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Numerical simulation of initial value problem of integro-differential equation models

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Abstract. In this article, the Volterra-Fredholm integral equation is derived from an initial value problem of kind integro-differential equation. We discuss the existence and uniqueness of the solution to the problem in Hilbert space. A numerical method is used to reduce this type of equation to the system of Fredholm integral equations of the second kind. In light of this, the Collocation method and the Galerkin method are used to solve the system of second-order Fredholm integral equations and calculate the error in each case. Finally, the approximate and exact solutions are plotted on the same coordinate plane using MATLAB code (2022).

2020 Mathematics Subject Classifications: 34K05,45E20
Key Words and Phrases: Integro-differential equations IDE, Volterra-Fredholm integral equation V-FIE, system of second-order Fredholm integral equations SFIEs, Collocation Method and Galerkin method

1. Introduction

Integro-differential equations **IDE** have garnered growing interest from the mathematical and physics communities. These equations appear often in a wide range of application domains including engineering, mechanics, elastic theory, probability theory, and mathematical physics. Also, arise in fluid dynamics, such as the glass-forming process and nano-hydrodynamics, falling condensation, biological models, chemical kinetics, ecology, and control theory in financial mathematics, space systems, and industrial mathematics. see [12][10][15]. Recently, the authors have used various methods to display the numerical or analytical solutions of **IDEs**. Therefore, many authors worked on semi-analytical methods such as sequential Taylor expansion method, see [2][13], Bessel collocation method, see[17], the variational iteration method, see[14], Haar functions method, see[5][3], Legendre-Spectral method, see[16] [11], multi-wave Legendre method, see [9], Legendre matrix method, see [18] and differential transform method, see [4].

In this article, we study the numerical solutions for **IDEs** of order two.

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2. Formulation of the problem

Consider the **IDE**,

$$\mu y''(t) + B_1 y'(t) + B_2 y(t) + \int_0^a k(t,\tau) y(\tau) d\tau = f(t) \tag{1}$$

with initial conditions:

$$y'(0) = q_1 \qquad y(0) = q_0 \tag{2}$$

where : $y''(t) = \frac{d^2y}{dt^2}$ and $y(t)$ is the unknown function in the Hilbert space and it is continuous with its derivatives. The parameters μ may have a physical meaning. The known function $f(t)$ is continuous. the $k(t,\tau)$ is the **IDE**'s kernel which is a continuous function or at least satisfies Fredholm's condition. B_1 and B_2 are known continuous functions in the class space $L_2[0,a]$ with their derivatives.

To solve the **IDE**, we must transform it into a Volterra-Fredholm integral equation **V-FIE**. Then the existence of a unique solution of the equation (2.1) under the given conditions (2.2) is provided. the essential requirements that ensure the result is unique. see[1].

After that, by using algebraic techniques, we reduce the **V-FIE** to an aliner system of Fredholm integral equation **SFIEs**. The Collocation and Galerkin methods are used to generate a numerical solution of a linear system of algebraic equations. which are then solved using numerical methods. Furthermore, in each method, the estimating error is computed and plotted. see[7]

Assume that,

$$y''(t) = g(t) \tag{3}$$

Integrating equation (2.3) two times. to follow,

$$y(t) = \int_0^t (t-x) g(x) dx + t q_1 + q_0 \tag{4}$$

Using the preceding results from equation (2.1),

$$\mu g(t) + \int_0^t \phi(t,x) g(x) dx + \int_0^a \psi(t,\tau) g(x) dx = F(t) \tag{5}$$

where,

$$\phi(t,x) = B_1 + (t-x) B_2 \tag{6}$$

$$\psi(t,\tau) = \int_0^t k(t,\tau)(\tau - x) d\tau \tag{7}$$

$$F(x) = f(x) - [B_1 q_1 + B_2 t q_1 + B_2 q_0] - [\int_0^a k(t,\tau)(t q_1 + q_0) d\tau] \tag{8}$$

In the space $L_2[0,a] \times C[0,T], T < \infty$ $t, \tau \in [0,a]$, equation (2.5) is known as the **V-FIE**. where the Fredholm integral term is positive and continuous kernel $\psi(t,\tau)$. while the Volterra integral term is considered in time with a positive continuous kernel $\phi(t,x)$ For all $t, x \in [0,T], T < \infty$. The free term $F(x,t)$ is the surface integral equation (2.5) and it is a known continuous function in the space $L_2[a,b]$

