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On a sixth-order solver for multiple roots of nonlinear equations

Young Hee Geum

Department of Mathematics, Dankook University, Cheonan, Korea 330-714

Abstract. A number of iterative schemes with high convergence order to solve nonlinear equations are presented in the literature. In this paper, a sixth-order multiple-zero finder has been developed and the dynamics of selected iterative schemes with uniparametric polynomial weight function are investigated using Möbius conjugacy map applied to the form $((z-A)(z-B))^m$. The complex dynamics on the Riemann sphere by analyzing the parameter spaces associated with the free critical points are studied, and the numerical experiments are carried out.

2020 Mathematics Subject Classifications: 65H05, 65H99

Key Words and Phrases: Conjugacy map, multiple root, nonlinear equation, parameter space

1. Introduction

The root-finding problems[1, 2] occur in various fields of artificial intelligence, science, biology and engineering. Finding the zeros of nonlinear equation [3–5, 16] is the process of finding the value of a variable that satisfies a given equation. A zero α of $h(x)=0$ is called a multiple root [14] with multiplicity m if $h^{(i)}(\alpha)=0,\ i=0,1,2,\cdots,m-1$ and $h^{(m)}(\alpha)\neq 0$.

It is known that the Newton's method is the most fundamental method for solving the equations, given by

$$x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)},\ n=0,1,2,\cdots. \tag{1}$$

Researchers [2, 7–9] are interested in solving roots of nonlinear equations[10, 13, 15, 17, 18] and investigating the dynamical analysis by exploring the relevant parameter plane and basins of attraction [12, 18, 19].

We study the dynamics of a class of sixth-order multiple-zero solvers developed by Geum-Kim-Neta[11] below.

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Email address: `conpana@empal.com` (Geum, Y.H.)

$$\begin{cases} y_n = x_n - m \cdot h(x_n),\ h(x_n) = \frac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot G_f(s_1) \cdot h(x_n),\ s_1 = (\frac{f(y_n)}{f(x_n)})^{\frac{1}{m}}, \\ x_{n+1} = x_n - m \cdot K_f(s_1, s_2) \cdot h(x_n),\ s_2 = (\frac{f(w_n)}{f(x_n)})^{\frac{1}{m}}, \end{cases} \tag{2}$$

where $G_f : C \to C$ is a analytic function in a small neighborhood of the origin 0 and $K_f : C^2 \to C$ is holomorphic in a small neighborhood of $(0, 0)$.

$$\begin{cases} G_f(s_1) = \frac{1+(a_1-a_2-1)s_1+a_1 s_1{}^2}{1+(a_1-a_2-2)s_1+a_2 s_1{}^2}, \\ K_f(s_1, s_2) = \frac{1+(a_1-a_2-1)s_1+a_1 s_1{}^2}{1+(a_1-a_2-2)s_1+a_2 s_1{}^2+[(1-a_1+a_2)s_1-1]s_2}, \end{cases} \tag{3}$$

where a_1 and a_2 are free parameter to be chosen. For brevity of analysis, we select $a_1 = 1$ and consider one parameter $a_2 = \kappa (\in C)$ here.

The numerical method in (2) is written as

$$x_{n+1} = I_f(x_n, a_1, a_2),$$

where $I_f(x_n, a_1, a_2) = x_n - m \cdot K_f(s_1, s_2) \cdot h(x_n)$ is a fixed point operator.

The process of solving the nonlinear equation of $f(z) = 0$ is regarded as a sequence of images of initial value x_0 under I_f below:

$$\{x_0, I_f(x_0), I_f^2(x_0), \cdots I_f^n(x_0), \cdots\}$$

We investigate the conjugacy map and stability surfaces of the selected iterative scheme in Section 2. The algorithm, related theorems and the parameter planes are shown in Section 3. Finally, conclusions are stated in the last section.

