1)\otimes Y$. If A is the infinitesimal generator of $\left(T(s)\overset{\alpha}{\otimes} I\right)_{s\geq 0}$, then $Az=\left(A_1\otimes I\right)z$. This means that $A_{|_{\mathfrak{D}(A_1)\otimes Y}}=A_1\otimes I$. In other words, A is an extension of $A_1\otimes I$ from the subspace $\mathfrak{D}(A_1)\otimes Y$ to the domain $\mathfrak{D}(A)$ of A. Being the infinitesimal generator of a one parameter C_0 semigroup, A is closed [9]. Thus A is a closed extension of $A_1\otimes I$. But $A_1\otimes I$ is closable [6]. Since the closure of an operator is its smallest closed extension, then $A\supset\overline{A_1\otimes I}\supset A_1\otimes I$. On the other hand, by [6], $\overline{A_1\otimes I}$ is the maximal extension of $A_1\otimes I$. Therefore $A\subset\overline{A_1\otimes I}$. Hence $A=\overline{A_1\otimes I}$. Similarly, one can show that $\overline{I\otimes A_2}$ generates $\left(I\overset{\alpha}{\otimes} S(t)\right)_{t\geq 0}$.

Theorem 2. *The infinitesimal generator of a C_0 T.P.S. $(T(s)\otimes S(t))_{s,t\geq 0}$ is the linear transformation $\mathfrak{L}:\mathfrak{R}^{+^2}\to\mathscr{L}\left(X\overset{\alpha}{\otimes} Y\right)$, $(a,b)\longmapsto\left(\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)(a,b)\right)=\left(a\overline{A_1\otimes I}+b\overline{I\otimes A_2}\right)$, where A_1, A_2 are the infinitesimal generators of the one parameter C_0 semigroups $\left(\widehat{T}(s)\right)_{s\geq 0}$, $\left(\widehat{S}(t)\right)_{t\geq 0}$ respectively.*

Proof. First, we should notice that $\mathfrak{L}(a,b)\left(x\otimes y\right) = \left(aA_1\otimes I, bI\otimes A_2\right)\left(x\otimes y\right)$ for all $x\in\mathfrak{D}\left(A_1\right), y\in\mathfrak{D}\left(A_2\right)$. Now, Let $\left(T\left(s\right)\otimes S\left(t\right)\right)_{s,t\geq 0}$ be a C_0 T.P.S., A its infinitesimal generator and let $z\in\mathfrak{D}\left(A\right)$. That is, $z\in X\overset{\alpha}{\otimes} Y$ such that $\left(T\left(s\right)\overset{\alpha}{\otimes} S\left(t\right)\right)z$ is differentiable at $(0,0)$. Thus $D\left(\left(T\left(s\right)\overset{\alpha}{\otimes} S\left(t\right)\right)z\right)_{|(s,t)=(0,0)}$ exists. In other words, there exist z_1, z_2 in $X\overset{\alpha}{\otimes} Y$ such that $\lim\limits_{(s,t)\to(0^+,0^+)}\frac{\left\|\{(T(s)\otimes S(t))-T(0)\otimes S(0)\}z-sz_1-tz_2\right\|}{\|(s,t)\|}=0$. In particular, choose (s,t) to be $(s,0)$ where $s\to 0^+$. Then $\lim\limits_{s\to 0^+}\frac{\left\|\{(T(s)\otimes S(0))-T(0)\otimes S(0)\}z-sz_1\right\|}{s}=0$. Therefore,

$$\lim_{s\to 0^+}\frac{\left\|\left\{\left(\widehat{T}\left(s\right)\otimes I\right)-I\otimes I\right\}z-sz_1\right\|}{s}=\lim_{s\to 0^+}\left\|\frac{\left\{\left(\widehat{T}\left(s\right)\otimes I\right)-I\otimes I\right\}}{s}z-z_1\right\|=0.$$

From Lemma 6, $z_1 = \left(\overline{A_1\otimes I}\right)z$, where A_1 generates the C_0 semigroup $\left(\widehat{T}\left(s\right)\right)_{s\geq 0}$. Similarly, one can show that $z_2 = \left(\overline{I\otimes A_2}\right)z$. Since z was arbitrarily chosen in $\mathfrak{D}(A)$, this shows that $\mathfrak{D}(A)$ is a subspace of $X\overset{\alpha}{\otimes} Y$. Further, $\mathfrak{D}(A)\subseteq\mathfrak{D}\left(\overline{A_1\otimes I}\right)\cap\mathfrak{D}\left(\overline{I\otimes A_2}\right)$. Now let $z\in\mathfrak{D}\left(\overline{A_1\otimes I}\right)\cap\mathfrak{D}\left(\overline{I\otimes A_2}\right)$ and $s,t>0$. Set

$$J(s,t)=\left(T\left(s\right)\otimes S\left(t\right)\right)-\left(T\left(0\right)\otimes S\left(0\right)\right)-\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)\begin{pmatrix}s\\ t\end{pmatrix}.$$