2.1. The main conditions

In order to ensure a unique solution to equation(2.1), we assume the following conditions must be satisfies :
(i) The kernel of the integral term $K(t,\tau)$ must be continuous or at least satisfy the Fredholm condition.

$$| \int_0^a \int_0^a k^2(t,\tau)dtd\tau \,|^{\frac{1}{2}} \leq \alpha \qquad \alpha \quad \text{is constant.}$$

(ii) For the constants A_1, A_2 the given continuous functions $B_1(t), B_2(t)$ satisfies the following conditions:

$$| B_1(t) | \leq A_1, | B_2(t) | \leq A_2$$

(iii) The norm of surtace function $f(x)$ is defined as:

$$\| f(x) \| = [\int_0^a f^2(x)dx]^{\frac{1}{2}} \leq \gamma, \qquad \gamma \quad \text{is constant.}$$

(iv) The unknown function $y(t)$ in the Hilbert space $L_2[0,a]$ behaves as the given function $f(t)$

2.2. The normality and continuity of the integral operator:

Theorem 1. *The integral equation (2.1) under the previous conditions (i)-(iv) has a distinct solution.*

proof:
To prove the existence of a unique solution of equation (2.5), we use the normality and continuity of the mixed integral equation. For this, the integral equation (2.5) can be written in the integral operator form:

$$\bar{W}g(t) = F(t) + Wg(t) \tag{9}$$

$$Wg(t) = \psi g(x) + \phi g(x) \tag{10}$$

$$\psi g(t) = -\frac{1}{\mu}\int_0^a | (t-x) | g(x)dx \tag{11}$$

$$\phi g(t) = -\frac{1}{\mu}\int_0^t \phi(t,x)g(x)dx \tag{12}$$

If conditions (i)-(iii) are satisfied, then we have respectively found:
(a) There exists a constant Ω_ψ, such that:

$$\left[\int_0^a \int_0^a \psi^2(|t-\tau|)d\tau dt\right]^{\frac{1}{2}} \leq \Omega_\psi \tag{13}$$

To obtain this, we take the norm of equation (2.7) and apply the Cauchy-Schwarz inequality.

$$\|\psi(t,\tau)\| \leq \frac{1}{\mu}\left\|\int_0^a \left(k^2(t,\tau)d\tau\right)^{\frac{1}{2}}\left(\int_0^a (\tau-x)^2 d\tau\right)^{\frac{1}{2}}\right\| \tag{14}$$

By using condition (i), we get:

$$\|\psi(t,\tau)\| \leq \frac{\alpha}{\mu}\left(\int_0^a \left[\frac{-x^3-3ax^2+3a^2x-a^3}{3}\right]dt\right)^{\frac{1}{2}} \tag{15}$$

Hence

$$\|\psi(|t-\tau|)\| \leq \frac{\alpha}{\mu}\left(\frac{a^4}{12}\right)^{\frac{1}{2}} = \Omega_\psi < 1 \tag{16}$$

(b) There exists a constant Ω_ϕ such that,

$$\|\phi(t,x)\| \leq \Omega_\phi, \tag{17}$$

Taking the norm of equation (2.6) and applying condition (ii), we get:

$$\begin{aligned}\|\phi(t,x)\| &\leq \frac{1}{\mu}[|B_1|+|B_2|\|(t-x)\|] \\ &\leq \frac{1}{\mu}(A_1+A_2\|t\|) \leq \frac{1}{\mu}(A_1+A_2(\frac{a^3}{3})^{\frac{1}{2}}) \leq \Omega_\phi < 1\end{aligned} \tag{18}$$

(c) There exists a constant Ω_F such that

$$\|F(x)\| \leq \Omega_F \tag{19}$$

Where

$$\frac{1}{\mu}\|f(x)\| \leq \gamma < 1 \tag{20}$$

$$\frac{1}{\mu}\|(B_1q_1+B_2q_0+B_2q_1t)\| \leq \epsilon < 1 \tag{21}$$

$$\frac{1}{\mu}\|[\int_0^a k(t,\tau)(tq_1+q_0)d\tau]\| \leq \frac{\alpha}{\mu}(q_0+q_1(\frac{a^3}{3})^{\frac{1}{2}}) = \sigma < 1 \tag{22}$$

$$\Omega_F = (\gamma+\epsilon+\sigma) \tag{23}$$

Theorem 2. *If conditions (a)-(c) are satisfied, and the integration factor (2.9) is normal and continuous, then equation (2.5) has a unique solution in Banach space $L_2\,[0,a]$, under the condition,*

$$|\lambda| \leq \frac{1-\Omega_\phi}{\Omega_\psi} \tag{24}$$

Proof:

We establish the normality and continuity of the integral operator (2.9).

