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On the Prime Radical of Nearrings Which is Kurosh-Amitsur

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Abstract. A prime radical of near-rings is introduced by defining a new class of prime modules of near-rings. It is a generalization of the Prime radical of rings. Properties of the radical are studied. It is established that this radical is a Kurosh-Amitsur radical of near-rings.

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1. Introduction

N is a near-ring and all near-rings are zero-symmetric. One may look for more definitions and results of near-rings in Pliz [4]. An additive group H is a right N-group if there is a mapping $(h, x) \to hx$ of $H \times N$ into H such that:

(i) $h(xy) = (hx)y$;

(ii) $h(x + y) = hx + hy$ for all $h \in H, x, y \in N$.

If K is a right ideal of N then K is a right N-group under the multiplication in N. Also the quotient group N/K is a right N-group under the operation $(x + K)y = xy + K$ for all $x, y \in N$.
A subgroup (normal subgroup) C of the right N-group H is a right N-subgroup (ideal) of H if $cx \in C$ for all $c \in C, x \in N$.

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An element $h_0 \in H$ is a distributive element if $h_0(x+y) = h_0x + h_0y$ for all $x, y \in N$. Since only right N-groups are considered, hereon we call a right N-group just an N-group and a right N-subgroup just an N-subgroup.
Unlike in rings the prime radical of near-rings is not a Kurosh-Amitsur radical [1]. In 1990, near-ringers could introduce a Kurosh-Amitsur prime radical of near-rings [2], called the equiprime radical. A characterization of the prime radical of near-rings was given in [5] using right modules of near-rings. With this motivation right representation of radicals of right near-ring was presented in [7] and a prime radical for near-rings, the right prime radical of type 1, was defined and studied in [6] which is a non-ideal hereditary Kurosh-Amitsur radical. This is the second known Kurosh-Amitsur prime radical of near-rings.
In this paper, using right modules, another prime radical is introduced for near-rings which is a Kurosh-Amitsur radical.

2. Prime N-groups of type 2

Let H be an N-group. The annihilator of H in N will be denoted by $\mathrm{An}(H) := \{x \in N \mid hx = 0 \text{ for all } h \in H\}$. The largest ideal of N contained in $\mathrm{An}(H)$, if it exists, will be denoted by $(H : 0)$.

Definition 1. *Let N be a near-ring and H be an N-group. H is a* prime N-group of type 2 *if :*

(i) $HN \neq \{0\}$;

(ii) for each $0 \neq h \in H$, *hN has a distributive element* $h_0 (\neq 0)$;

(iii) for each $0 \neq h \in H$, $An(hN) = An\ (H)$.

If the near-ring N is a ring then from the conditions (i) and (iii) of the prime N-group of type 2 it follows that it is a prime module [3].
It is clear that a non-zero N-subgroup of a prime N-group of type 2 is also a prime N-group of type 2.

Example 1. *We give an example of a prime N-group of type 2. Let $H := \{0, a, b, c\}$ be the additive non-cyclic group of order 4. Consider the near-ring $M_0(H)$ of mappings of H into H fixing 0. We claim that the $M_0(H)$-group $M_0(H)$ is a prime $M_0(H)$-group of type 2. It is clear that the distributive elements of the near-ring $M_0(H)$ are precisely the endomorphisms of group H and are also the distributive elements of the $M_0(H)$-group $M_0(H)$.*
(i) we have $M_0(H)M_0(H) \neq \{0\}$;
(ii) let $0 \neq f \in M_0(H)$. We suppose without loss of generality that $f(a) \neq 0$ and $f(a) = x, x \in \{a, b, c\}$. We choose $g \in M_0(H)$ such that $g(a) = a = g(c), g(b) = 0$. Now fg is an endomorphism of H and hence a distributive element in $fM_0(H)$;
(iii) let $0 \neq f \in M_0(H)$. It is clear that $An(fM_0(H)) = \{0\} = An(M_0(H))$.
Therefore $M_0(H)$ is a prime $M_0(H)$-group of type 2.

We give an example of a prime N-group of type 1 [6] which is not of type 2.

Definition 2. *Let H be an N-group with $HN \neq \{0\}$. Then H is* a prime N-group of type 1 *if:*

(i) every non-zero N-subgroup of H has a non-zero distributive element;

(ii) $hNx = \{0\}$, $0 \neq h \in H, x \in N$ implies $Hx = \{0\}$.

Example 2. *Let H be a cyclic group of order p, where p is a prime number greater than 2. Clearly $M_0(H)$ is a $M_0(H)$-group. Since H has exactly two subgroups, $\{0\}$ and $M_0(H)$ are the only $M_0(H)$-subgroups of $M_0(H)$. Therefore $M_0(H)$ is a prime $M_0(H)$-group of type 1. Note that any non-zero endomorphism of H is an automorphism of H. Choose a non-zero function $f \in M_0(H)$ such that the image of f is not equal to H. We have no $g \in M_0(H)$ such that fg is an automorphism of H, that is, a non-zero endomorphism of H, that is, a non-zero distributive element of $M_0(H)$. This shows that $fM_0(H)$ has no non-zero distributive element. Therefore $M_0(H)$ is not a prime $M_0(H)$-group of type 2.*

Now we study some properties of prime N-groups of type 2.