Then

$$\begin{aligned}
\|J\left(s,t\right)z\| &= \left\|\begin{array}{c}\left(T\left(s\right)\overset{\alpha}{\otimes}S\left(t\right)\right)(z)-(z)\\ -\left(s\overline{A_1\otimes I}\right)(z)-\left(t\overline{I\otimes A_2}\right)(z)\end{array}\right\| \\
&\leq \left\|\begin{array}{c}\left(\widehat{T}\left(s\right)\overset{\alpha}{\otimes}I\right)\left(I\overset{\alpha}{\otimes}\widehat{S}\left(t\right)\right)(z)\\ -\left(\widehat{T}\left(s\right)\overset{\alpha}{\otimes}I\right)(z)-\left(t\overline{I\otimes A_2}\right)(z)\end{array}\right\| \\
&\quad+\left\|\left(\widehat{T}\left(s\right)\overset{\alpha}{\otimes}I\right)(z)-(z)-\left(s\overline{A_1\otimes I}\right)(z)\right\| \\
&\leq t\left\|\begin{array}{c}\left(\widehat{T}\left(s\right)\overset{\alpha}{\otimes}I\right)\left(\frac{\left(I\overset{\alpha}{\otimes}\widehat{S}(t)\right)(z)-\left(I\overset{\alpha}{\otimes}I\right)(z)}{t}\right)\\ -\left(\overline{I\otimes A_2}\right)(z)\end{array}\right\| \\
&\quad+s\left\|\left(\frac{\widehat{T}\left(s\right)\overset{\alpha}{\otimes}I-I\overset{\alpha}{\otimes}I}{s}\right)(z)-\left(\overline{A_1\otimes I}\right)(z)\right\|.
\end{aligned}$$

Divide both sides by $\|(s,t)\|=\sqrt{s^2+t^2}$ to get

$$\frac{\|J\left(s,t\right)z\|}{\|(s,t)\|}\ \leq\ \psi_{s,t}\left\|\left(\widehat{T}\left(s\right)\overset{\alpha}{\otimes}I\right)\left[\left\{\frac{1}{t}\left(I\overset{\alpha}{\otimes}\widehat{S}\left(t\right)-\left(I\overset{\alpha}{\otimes}I\right)\right)-\left(\overline{I\otimes A_2}\right)\right\}(z)\right]\right\|$$

$$+\phi_{s,t}\left\|\left\{\frac{1}{s}\left(\widehat{T}(s)\overset{\alpha}{\otimes}I-I\overset{\alpha}{\otimes}I\right)-\left(\overline{A_1\otimes I}\right)\right\}(z)\right\|,$$

where $\psi_{s,t}=\frac{t}{\sqrt{s^2+t^2}}$, $\phi_{s,t}=\frac{s}{\sqrt{s^2+t^2}}$. But $\psi_{s,t}\leq 1$, $\phi_{s,t}\leq 1$ for all $s,t>0$. Therefore,

$$\begin{aligned}\frac{\|J(s,t)z\|}{\|(s,t)\|} \leq & \left\|\left(\widehat{T}(s)\overset{\alpha}{\otimes}I\right)\left[\left\{\frac{1}{t}\left(I\overset{\alpha}{\otimes}\widehat{S}(t)-\left(I\overset{\alpha}{\otimes}I\right)\right)-\left(\overline{I\otimes A_2}\right)\right\}(z)\right]\right\| \\ & +\left\|\left\{\frac{1}{s}\left(\widehat{T}(s)\overset{\alpha}{\otimes}I-I\overset{\alpha}{\otimes}I\right)-\left(\overline{A_1\otimes I}\right)\right\}(z)\right\|,\end{aligned}$$

As $(s,t)\to\left(0^+,0^+\right)$, the second norm in the right hand side converges to zero, whereas the first norm converges to zero by Lemma 6, the strong continuity of $\left(\widehat{T}(s)\overset{\alpha}{\otimes}I\right)_{s\geq 0}$, and the uniform boundedness principle. Therefore, $\frac{\|J(s,t)z\|}{\|(s,t)\|}\to 0$ as $(s,t)\to(0^+,0^+)$. Now, define $\mathfrak{L}:\mathfrak{R}^{+^2}\to\mathscr{L}(X\overset{\alpha}{\otimes}Y)$ by $(\mathfrak{L}(s,t))z=\left(\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)\binom{s}{t}\right)z$ for every $z\in\mathfrak{D}\left(\left(\overline{A_1\otimes I}\right)\binom{s}{t}\right)\cap\mathfrak{D}\left(\left(\overline{I\otimes A_2}\right)\binom{s}{t}\right)=\mathfrak{D}\left(\left(\overline{A_1\otimes I}\right)\right)\cap\mathfrak{D}\left(\left(\overline{I\otimes A_2}\right)\right)$. Then
$D(T(s)\otimes S(t))\Big|_{(s,t)=(0,0)}=(\overline{A_1\otimes I},\overline{I\otimes A_2})$, as a linear transformation from $\mathfrak{R}^{+^2}$ to $\mathscr{L}(X\overset{\alpha}{\otimes}Y)$ is the derivative of the C_0 T.P.S.$(T(s)\otimes S(t))_{s,t\geq 0}$ at $(0,0)$. Hence the linear transformation $\mathfrak{L}=\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)$ is the infinitesimal generator of the C_0 T.P.S. $(T(s)\otimes S(t))_{s,t\geq 0}$.