(a) For the normality of the integral operator Wg, we write:

$$\|Wg\| \leq \frac{-1}{\mu}[\left\|\int_0^t \phi(t,x)g(x)dx\right\| + \left\|\int_0^a \psi(|t-x|)g(x)dx\right\|]. \tag{25}$$

Then,

$$\|\phi g(x)\| \leq \frac{1}{\mu}(A_1 + A_2\|t\|)\|g(x)\| \leq \frac{1}{\mu}(A_1 + A_2(\frac{a^3}{3})^{\frac{1}{2}})\|g(x)\|, \tag{26}$$

which can be adapted as:

$$\|\phi g(x)\| \leq \Omega_\phi \|g(x)\|, \quad . \tag{27}$$

also,

$$\|\psi g(x)\| \leq \left\|\int_0^a \psi(|t-x|)g(x)dx\right\|. \tag{28}$$

By using condition (a), we get:

$$\|\psi g(x)\| \leq \Omega_\psi \| g(x)\|, \tag{29}$$

and them,

$$\|Wg(x)\| \leq \chi \|g(x)\|, \quad \chi = (\Omega_\phi + \Omega_\psi). \tag{30}$$

So, W is a norm operator that leads straight to the normality of the operator $\bar{W}$ after applying condition (c).

(b) We assume that the two potential functions $g1(x), g2(x)$ in the Helbert space, then,

$$\begin{aligned}\left\|\bar{W}(g_1 - g_2)\right\| \leq & \left\|\int_0^t \phi(t,x)(g_1(x) - g_2(x))\,dx\right\| \\ & + \left\|\int_0^a \psi(|t-x|)(g_1(x) - g_2(x))\,dx\right\|.\end{aligned} \tag{31}$$

Using conditions (a) and (b), we get:

$$\left\|\bar{W}(g_1 - g_2)\right\| \leq \chi \|g_1(x) - g_2(x)\|. \tag{32}$$

That proves $\bar{W}$ is a continuous operator,then, by using the condition $\chi < 1$, we deduce that $\bar{W}$ is a contraction operator, and it has a unique solution.

2.3. System of Fredholm integral equations SFIEs

The quadratic method has wide application in mathematical and physics problems, where the eigenvalues and eigenfunctions of integral equations are often studied and discussed. It also has wide applications in applied sciences, especially in the theory of elasticity, mixed problems in the fluid of mechanics, and communication problems. This numerical technique will be applied in this section to reduce **V-FIEs** to linear **SFIEs**.

Consider,

$$\mu g(t) = F(t) + \int_0^t \phi(t,x)g(x)dx + \int_0^a \psi(t,\tau)g(x)dx \tag{33}$$

Divide the interval $[0,T]$ as $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$. Using the quadrature formula, equation (4.1) becomes:

$$\int_0^t \phi(t,x)g(x)\mathrm{d}x = \sum_{m=0}^{n} u_m \phi(t_n, x_m)g(x_m) \tag{34}$$

Where $n = 0, 1, 2, \cdots, N-1$, and $u_0 = \frac{1}{2}h_0, u_n = \frac{1}{2}h_n, u_i = h_i, (i \neq 0, n)$.
Using (2.34) in (2.33), we have:

$$g(t) = F(t) + \int_0^a \psi(t,\tau)g(x)dx + \sum_{m=0}^{n} u_m \phi\left(t_n, x_m\right) g(x_m) \tag{35}$$

Then:

$$g_n = Y_n(t) + \int_0^a \psi(t,\tau)g_n(x)dx \tag{36}$$

Where $Y_n(t) = F_n(t) + \sum_{m=0}^{n} u_m \phi\left(t_n, x_m\right) g(x_m), n = 0, 1, \cdots, N$.
Formula (2.36) represents a **SFIEs** , and we have N unknown functions $g_n(t)$ corresponding to time interval $[0,T]$.