2. Conjugacy map and analysis

Via Möbius conjugacy map [6] $T(z) = (z - A)/(z - B)$ when applied to a polynomial $f(z) = ((z - A)(z - B))^m$, I_f is conjugated to $J(z, \kappa)$ satisfying

$$J(z, A, B, \kappa) = \frac{H(z, A, B, \kappa)}{D(z, A, B, \kappa)}, \tag{4}$$

where $z, A, B \in C \bigcup \{\infty\}$, $A \neq B$ and H and D are polynomials whose coefficients are dependent upon parameters A, B and κ.

Using Mathematica [20] computation with $T^{-1}(z) = (Bz - A)/(z - 1)$, we obtain $J(z, \kappa)$ as follows

$$J(z, \kappa) = -\frac{z^6(3 + 3z + z^2 - \kappa - z\kappa)H_1(z)}{(-1 - 3z - 3z^2 + z\kappa + z^2\kappa)D_1(z)}, \tag{5}$$

where $H_1(z) = (2 + 7z + 8z^2 + 4z^3 + z^4 - 2z\kappa - 2z^2\kappa - z^3\kappa)$ and $D_1(z) = (1 + 4z + 8z^2 + 7z^3 + 2z^4 - z\kappa - 2z^2\kappa - 2z^3\kappa)$. We find out that J is dependent only on κ but independent of parameter A and B.

We figure out that the fixed points of the iterative scheme $J(z,\kappa)$. Let $\phi(z,\kappa) = z - J(z,\kappa)$ where roots are the desired fixed points of $J(z,\kappa)$. After a lengthy computation, we find that $z=0$ and $z=1$ are the zeros of $\phi(z,\kappa)$ and we have the following expression of $\phi(z,\kappa)$:

$$\phi(z,\kappa) = \frac{(-1+z)z(1+z+z^2)^2 W(z)}{w(z)}, \tag{6}$$

$$\begin{aligned} W(z) =& 1+z^6+z^2(-4+\kappa)^2+z^4(-4+\kappa)^2-2z(-3+\kappa)-2z^5(-3+\kappa)+z^3(22-13\kappa+2\kappa^2),\\ w(z) =& -1+2z^6(-3+\kappa)+z(-7+2\kappa)+z^4(-47+27\kappa-4\kappa^2)+z^3(-43+23\kappa-3\kappa^2)\\ &+z^5(-27+15\kappa-2\kappa^2)-z^2(23-10\kappa+\kappa^2). \end{aligned}$$

Theorem 1. *(1) If $\kappa=0$, then*

$$\phi(z,\kappa) = -\frac{(-1+z)z(1+z)(1+z+z^2)^2(1+4z+7z^2+4z^3+z^4)}{1+6z+17z^2+26z^3+21z^4+6z^5},$$

and the strange fixed points z are given by $z=\pm 1, z=-0.5\pm 0.866025i, z=-1.62481\pm 1.30024i, z=-0.375189\pm 0.300243i$.
(2) If $\kappa=2$, then

$$\phi(z,\kappa) = -\frac{z(1+z+z^2)^2(-1+z-z^2+z^3)}{1+z+2z^2},$$

and the strange fixed points z are given by $z=\pm i, z=1, z=-0.5\pm 0.866025i$.
(3) If $\kappa=\frac{7}{2}$, then

$$\phi(z,\kappa) = \frac{z(1+z+z^2)^2(4-4z+z^2+4z^3+z^4-4z^5+4z^6)}{4+4z+5z^2+2z^3+8z^4+4z^5},$$