Remark 2. *One can show that for any nonzero* $(a,b)\in\mathfrak{R}^{+^2}$, $\mathfrak{D}\left(A_1\right)\otimes\mathfrak{D}\left(A_2\right)$ *is a core for* $\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)\binom{a}{b}$.

1. *In general, if A,B are closable, or even, closed linear operators on the Banach space X, then $A+B$ need not be closed. But, Theorem 1.1 in [5] ensures that $aA_1\otimes I+b,I\otimes A_2$, $a,b\neq 0$ is closable. Moreover, its closure is $a\overline{A_1\otimes I}+b\overline{I\otimes A_2}$.*

2. *Since the restriction of $\mathfrak{L}(a,b)$ to $X\otimes Y$ is defined by*

$$\mathfrak{L}(a,b)\left(x\otimes y\right)=\left(aA_1\otimes I+bI\otimes A_2\right)\left(x\otimes y\right)=\left(A_1\otimes I+I\otimes A_2\right)\begin{pmatrix}a\\b\end{pmatrix}\left(x\otimes y\right),$$

for all $x\in\mathfrak{D}\left(A_1\right), y\in\mathfrak{D}\left(A_2\right)$, and since $X\otimes Y$ is dense in $X\overset{\alpha}{\otimes}Y$, it is enough to study $T(s)\otimes S(t)$ instead of its extension $T(s)\overset{\alpha}{\otimes}S(t)$, and $\left(A_1\otimes I+I\otimes A_2\right)\binom{a}{b}$ instead of its closure $\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)\binom{a}{b}$.

From now on, the infinitesimal generator of $(T(s)\otimes S(t))_{s,t\geq 0}$ will be denoted by $\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)$.

Lemma 7. *If $(T(s)\otimes I)_{s\geq 0}$ is a C_0 semigroup on $X\overset{\alpha}{\otimes}Y$ with infinitesimal generator $\overline{A_1\otimes I}$ where A_1 is a linear operator on X, then $(T(s))_{s\geq 0}$ is a C_0 semigroup on X with infinitesimal generator A_1.*

The proof follows from general functional analysis arguments and will be omitted.

Lemma 8. *If $(T(s)\otimes S(t))_{s,t\geq o}$ is a C_0 T.P.S. on $X \overset{\alpha}{\otimes} Y$, then for every $(a,b)\in\Re^{+^2}$, the family $(T(as)\otimes S(bs))_{s\geq o}$ is a one parameter C_0 semigroup on the Banach space $X\overset{\alpha}{\otimes} Y$.*

Proof. Let $Q(h)=T(ah)\otimes S(bh)$. Then $Q(0)=I$, where I is the identity on $X\otimes Y$, and

$$\begin{aligned} Q(h_1+h_2) &= \left(T\left(ah_1\right)\otimes S\left(bh_1\right)\right)\left(T\left(ah_2\right)\otimes S\left(bh_2\right)\right) \\ &= Q(h_1)Q(h_2). \end{aligned}$$

Put $bh=t$, $ah=s$. Since

$$h\to 0^+ \text{ if and only if } s=ah\to 0^+ \text{ if and only if } t=bh\to 0^+,$$

then the function $Q(h)=T(s)\otimes S(t)$ converges to I as $h\to 0^+$ in the strong operator topology.

Lemma 9. *Let $0\neq(a,b)\in\Re^{+^2}$. Then the infinitesimal generator of the one parameter C_0 semigroup $(T(as)\otimes S(bs))_{s\geq o}$ is the linear operator*

$$a\overline{A_1\otimes I}+b,\overline{I\otimes A_2}.$$

Proof. The generator of the one parameter semigroup $(T(as)\otimes S(bs))_{s\geq o}$ is given by

$$\begin{aligned} \frac{d^+}{ds}\left(T(as)\otimes S(bs)\right)_{|_{s=0}} &= \frac{d^+}{ds}\left(T(as)\otimes I\right)\left(I\otimes S(bs)\right)_{|_{s=0}} \\ &= \frac{d^+}{ds}\left(\widehat{T}(as)\otimes I\right)\left(I\otimes\widehat{S}(bs)\right)_{|_{s=0}}. \end{aligned}$$

Being the derivative of a function of one variable at $s=0$, the derivative is

$$\begin{aligned} &\left(\frac{d^+}{ds}\left(\widehat{T}(as)\otimes I\right)_{|_{s=0}}\right)\left(I\otimes\widehat{S}(0)\right)+\left(\widehat{T}(0)\otimes I\right)\left(\frac{d^+}{ds}\left(I\otimes\widehat{S}(bs)\right)_{|_{s=0}}\right) \\ =\ & a\left(\frac{d^+}{d(as)}\left(\widehat{T}(as)\otimes I\right)_{|_{s=0}}\right)(I\otimes I)+(I\otimes I)\,b\left(\frac{d^+}{d(as)}\left(I\otimes\widehat{S}(bs)\right)_{|_{s=0}}\right) \end{aligned} \tag{1}$$

and this is just $a\overline{A_1\otimes I}+b,\overline{I\otimes A_2}$.