3. Numerical Methods

In this section, the Collocation method and Galerkin method are used to solve **V-FIE** of the second kind.

3.1. Collocation Method

To find the solution to equation (2.5), use the Collocation method. We approximate the unidentified function $g(x)$ by using the function $L_i(x)$.

$$S(t, x_i) = \sum_{i=1}^{n} c_i L_i(x) \tag{37}$$

Given a set of n linearly independent functions $L_1(x), L_2(x), \ldots, L_N(x)$ defined on the interval $(0, a)$. Therefore, we have:

$$\begin{aligned} \mu S_n(t) \approx F\left(t, x_i\right) + \int_0^a \psi(t,x)S_n(x)dx + \sum_{m=0}^{n-1} u_m \phi_{jm} S_n(x) \\ + \epsilon\left(x, c_1(t), c_2(t), \ldots, c_N(t) + R\left(h_i^{p+1}\right)\right) .j = 0, 1, \ldots, n \end{aligned} \tag{38}$$

Of course, if the approximate solution (3.1) is substituted with (2.36) for the function $g(t,x)$, there would inevitably be an error denoted as $\varepsilon\left(x,c_1(t),c_2(t),\ldots,c_N(t)\right)$. The extent of this error is dependent on the selection of coefficients in the formula. The value of x is selected. Given that x is equal to x_i for i ranging from 0 to N, we may express this as:

$$\begin{aligned}\mu\sum_{i=1}^{n}c_iL_i(x)&\approx F\left(t,x_i\right)+\int_0^a\psi(t,x)\sum_{i=1}^{n}c_iL_i(x)dx\\&+\sum_{m=0}^{n-1}u_m\phi_{jm}\sum_{i=1}^{n}c_iL_i(x)\\&+\epsilon\left(x,c_1(t),c_2(t),\ldots,c_N(t)+R\left(h_i^{p+1}\right)\right),j=1,\ldots,n\end{aligned}\tag{39}$$

For determining the coefficients $c_1\left(t_i\right),c_2\left(t_i\right),\ldots,c_n\left(t_i\right)$ of the approximate solution $S_n\left(x_j\right)$, as given in equation (3.1), using n linearly independent functions $L_1(x),L_2(x),\ldots,L_n(x)$ defined on the numerical interval $[0,a]$. Hence, we need to carry out the process of integration and subsequently replace x with $x_1,x_2,\ldots,x_N$ to identify the points at which the error $\epsilon\left(x,c_1(t),c_2(t),\ldots,c_N(t)\right)$ becomes zero. By substituting the equation (3.1) into equation (3.2), we obtain:

$$\begin{aligned}\mu\sum_{i=1}^{n}c_iL_i(x)&\approx F\left(t,x_i\right)+\int_0^a\psi(t,x)\sum_{i=1}^{n}c_iL_i(x)dx+\sum_{m=0}^{n-1}u_m\phi_{jm}\sum_{i=1}^{n}c_iL_i(x)\\&+\epsilon\left(x,c_1(t),c_2(t),\ldots,c_N(t)+R\left(h_i^{p+1}\right)\right),j=1,\ldots,n\end{aligned}$$

This gives us:

$$\begin{aligned}\mu\sum_{i=1}^{n}c_iL_i(x)&\approx F\left(t,x_i\right)+\sum_{i=1}^{n}c_i\int_0^a\psi(t,x)L_i(x)dx\\&+\sum_{m=0}^{n-1}\sum_{i=1}^{n}u_m\phi_{jm}c_iL_i(x)\\&+\epsilon\left(x,c_1(t),c_2(t),\ldots,c_N(t)+R\left(h_i^{p+1}\right)\right),j=0,\ldots,n\end{aligned}$$

see[6, 7]

3.2. Galerkin method

The Galerkin method is used to obtain an approximate solution to equation (2.36). The approach sets the necessary conditions for calculating a set of n coefficients, as stated in Equation (3.1). By introducing the error $\varepsilon\left(x,c_1(t),c_2(t),\ldots,c_N(t)\right)$ in equation (3.2) which is perpendicular to n linearly independent functions $L_1(x),L_2(x),\ldots,L_N(x)$ on the

interval $(0, a)$, i.e.