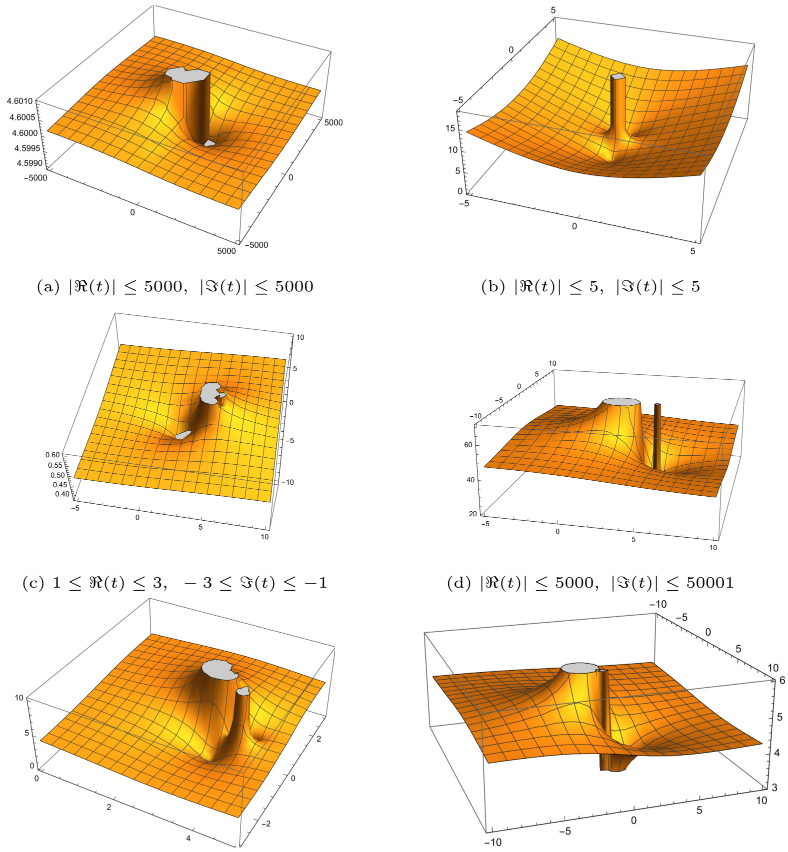
and the strange fixed points z are given by $z=-0.5\pm 0.866025i, z=0.75\pm 0.661438i, z=-0.84307\pm 0.537803i, z=0.59307\pm 0.805151i$.
(4) If $\kappa=\frac{22}{5}$, then

$$\phi(z,\kappa) = \frac{z(1+z+z^2)^2(25-70z+4z^2+88z^3+4z^4-70z^5+25z^6)}{25-20z-61z^2-64z^3+77z^4+70z^5},$$

and the strange fixed points z are given by $z=-0.5\pm 0.866025i, z=0.536675, z=1.86, z=0.67, z=1.48, z=-0.874796\pm 0.475352i$.
(5) Let $\kappa\notin\{0,2,\frac{7}{2},\frac{22}{5}\}$. Then $W(\frac{1}{z})=z^{-6}W(z)$.

Proof. (1)-(4) Suppose $W(z)=0$ and $w(z)=0$ for z. Eliminating κ from the two polynomials, we have the relation $F(z)=(z+1)(z^2+z+1)=0$. Substituting all the roots of F into $W(z)=0$ and $w(z)=0$, we get the relations for κ and solving them for κ, we have $\kappa=0,2$. The remaining part is straightforward. If $(z-1)$ is a divisor of $w(z)$,

then $w(1) = -154 + 79\kappa - 10\kappa^2 = 0$, yielding $\kappa = \frac{7}{2},\ \frac{22}{5}$. Then remaining proof is trivial.
(5) By direct computation, we have $W(1/z) = z^{-6}w(z)$. □



(a) $|\Re(t)| \le 5000,\ |\Im(t)| \le 5000$ (b) $|\Re(t)| \le 5,\ |\Im(t)| \le 5$

(c) $1 \le \Re(t) \le 3,\ -3 \le \Im(t) \le -1$ (d) $|\Re(t)| \le 5000,\ |\Im(t)| \le 50001$

(e) $|\Re(t)| \le 5000,\ |\Im(t)| \le 5000$ (f) $0 \le \Re(t) \le 500,\ -500 \le \Im(t) \le 0$

Figure 1: Stability surfaces .