Corollary 1. *The linear operator $a\overline{A_1\otimes I}+b,\overline{I\otimes A_2}$ is closed and densely defined.*

Corollary 2. *For every $0\neq(a,b)\in\Re^{+^2}$ the linear operator $aA_1\otimes I+b,I\otimes A_2$ is closable, densely defined and its closure is $a\overline{A_1\otimes I}+b,\overline{I\otimes A_2}$.*

Proof. Being closable is proved in [6]. Since

$$\begin{aligned} \mathfrak{D}\left(aA_1\otimes I+b,I\otimes A_2\right) &= \left(\mathfrak{D}\left(A_1\right)\otimes Y\right)\cap\left(X\otimes\mathfrak{D}\left(,A_2\right)\right) \\ &= \mathfrak{D}\left(A_1\right)\otimes\mathfrak{D}\left(,A_2\right), \end{aligned}$$

and since A_1,A_2 are densely defined in X, Y respectively, one can show that the subspace $\mathfrak{D}\left(A_1\right) \otimes \mathfrak{D}\left(,A_2\right)$ is dense in $X \otimes Y$, which is in turn dense in $X \overset{\alpha}{\otimes} Y$. Thus $aA_1 \otimes I + b, I \otimes A_2$ is densely defined on $X \overset{\alpha}{\otimes} Y$. So

$$\left(a\overline{A_1 \otimes I} + b, \overline{I \otimes A_2}\right)\left(x \otimes y\right) = \left(aA_1 \otimes I + b, I \otimes A_2\right)\left(x \otimes y\right),$$

for all $x \otimes y \in \mathfrak{D}\left(A_1\right) \otimes \mathfrak{D}\left(,A_2\right)$. *That is*

$$\left(a\overline{A_1 \otimes I} + b, \overline{I \otimes A_2}\right)_{|\left(X \otimes Y\right) \cap \mathfrak{D}\left(aA_1 \otimes I + b, I \otimes A_2\right)} = aA_1 \otimes I + b, I \otimes A_2.$$

Therefore, $B = a\overline{A_1 \otimes I} + b, \overline{I \otimes A_2}$ is an extension of $A = aA_1 \otimes I + b, I \otimes A_2$ from the subspace $\mathfrak{D}\left(A_1\right) \otimes \mathfrak{D}\left(,A_2\right)$ to $\mathfrak{D}(B)$. From Corollary 1, B is a closed extension of A. Since A is closable, and the closure is the smallest closed extension, $\overline{A} \subset B$. On the other hand, A is closable, and the closure of a closable operator is its maximal extension. Thus $B \subset \overline{A}$. Hence $\overline{A} = B$ completes the proof of the corollary.

Corollary 3. *Let* $(T(s) \otimes S(t))_{s,t \geq 0}$ *be a* C_0 *T.P.S. on* $X \overset{\alpha}{\otimes} Y$*, with infinitesimal generator* $\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)$ *and* $0 \neq (a,b) \in \mathfrak{R}^{+^2}$. *Then the infinitesimal generator of the one parameter* C_0 *semigroup* $(T(as) \otimes S(bs))_{s \geq o}$ *is the linear operator* $a\overline{A_1 \otimes I} + b, \overline{I \otimes A_2} = \overline{a\left(A_1 \otimes I\right) + b\left(I \otimes A_2\right)}$.

As a consequence of Corollary 3, we obtain Nagel's result [1, Proposition, Sec. 3.7].

Corollary 4. *The infinitesimal generator of the one parameter* C_0 *T.P.S.* $(T(t) \otimes S(t))_{t \geq 0}$, *is* $\overline{(A_1 \otimes I) + (I \otimes A_2)}$ *defined on the core* $\mathfrak{D}\left(A_1\right) \otimes \mathfrak{D}\left(A_2\right)$ *of the generator.*

Proof. From Corollary 3 the operator

$$a\overline{\left(A_1 \otimes I\right)} + b, \overline{\left(I \otimes A_2\right)} = \overline{a\left(A_1 \otimes I\right) + b\left(I \otimes A_2\right)}$$

generates $(T(at) \otimes S(bt))_{t \geq 0}$. As a particular case take $(a,b) = (1,1)$. Then

$$\overline{\left(A_1 \otimes I\right) + \left(I \otimes A_2\right)}$$

is the infinitesimal generator of the one parameter C_0 semigroup $(T(t) \otimes S(t))_{t \geq 0}$. But $\left(A_1 \otimes I\right) + \left(I \otimes A_2\right)$ is defined on $\mathfrak{D}\left(A_1\right) \otimes \mathfrak{D}\left(A_2\right)$, which is a core for the infinitesimal generator $\overline{\left(A_1 \otimes I\right) + \left(I \otimes A_2\right)}$