$$\int_0^a L_j(x)\epsilon(x, c_1(t), c_2(t)), \ldots, c_N(t)dx = 0 \tag{40}$$

Then from (2.34), we have

$$\begin{aligned}\mu \sum_{i=1}^{n} c_i L_i(x) - \int_0^a \psi(t,x) \sum_{i=1}^{n} c_i L_i(x)dx - \sum_{m=0}^{n-1} u_m \phi_{jm} \sum_{i=1}^{n} c_i L_i(x) \\ + \epsilon \left(x, c_1(t), c_2(t), \ldots, c_N(t) + R\left(h_i^{p+1}\right)\right) = F(t)\end{aligned} \tag{41}$$

Then the Galerkin equations are obtained by multiplying both sides of (3.5) by $L_j(x)$ and then integrating with respect to x from 0 to a, we obtain

$$\sum_{i=1}^{n} c_i \int_0^a \left[\mu L_i(x) - \int_0^a \psi(t,x) L_i(x)dx - \sum_{m=0}^{n-1} u_m \phi_{jm} L_i(x)\right] L_i(x)dx \tag{42}$$

$$= \int_0^a F(t) L_i(x)dx \tag{43}$$

The unknown parameters c_i are determined by solving the system of equations mentioned above and inserting these values of parameters in trial functions. We obtain an approximate solution, denoted as g_h, of the **V-FIE** .see [7, 8]

4. Applications.

Consider the following applications:

Application (1)

Consider the IVP,

$$y''(t) = 2 - \cos t + \int_0^{\pi} ty(t)dt \tag{44}$$

under the initial conditions

$$y'(0) = 0 \quad y(0) = 0 \tag{45}$$

The exact solution is $y(t) = |\cos t|$.
After converting it to **V-FIE** we get:

$$\phi(t) - \int_0^{\pi} \psi(x)\phi(x)dx = F(t)$$

where,

$$\begin{aligned}\psi(x) &= \int_0^t t(t-x)dt \\ F(x) &= 2 - \cos t.\end{aligned} \tag{46}$$

- Using **Collocation** method

Let us consider the approximate answer the equation (4.1), as the three independent functions,
$L_0(t) = 1, L_1 = t, L_2(t) = |cos(t)|$. By substituting these functions into equation (4.3) and then solving the resulting equation when $t = 0, \frac{\pi}{2}, \pi$, we get:

$$c_0 = -1.34404356$$
$$c_1 = 0.382981562$$
$$c_2 = 2.34404356$$

Therefore, the approximate solution is

$$S_1(t) = -1.34404356 + 0.382981562t + 2.34404356|cos(t)|.$$

- Using **Galerkin** method

As the same consideration independent approximate solution and the same arbitrary points in the collocation method, we get,

$$c_0 = -1.32446935$$
$$c_1 = 0.332647197$$
$$c_2 = 2.18043309$$

Therefore, the approximate solution :

$$S_2(t) = -1.32446935 + 0.332647197t + 2.18043309|cos(t)|.$$

The following figures discuss the shape of the numerical solution of the two methods and the relation between the estimating errors that were obtained.

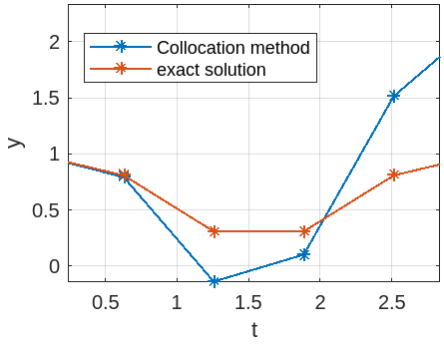

Figure 1: The relation between the exact solution and numerical solution in Collocation approximate.

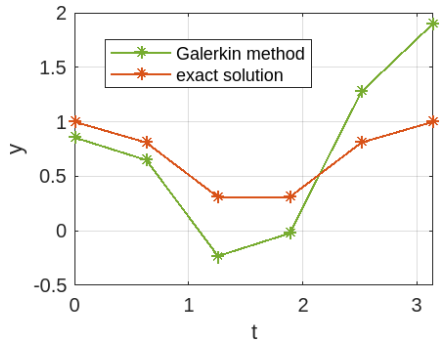

Figure 2: relation between the exact solution and numerical solution in Galerkin approximate.