To find the stability of fixed points, we compute the derivative of $J(z,\kappa)$ as follows:

$$J'(z,\kappa) = \frac{z^5 \cdot Q(z)}{q(z)^2}, \tag{7}$$

where

$$\begin{aligned} Q(z) =& a_0 + a_2 z + a_3 z^2 + a_4 z^3 + 2a_5 z^4 + a_7 z^5 + 2a_6 z^6 + a_7 z^7 \\ &+ 2a_5 z^8 + a_4 z^9 + a_3 z^{10} + a_2 z^{11} + a_0 z^{12}, \\ q(z) =& (-1 + 2z^4 + z^3(7-2\kappa) - z(-4+\kappa) - 2z^2(-4+\kappa)(-1 + z(-3+\kappa) + z^2(-3+\kappa)), \\ a_0 =& -12(-3+\kappa),\ a_2 = 399 - 235\kappa + 34\kappa^2,\ a_3 = 2062 - 1594\kappa + 400\kappa^2 - 32\kappa^3, \end{aligned}$$

$$a_4 = 6569 - 5935\kappa + 1908\kappa^2 - 249\kappa^3 + 10\kappa^4,\ a_5 = 7166 - 7099\kappa + 2602\kappa^2 - 415\kappa^3 + 24\kappa^4,$$
$$a_6 = 13061 - 13800\kappa + 5508\kappa^2 - 986\kappa^3 + 67\kappa^4,$$
$$a_7 = 22523 - 23435\kappa + 9170\kappa^2 - 1600\kappa^3 + 105\kappa^4.$$

Theorem 2. *(1) If* $\kappa = -1$*, then*

$$J'(z,\kappa) = \frac{z^5(2+z)^2 Q_1(z)}{(1+2z)^4(1+3z+4z^2+z^3)^2},$$

where $Q_1(z) = 12 + 107z + 388z^2 + 785z^3 + 980z^4 + 785z^5 + 388z^6 + 107z^7 + 12z^8$.

(2) If $\kappa = 0$*, then*

$$J'(z,\kappa) = \frac{z^5 Q_2(z)}{(1+3z+3z^2)^2(1+3z+5z^2+2z^3)^2},$$

where $Q_2(z) = 36 + 327z + 1372z^2 + 3498z^3 + 5964z^4 + 7097z^5 + 5964z^6 + 3498z^7 + 1372z^8 + 327z^9 + 36z^{10}$.

(3) If $\kappa = 2$*, then*

$$J'(z,\kappa) = \frac{z^5(12 + 17z + 30z^2 + 17z^3 + 12z^4)}{(1+z+2z^2)^2}.$$

(4) If $\kappa = \frac{7}{2}$*, then*

$$J'(z,\kappa) = \frac{-z^5 Q_4(z)}{(2+z)^2(2+z+2z^2+4z^4)^2},$$

where $Q_4(z) = (96 + 304z + 336z^2 + 460z^3 + 508z^4 + 723z^5 + 508z^6 + 460z^7 + 336z^8 + 304z^9 + 96z^{10})$.

(5) If $\kappa = \frac{22}{5}$*, then*

$$J'(z,\kappa) = \frac{z^5 Q_5(z)}{(-5+7z+7z^2)^2(-5-3z+z^2+10z^3)^2},$$

where $Q_5(z) = 10500 + 6475z - 39120z^2 - 41690z^3 + 19252z^4 + 79689z^5 + 19252z^6 - 41690z^7 - 39120z^8 + 6475z^9 + 10500z^{10}$.
(6) Let $\kappa \notin \{-1, 0, 2, \frac{7}{2}, \frac{22}{5}\}$*. Then* $Q(\frac{1}{z}) = z^{12}Q(z)$.

Proof. (1)-(5) Suppose that $Q(z) = 0, q(z) = 0$ for z. By eliminating κ from $Q(z) = 0$, and $q(z) = 0$, we get the relation: $T(z) = (z-1)(z+1)(1+2z)(1+z+z^2)(-1-4z-$

$6z^2 - 2z^3 + 4z^4 + 8z^5 + 4z^6) = 0$. Substituting all the roots of $T(z)$ into $Q(z) = 0$ and $q(z) = 0$, we find $\kappa = -1, 0, 2, \frac{7}{2}, \frac{22}{5}$. $\square$

The stability surfaces are shown by illustrative conical surfaces in Figure 1.