Definition 3. *Let* $(T(s))_{s \geq 0}$ *and* $(S(t))_{t \geq 0}$ *be one parameter* C_0 *semigroups on the Banach spaces* X *and* Y *respectively. For* $u = (a,b) \in \mathfrak{R}^{+^2}$, *the **almost directional derivative*** $\mathfrak{a}.D_u$ *of* $T(s) \otimes S(t)$ *at* $(0,0)$ *is defined by*

$$\mathfrak{D}\left(\mathfrak{a}.D_u\left(T(s) \overset{\alpha}{\otimes} S(t)\right)_{|(s,t)=(0,0)}\right) = \left\{ z \in X \overset{\alpha}{\otimes} Y : \lim_{h \to 0^+} \frac{T(ah) \overset{\alpha}{\otimes} S(bh) z - z}{h} \text{ exists} \right\}$$

and

$$\left(\mathfrak{a}.D_u\left(T(s)\overset{\alpha}{\otimes}S(t)\right)\Big|_{(s,t)=(0,0)}\right)z=\lim_{h\to 0^+}\frac{(T(ah)\overset{\alpha}{\otimes}S(,bh))z-z}{h}$$

It follows from the definition that the almost directional derivative $\mathfrak{a}.D_u\left(T(s)\overset{\alpha}{\otimes}S(t)\right)\Big|_{(s,t)=(0,0)}$ is the infinitesimal generator of the one parameter C_0 semigroup $(T(at)\otimes S(,bt))_{t\geq 0}$. Further, for $u=(a,b)\in\mathfrak{R}^{+^2}$, $\mathfrak{a}.D_u(T(s)\overset{\alpha}{\otimes}S(t))\Big|_{(s,t)=(0,0)}=\left(a\overline{A_1\otimes I}+b\overline{I\otimes A_2}\right)=\overline{a\left(A_1\otimes I\right)+b\left(I\otimes A_2\right)}$. Also, since $\nabla\left(T(s)\otimes S(t)\right)=\frac{\partial}{\partial s}T(s)\otimes S(t)i+\frac{\partial}{\partial t}T(s)\otimes S(t)j$, then for $u=(a,b)\in\mathfrak{R}^{+^2}$

$$\mathfrak{a}.D_u(T(s)\overset{\alpha}{\otimes}S(t))\Big|_{(s,t)=(0,0)}=\nabla T(s)\otimes S(t)\Big|_{(s,t)=(0,0)}.u.$$

Theorem 3. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be one parameter C_0 semigroups on Banach spaces X and Y with infinitesimal generators A_1 and A_2 respectively. Then*

$$D\left(T(s)\otimes S(t)\right)\begin{pmatrix}a\\b\end{pmatrix}\left(x\otimes y\right)=\left(A_1\otimes I,I\otimes A_2\right)\begin{pmatrix}a\\b\end{pmatrix}\left(T(s)\otimes S(t)\right)\left(x\otimes y\right)\tag{2}$$

for all $(a,b)\in\mathfrak{R}^{+^2}$, all $x\in\mathfrak{D}\left(A_1\right)$ and $y\in\mathfrak{D}\left(A_2\right)$.

Proof. Let $(a,b)\in\mathfrak{R}^{+^2}$, $x\in\mathfrak{D}\left(A_1\right)$, and $y\in\mathfrak{D}\left(A_2\right)$. Then $D(T(s)\otimes S(t))$ as a function of two variables is given by

$$\begin{aligned}&(D(T(s)\otimes S(t)))\begin{pmatrix}a\\b\end{pmatrix}\left(x\otimes y\right)\\=&\left(\frac{\partial}{\partial s}\left(T(s)\otimes S(t)\right),\frac{\partial}{\partial t}\left(T(s)\otimes S(t)\right)\right)\begin{pmatrix}a\\b\end{pmatrix}\left(x\otimes y\right)\\=&\left(a\frac{\partial}{\partial s}\left(T(s)\otimes S(t)\right)+b\frac{\partial}{\partial t}\left(T(s)\otimes S(t)\right)\right)\left(x\otimes y\right)\\=&\left(a\frac{d\left(T(s)\otimes I\right)}{ds}\left(I\otimes S(t)\right)+b\frac{d\left(I\otimes S(t)\right)}{dt}\left(T(s)\otimes I\right)\right)\left(x\otimes y\right).\end{aligned}$$

Then, by Lemma 2-c, Theorem 2 and Lemma 6 we have

$$\begin{aligned}&(D(T(s)\otimes S(t)))\begin{pmatrix}a\\b\end{pmatrix}\left(x\otimes y\right)\\=&a\left[\left(\overline{A_1\otimes I}\right)\left(T(s)\otimes I\right)\right]\left(x\otimes S(t)y\right)\\&+b\left[\left(,\overline{I\otimes A_2}\right)\left(I\otimes S(t)\right)\right]\left(T(s)x\otimes y\right)\\=&a\left(A_1\otimes I\right)\left(T(s)x\otimes S(t)y\right)+b\left(,I\otimes A_2\right)\left(T(s)x\otimes S(t)y\right)\\=&\left(A_1\otimes I,I\otimes A_2\right)\begin{pmatrix}a\\b\end{pmatrix}\left(T(s)\otimes S(t)\right)\left(x\otimes y\right).\end{aligned}$$

which is (2).