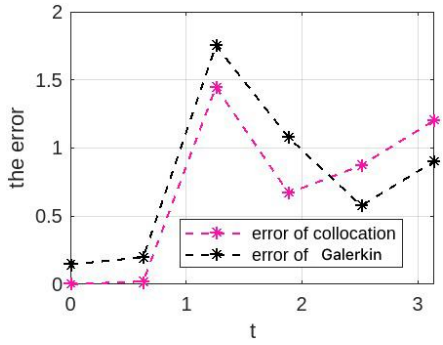

Figure 3: The relation between estimating error of Collocation approximate and Galerkin approximate.

Application (2)

Consider the **IVP**,

$$y''(t) = 1 - e + e^t + \int_0^1 y(t)dt \tag{47}$$

under the initial conditions

$$y'(0) = 1 \quad y(0) = 1 \tag{48}$$

The exact solution is $y(t) = e^t$
After converting it to **F-VIE** We get:

$$\phi(t) - \int_0^1 \psi(x)\phi(x)dx = F(x)$$

where,

$$\psi(x) = \int_0^t (t-x)dtF(x) = \frac{5}{2} - e + e^t. \tag{49}$$

- Using **Collocation** method

Let us consider the approximate answer the equation (4.1), as the three independent functions,
$L_0(t) = 1, L_1 = t, L_2(t) = t^2$. By substituting these functions into equation (4.6) and then solving the resulting equation when $t = 0, \frac{\pi}{2}, \pi$, we get:

$$c_0 = 0.7817181715$$
$$c_1 = -0.267748300$$
$$c_2 = 1.98603013$$

Therefore, the approximate solution is

$$S_2(t) = 0.7817181715 - 0.267748300t + 1.98603013t^2.$$

- Using **Galerkin** method

As the same consideration independent approximate solution and the same arbitrary points in the collocation method, we get,

$$c_0 = 0.844299425$$
$$c_1 = -0.370315407$$
$$c_2 = 1.90695998$$

Therefore, the approximate solution:

$$S_2(t) = 0.844299425 - 0.370315407t + 1.90695998t^2.$$

The results are shown in figure 4 and 5 for $n = 6$.

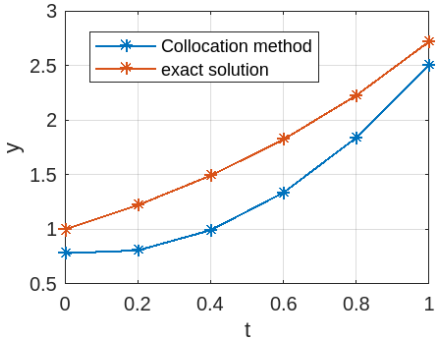

Figure 4: The relation between the exact solution and numerical solution in Collocation approximate.

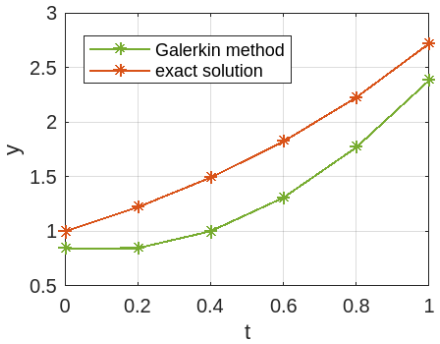

Figure 5: relation between the exact solution and numerical solution in Galerkin approximate.

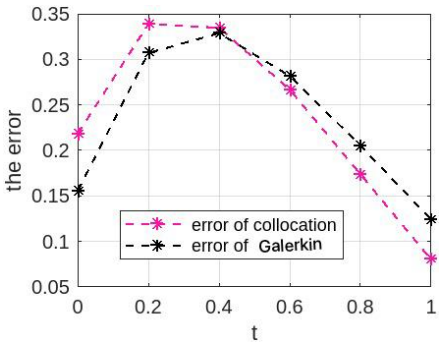

Figure 6: The relation between estimating error of Collocation approximate and Galerkin approximate.

5. Conclusion

In this study, we focus on the initial value problem of a linear integro-differentail equation with a continuous kernel, or at least satisfy the Fredholm condition. By converting the studding equation to a Volterra-Fredholm integral equation of the second kind it transforms into a linear system of Fredholm integral equations. The collocation method and the Galerkin method are used to efficiently discretize the integral equations using a set of points. We discuss the error estimates associated with both methods.

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