3. Experiment

According to Algorithm 1, the numerical parameter spaces are constructed in Figure 2. The systematic color palette in Table 1 is utilized to paint a value according to the orbital period of the point z of $J(z, \kappa)$. The tolerance of 10^{-4} after up to 1000 iterations is assigned [**?**].

Algorithm 1

(1) Set $i = 1$
(2) Choose a region $B \in C$ and select a point $v = (Re(v), Im(v))$ in B
(3) For the v, find the free critical point.
(4) Compute the orbit of $J(z, t)$ within the maximal iterative number.
(5) If the orbit converges to one cycle within the given error, then color the point v according to the color palette in Table 1.
(6) Choose the next value in B
(7) Repeat steps (2)-(6) until desired result is obtained.
(8) Set $i = i + 1$ and if $i \leq w$, then repeat steps (2)-(8)
(9) If $i = w$, then stop the process.

Table 1: Color palette for a n-periodic orbit with $n \in N \cup \{0\}$

n	C_n
$n = 1$	$C_1 = \begin{cases} \text{magenta, for fixed point } \infty \\ \text{cyan, for fixed point } 0 \\ \text{yellow, for fixed point } 1 \\ \text{red, for other strange fixed point ,} \end{cases}$
$2 \leq n \leq 68$	$C_2 =$ orange, $C_3 =$ light green, $C_4 =$ dark red, $C_5 =$ dark blue, $C_6 =$ dark green, $C_7 =$ dark yellow, $C_8 =$ floral white, $C_9 =$ light pink, $C_{10} =$ khaki, $C_{11} =$ dark orange, $C_{12} =$ turquoise, $C_{13} =$ lavender, $C_{14} =$ thistle, $C_{15} =$ plum, $C_{16} =$ orchid, $C_{17} =$ medium orchid, $C_{18} =$ blue violet, $C_{19} =$ dark orchid, $C_{20} =$ purple, $C_{21} =$ power blue, $C_{22} =$ sky blue, $C_{23} =$ deep sky blue, $C_{24} =$ dodger blue, $C_{25} =$ royal blue, $C_{26} =$ medium spring green, $C_{27} =$ spring green, $C_{28} =$ medium sea green, $C_{29} =$ sea green, $C_{30} =$ forest green, $C_{31} =$ olive drab, $C_{32} =$ bisque, $C_{33} =$ moccasin, $C_{34} =$ light salmon, $C_{35} =$ salmon, $C_{36} =$ light coral, $C_{37} =$ Indian red, $C_{38} =$ brown, $C_{39} =$ fire brick, $C_{40} =$ peach puff, $C_{41} =$ wheat, $C_{42} =$ sandy brown, $C_{43} =$ tomato, $C_{44} =$ orange red, $C_{45} =$ chocolate, $C_{46} =$ pink, $C_{47} =$ pale violet red, $C_{48} =$ deep pink, $C_{49} =$ violet red, $C_{50} =$ gainsboro, $C_{51} =$ light gray, $C_{52} =$ dark gray, $C_{53} =$ gray, $C_{54} =$ charteruse, $C_{55} =$ electric indigo, $C_{56} =$ electric lime, $C_{57} =$ lime, $C_{58} =$ silver, $C_{59} =$ teal, $C_{60} =$ pale turquoise, $C_{61} =$ sandy brown, $C_{62} =$ honeydew, $C_{63} =$ misty rose, $C_{64} =$ lemon chiffon, $C_{65} =$ lavender blush, $C_{66} =$ gold, $C_{67} =$ crimson, $C_{68} =$ tan.
$n = 0^*$ or $n > 69$	$C_n =$ black.

$*$: $n = 0$: the orbit is non-periodic but bounded.