As in the classical case, one can show the existence of constants $\omega\geq 0$ and $M\geq 1$ such that $\left\|T(s)\overset{\alpha}{\otimes}S(t)\right\|\leq Me^{\omega(t+s)}$ for $s,t\geq 0$.

4. The Hille-YosidaTheorem for T.P.S'.

Definition 4. *Let X and Y be Banach spaces and A be a linear transformation that maps $\Re^{+^2}$ into $\mathscr{L}\left(X \overset{\alpha}{\otimes} Y\right)$ given by $A = (\overline{A_1 \otimes I}, \overline{I \otimes A_2})$, where $A_1, , A_2$ are linear operators on X and Y respectively, satisfying:*

a. For any $(a,b) \in \Re^{+^2}$

$$A\begin{pmatrix} a \\ b \end{pmatrix}(x \otimes y) = (aA_1 \otimes I, +bI \otimes A_2)(x \otimes y),$$

$x \in \mathfrak{D}(A_1)$, $y \in \mathfrak{D}(A_2)$.

b. A is the infinitesimal generator of a C_0 T.P.S. $(T(s) \otimes S(t))_{s,t \geq 0}$.

Then we call the linear transformation $B = (A_1 \otimes I, I \otimes A_2)$ the ***pseudo-infinitesimal generator*** *of $(T(s) \otimes S(t))_{s,t \geq 0}$.*

We should remark that uniqueness of the closure of a linear operator, and uniqueness of the infinitesimal generator of a T.P.S. imply that the pseudo-infinitesimal generator of a T.P.S. is unique.

Now, we are ready to prove one of the main results of this section (A Hille-Yosida Theorem for T.P.S.').

Theorem 4. *Let X, Y be Banach spaces. A linear transformation A from $\Re^{+^2}$ into $\mathscr{L}\left(X \overset{\alpha}{\otimes} Y\right)$ is the pseudo-infinitesimal generator of a C_0 T.P.S. $(T(s) \otimes S(t))_{s,t \geq 0}$ on $X \overset{\alpha}{\otimes} Y$ satisfying $\|T(s) \otimes S(t)\| \leq Me^{\omega(s+t)}$, for all $s, t \geq 0$, for some constants $M \geq 1, \omega \geq 0$, if and only if the followings hold*

(i) $\left(A\binom{0}{1}\right)(x \otimes y) = (A_1 x) \otimes y$, and $\left(A\binom{1}{0}\right)(x \otimes y) = x \otimes (A_2 y)$, $x \in \mathfrak{D}(A_1)$, $y \in \mathfrak{D}(A_2)$ for some linear operators A_1, A_2 (not necessarily bounded) on X, Y respectively.

(ii) A_1, A_2 in part (i) are closed and densely defined on X, Y respectively.

(iii) $\rho(A_i)$ contains (ω, ∞), $i = 1, 2$, and for every $\lambda > \omega$

$$\left\| \left[R_\lambda(A_i)\right]^n \right\| \leq \frac{M_i}{(\lambda - \omega)^n}, n = 1, 2, 3, \ldots, \text{ for some } M_i \geq 1, i = 1, 2.$$

Proof. Let the conditions (i), (ii) and (iii) hold. From (ii), (iii) and Theorem 1.7 in [1] A_1, A_2 are the infinitesimal generators of one parameter C_0 semigroups say $(T(s))_{s \geq 0}$, $(S(t))_{t \geq 0}$ on X, Y respectively, satisfying

$$\|T(s)\| \leq M_1 e^{\omega s} \text{ for all } s \geq 0, \text{ and } \|S(t)\| \leq M_2 e^{\omega t} \text{ for all } t \geq 0.$$

By Theorem 1 $(T(s) \otimes S(t))_{s,t \geq 0}$ is a C_0 T.P.S. on $X \overset{\alpha}{\otimes} Y$ satisfying

$$\|T(s) \otimes S(t)\| \leq \|T(s)\| \, \|S(t)\|$$

$$\begin{aligned} &\leq M_1 M_2 e^{\omega(s+t)} \\ &= M e^{\omega(s+t)}, \text{ for all } s,t \geq 0. \end{aligned}$$

By Theorem 2, the transformation $\left(A_1 \otimes I, I \otimes A_2\right)$ is the pseudo-infinitesimal generator of $T(s) \otimes S(t)$. Let $(a,b) \in \Re^{+^2}$, $x \in \mathfrak{D}\left(A_1\right)$, $y \in \mathfrak{D}\left(A_2\right)$. Then

$$\left(\left(A_1 \otimes I, I \otimes A_2\right)\begin{pmatrix} a \\ b \end{pmatrix}\right)(x \otimes y) = \left(aA_1 \otimes I + bI \otimes A_2\right)(x \otimes y),$$

which is by (i),

$$\left(aA\begin{pmatrix} 0 \\ 1 \end{pmatrix} + bA\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)(x \otimes y) = \left(A\begin{pmatrix} a \\ b \end{pmatrix}\right)(x \otimes y).$$

Therefore $A\binom{a}{b}$ coincides with $\left(A_1 \otimes I, I \otimes A_2\right)\binom{a}{b}$ on $\mathfrak{D}\left(A_1\right) \otimes \mathfrak{D}\left(A_2\right)$ for every $(a,b) \in \Re^{+^2}$, thus their closures coincide.