Theorem 3. *Let $\psi : C^2 \to C$ be defined by $\psi(\kappa, z) = l_0(z) + l_1(z)\kappa + l_2(z)\kappa^2$, where $l_i(z)(0 \leq i \leq 2)$ are complex polynomials with real coefficients. Suppose $\psi(\kappa, z) = 0$. Let $\bar{z}$*

be a complex conjugate of z.
(1) $\kappa(\bar{z}) = \overline{\kappa(z)}$.
(2) If $z(\kappa)$ *is a zero of* ψ*, then so is* $\bar{z}(\bar{\kappa})$.

Proof. (1) Solving $\psi(\kappa, z) = 0$ for κ, we have

$$\kappa(z) = \frac{-l_1(z) \pm \sqrt{l_1(z)^2 - 4l_1(z)l_2(z)}}{2l_1(z)}. \tag{8}$$

Since $l_i(z), (0 \le i \le 2)$ are complex polynomials with real coefficients, we have $l_i(\bar{z}) = \overline{l_i(z)}$. From 8, we get

$$\begin{aligned} \kappa(\bar{z}) =& \frac{-l_1(\bar{z}) \pm \sqrt{l_1(\bar{z})^2 - 4l_1(\bar{z})l_2(\bar{z})}}{2l_1(\bar{z})} \\ =& \frac{-\overline{l_1(z)} \pm \sqrt{\overline{l_1(z)}^2 - 4\overline{l_1(z)l_2(z)}}}{2\overline{l_1(z)}}, \end{aligned}$$

$$\overline{\psi(\kappa, z)} = \overline{l_0(z)} + \overline{l_1(z)}\bar{\kappa} + \overline{l_2(z)}\bar{z}^2 = 0.$$

Therefore $\overline{\kappa(z)} = \kappa(\bar{z})$.
(2) Let $z(\kappa)$ be a zero of $\psi(\kappa, z)$. Then

$$\begin{aligned} \psi(\kappa, z) =& 0 = \overline{\psi(\kappa, z)} \\ =& \overline{l_0(z) + l_1(z)\kappa + l_2(z)\kappa^2} \\ =& \overline{l_1(z)} + \overline{l_1(z)}\bar{\kappa} + \overline{l_2(z)}\bar{\kappa}^2 \\ =& l_0(\bar{z}) + l_1(\bar{z})\bar{\kappa} + l_2(\bar{z})\bar{\kappa}^2 \\ =& \psi(\bar{\kappa}, \bar{z}), \end{aligned}$$

implying $\bar{z}(\bar{\kappa})$ is a zero of $\psi(\kappa, z)$. □

Let $\mathcal{P} = \{\eta \in C :$ a critical orbit of z converges to a number $w_p \in \overline{\mathbb{C}}\}$. It is called the parameter space. There are finite periods in the orbit if the number w_p is a finite constant. Otherwise, the orbit is not periodic however it is bounded or goes to infinity.

Theorem 4. *Let* $z(\kappa)$ *be a free critical point of* $J(z, \kappa)$*. Then the parameter space is symmetric with respect to its horizontal axis.*

Proof. Let $z(\kappa)$ is a root of $\psi(\kappa, z)$. Then $\bar{z}(\bar{\kappa})$ is a root of $Q(z)$ at $\bar{\kappa}$. From conjugated map $J(z, \kappa)$, we obtain

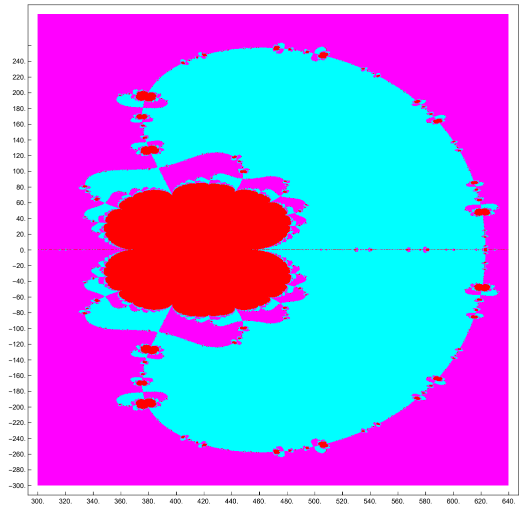
$$\begin{aligned} |J(z, \kappa)| =& |J(z(\kappa), \kappa)| = |\overline{J(z(\kappa), \kappa)}| \\ =& |J(\overline{z(\kappa)}, \bar{\kappa})| = |J(\bar{z}(\bar{\kappa}), \bar{\kappa})|. \end{aligned}$$