But the transformation mapping $(a,b) \in \Re^{+^2}$ into $\overline{A\binom{a}{b}} = \overline{\left(A_1 \otimes I, I \otimes A_2\right)\binom{a}{b}}$ is the infinitesimal generator of $(T(s) \otimes S(t))_{s,t \geq 0}$ (See Theorem 2, and Corollary 2). In other words, $A\binom{a}{b}$ is the pseudo-infinitesimal generator of $(T(s) \otimes S(t))$.

Conversely, let A be as in the statement. Since $T(s) \otimes S(t)$ is a C_0 T.P.S., then by Theorem 1 there exist unique $\beta \neq 0$, and unique one parameter C_0 semigroups $\left(\widehat{T}(s)\right)_{s \geq 0}$, $\left(\widehat{S}(t)\right)_{t \geq 0}$ on X, Y respectively, such that (1) holds. Let A_1, A_2 be their generators. Then by Theorem 2, A_1, A_2 satisfy (i) and (ii). By Theorem 2 and Corollary 2, the transformation

$$(a,b) \mapsto \overline{\left(A_1 \otimes I, I \otimes A_2\right)\begin{pmatrix} a \\ b \end{pmatrix}} = \left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)\begin{pmatrix} a \\ b \end{pmatrix}$$

is the infinitesimal generator of $\widehat{T}(s) \otimes \widehat{S}(t)$. But $\widehat{T}(s) \otimes \widehat{S}(t) = T(s) \otimes S(t)$. Thus $\left(A_1 \otimes I, I \otimes A_2\right)$ is the pseudo-infinitesimal generator of $T(s) \otimes S(t)$.

Uniqueness of the pseudo-infinitesimal generator, implies that the linear transformation $\left(A_1 \otimes I, I \otimes A_2\right) = A$. That is $A\binom{a}{b} = \left(A_1 \otimes I, I \otimes A_2\right)\binom{a}{b}$ for all (a,b) in $\Re^{+^2}$. In particular, for $(a,b) = (0,1)$, and $(a,b) = (1,0)$. Hence (i) is fulfilled.

Theorem 5. *Let X, Y be Banach spaces, and $(T(s) \otimes S(t))_{s,t \geq 0}$ be a C_0 T.P.S. on the Banach space $X \overset{\alpha}{\otimes} Y$ with infinitesimal generator $A = (\overline{A_1 \otimes I}, \overline{I \otimes A_2})$. If $\lambda \in \rho\left((A_1 \otimes I, I \otimes A_2)\binom{a}{b}\right)$, where $(a,b) \in \Re^{+^2}$, and $\lambda > (a+b)\max_{i=1,2}\left(\omega(A_i)\right)$, where $0 < \omega(A_i) \in \rho(A_i)$, for $i = 1,2$, then*

$$\left(R_\lambda\left((\overline{A_1 \otimes I},, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right)\right)(x \otimes y) = \int_0^\infty e^{-\lambda t}\left(T(at) \otimes S(bt)\right)(x \otimes y)\, dt. \quad (3)$$

Proof. Let $x \in X$, $y \in Y$, $(a,b) \in \Re^{+^2}$, and λ be as given. Define

$$R(\lambda)(x \otimes y) = \int_0^\infty e^{-\lambda t}\left(T(at) \otimes S(bt)\right)(x \otimes y)\, dt.$$

Since the map $t \mapsto (T(at) \otimes S(bt))\left(x \otimes y\right)$ is continuous and $\lambda > (a+b)\max\limits_{i=1,2}\left(\omega(A_i)\right)$, the integral exists as an improper Riemann integral and defines a bounded linear operator on $X \otimes Y$. Further, for $h > 0$

$$\frac{T\,(ah) \otimes S\,(bh) - I \otimes I}{h} R(\lambda)\left(x \otimes y\right)$$

$$= \frac{1}{h}\int\limits_0^\infty e^{-\lambda t}\left[(T(a(t+h) \otimes S\,(b\,(t+h))\left(x \otimes y\right) - (T\,(at) \otimes S\,(bt))\left(x \otimes y\right)\right] dt$$

$$= \frac{1}{h}\left(\begin{array}{l} \int\limits_h^\infty e^{-\lambda(t-h)}\,(T\,(at) \otimes S\,(bt))\left(x \otimes y\right) dt \\ -\int\limits_0^\infty e^{-\lambda t}\,(T\,(at) \otimes S\,(bt))\left(x \otimes y,\right) dt \end{array}\right)$$

$$= \frac{e^{\lambda h}}{h}\int\limits_h^\infty e^{-\lambda t}\,(T\,(at) \otimes S\,(bt))\left(x \otimes y\right) dt - \frac{1}{h}\int\limits_0^\infty e^{-\lambda t}\,(T\,(at) \otimes S\,(bt))\left(x \otimes y,\right) dt$$

$$= \frac{e^{\lambda h}-1}{h}\int\limits_0^\infty e^{-\lambda t}\,(T\,(at) \otimes S\,(bt))\left(x \otimes y\right) dt - \frac{e^{\lambda h}}{h}\int\limits_0^h e^{-\lambda t}\,(T\,(at) \otimes S\,(bt))\left(x \otimes y\right) dt.$$

Taking the limit of both sides as $h \to 0^+$ yields

$$\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right)\left(R(\lambda)\left(x \otimes y\right)\right) = \lambda R(\lambda)\left(x \otimes y\right) - \left(x \otimes y\right).$$

This implies that

$$R(\lambda)\left(x \otimes y\right) \in \mathfrak{D}\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right) \text{ for all } x \otimes y \in X \otimes Y,$$

and

$$\left(\lambda I \otimes I - (\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right) R(\lambda) = I \otimes I \text{ on } X \otimes Y.$$

Now, for $x \otimes y \in \mathfrak{D}\left((A_1 \otimes I, I \otimes A_2)\binom{a}{b}\right) \subseteq \mathfrak{D}\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\binom{a}{b}\right)$ we have

$$R(\lambda)\left[\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right)\left(x \otimes y\right)\right]$$

$$= \int\limits_0^\infty e^{-\lambda t}\,(T\,(at) \otimes S\,(bt))\left[\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right)\left(x \otimes y,\right)\right] dt$$

$$= \int\limits_0^\infty e^{-\lambda t}\left(\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)\begin{pmatrix} a \\ b \end{pmatrix}\right)\left[(T\,(at) \otimes S\,(bt))\left(x \otimes y\right)\right] dt,$$

by Theorem 3. Since $(\overline{A_1 \otimes I}, \overline{I \otimes A_2})\binom{a}{b}$ is closed by Corollary 2, it follows that the right-hand side of (3) is

$$\left(\left(\overline{A_1 \otimes I},, \overline{I \otimes A_2}\right)\begin{pmatrix} a \\ b \end{pmatrix}\right)\int_0^\infty e^{-\lambda t}\left(T\left(at\right)\otimes S\left(bt\right)\right)\left(x\otimes y\right)dt$$

$$= \quad (\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\left(R(\lambda)\left(x\otimes y\right)\right).$$

Hence,

$$R(\lambda)\left(\lambda I - \left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right)\right)\left(x\otimes y\right) = \left(x\otimes y\right),$$

for all $x \otimes y \in \mathfrak{D}\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\binom{a}{b}\right) \cap (X \otimes Y)$. Since by Corollary 1 the domain $\mathfrak{D}\left((\overline{A_1 \otimes I}, \overline{I \otimes A_2})\binom{a}{b}\right)$ is dense in $X \overset{\alpha}{\otimes} Y$, then

$$R(\lambda)\left(\lambda I - (\overline{A_1 \otimes I}, \overline{I \otimes A_2})\begin{pmatrix} a \\ b \end{pmatrix}\right) = I,$$

where I is the identity map on $X \overset{\alpha}{\otimes} Y$.

References

[1] W. Arendt, Grabosch, A. Greiner, G., Groh, U., Lotz, H. P., Moustakas, U., Nagel, R., Neubrander, F., and Schlotterbeck, U. One Parameter Semigroups of Positive Operators (edited by R. Nagel). Lecture Notes in Mathematics, 1184. Springer-Verlag. 1986.

[2] K.J. Engel, and R. Nagel. One Parameter Semigroups for Linear Evolution Equations. New York: Springer-Verlag. 2000.

[3] J. Goldstein. Semigroups of Linear Operators and Applications. New York: Oxford University Press. 1985.

[4] E. Hille, and R. S. Phillips. Functional Analysis and Semigroups. Rhode Island: Amer. Math. Soc. Colloq. Publi. 31, Providence. 1957.

[5] T. Ichinose. On the spectra of tensor products of linear operators in Banach spaces. J. Reine Angew. Math., (244): 119 - 153 .1970.

[6] T. Ichinose. Operators on tensor product of Banach spaces. Transactions of The American Mathematical Socociety, 1(70): 197 - 219. 1972.

[7] R. Khalil, and S. Al-Sharif. Two parameter semigroups. Journal of Applied Mathematics and Computation, (156): 403 - 414. 2004.

[8] W. A. Light, and E. W. Cheney. Approximation theory in tensor product spaces, Lecture notes in maths., 1169. Springer-Verlag .1985.

[9] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations, Lecture notes, University of Maryland. 1974.

[10] A.Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations. New York: Springer-Verlag. 1983.