Then the parameter space with $J(z, \kappa)$ is symmetric with respect to its horizontal axis.
□

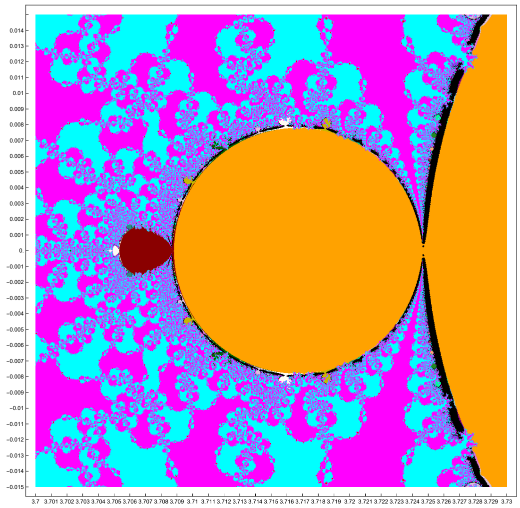
The parameter spaces $\mathcal{P}$ are illustrated in Figures 2. A point $\epsilon \in \mathcal{P}$ is painted using the color palette shown in Table 1. In terms of numerical phenomena, every point of the parameter space $\mathcal{P}$ whose color is none of cyan(root $z = A$), magenta(root $z = B$), red or yellow is not a better choice of t. The complicated patterns are found and for $n \in N - \{1\}$, n-periodic orbit is budding at period-1 component and 4-periodic component is budding at period-2 component.

4. Conclusion

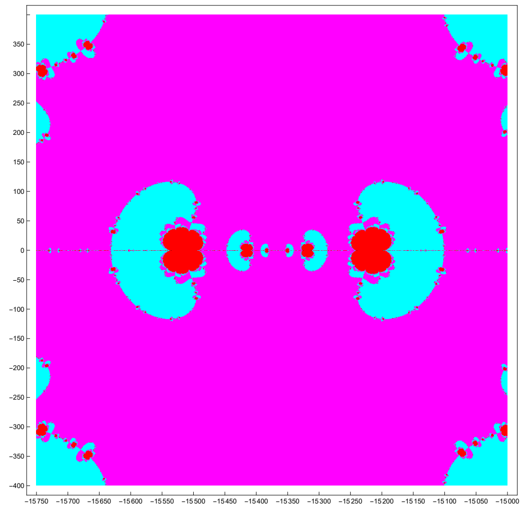
We have investigated the complex dynamical analysis on the Riemann sphere by drawing the parameter spaces associated with the free critical points for the uniparametric family of sixth-order multiple-root finders. Such research from a viewpoint of complex dynamics may limit us from treating the real dynamical phenomenon for real nonlinear equations. However, this research for investigating the relevant complex dynamics lies is finding the dynamical behavior of a family of iterative schemes via Möbius conjugacy map by showing the parameter spaces. As the next work, we visualize different types of higher order numerical methods by improving the current work. In addition, we draw the convergent region and the basins of attraction of the developed multiple-root finder in more detail.



(a) $2.6 \leq \Re(t) \leq 4.6,\ |\Im(t)| \leq 1$



(a) $2.6 \leq \Re(t) \leq 4.6,\ |\Im(t)| \leq 1$



(a) $2.6 \leq \Re(t) \leq 4.6,\ |\Im(t)| \leq 1$

Figure 2: Parameter spaces .

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