Proposition 1. *Let H be a prime N-group of type 2. Then $(H : 0)_N$ exists.*

Proof. Suppose that H is a prime N-group of type 2. Now H has a distributive element $0 \neq h_0$. It is clear that $\{0\} \neq h_0N$ is an N-subgroup of H. We have $(h_0x)0 = h_0(x0) = h_00 = 0$ for all $x \in N$, that is, $(h_0N)0 = \{0\}$. So $H0 = \{0\}$. Let K, L be ideals of N contained in An (H). We have $(h_0x)(k + l) = h_0(x((k + l) - xk + xk) = h_0(x((k + l) - xk) + h_0(xk) = 0 + 0 = 0$ for all $x \in N, k \in K, l \in L$. Therefore $(h_0N)(K + L) = \{0\}$. Since $\{0\} \neq h_0N$ is an N-subgroup of H, $H(K + L) = \{0\}$. Hence there is a largest ideal of N contained in An (H), that is, $(H : 0)_N$ exists.

Proposition 2. *Let H be a prime N-group of type 2 and K be an ideal of N and $HK = \{0\}$. Then h_0N is a prime N/K-group of type 2 for any distributive element $0 \neq h_0 \in H$. Moreover $(h_0N : 0)_{N/K} = (H : 0)_N/K$.*

Proof. K is an ideal of N and H is a prime N-group of type 2 and $HK = \{0\}$. Let $0 \neq h_0 \in H$ be a distributive element. Clearly $h_0N = \{h_0x \mid x \in N\}$ is a subgroup of $(H, +)$ and is an N-subgroup of H. Let $h_0x \in h_0N, x, y, z \in N$. Define $(h_0x)(y + N) := (h_0x)y$. This operation is well-defined. For this suppose that let $y + K = z + K$. Now $-z + y \in K$. We have $(h_0x)y = (h_0x)[z+(-z+y)]-(h_0xz)+(h_0xz) = h_0[(x(z+(-z+y))-xz]+(h_0xz) = 0+(h_0xz) = (h_0x)z$. Therefore the above operation is well defined. It can be easily verified that with this operation h_0N is an N/K-group. Moreover h_0N is a prime N-group of type 2 being N-subgroup of H. From this it follows that h_0N is a prime N/K-group of type 2. Let M be the largest ideal of N contained in An (H), that is, $(H : 0)_N = M$. Clearly $K \subseteq M$. Since $HM = \{0\}$, $(h_0N)M = \{0\}$. So $(h_0N)(M/K) = \{0\}$, that is, $M/K \subseteq (h_0N : 0)_{N/K}$, that is, $(H : 0)_N/K \subseteq (h_0N : 0)_{N/K}$. Similarly if $(h_0N : 0)_{N/K} = T/K$, then the ideal T of N is contained in $(H : 0)_N$ and hence $(h_0N : 0)_{N/K} \subseteq (H : 0)_N/K$. Therefore $(h_0N : 0)_{N/K} = (H : 0)_N/K$.

Proposition 3. *Let H be a prime N/K-group of type 2, K an ideal of N. Then H is a prime N-group of type 2 and $(H:0)_N/K = (H:0)_{N/K}$.*

Proof. Suppose that H is a prime N/K-group of type 2, K is an ideal of N. For $h \in H, x \in N$ define $hx := h(x+K)$. Note that $HK = \{0\}$. Clearly H is an N-group under the above operation. It is an easy observation that this operation also satisfies all the three conditions of a prime N-group of type 2. So H is a prime N-group of type 2. Let M/K be the largest ideal of N/K contained in An $_{N/K}(H)$, that is, $(H:0)_{N/K} = M/K$. Since $H(M/K) = \{0\}$, we have $HM = \{0\}$. So $(H:0)_{N/K} \subseteq (H:0)_N/K$. Let T be the largest ideal of N contained in An $_N(H)$, that is $(H:0)_N = T$. Now $K \subseteq T$ and $H(T/K) = \{0\}$ as $HT = \{0\}$. So $(H:0)_N/K \subseteq (H:0)_{N/K}$. Therefore $(H:0)_N/K = (H:0)_{N/K}$.

3. The right prime radical of type 2

$\mathcal{N}$ denotes the class of zero-symmetric near-rings. An *ideal-mapping* on $\mathcal{N}$ is a mapping $\mathcal{R}$ from $\mathcal{N}$ into itself such that $\mathcal{R}(N)$ is an ideal of N for all $N \in \mathcal{N}$. An ideal-mapping is a *Hoehnke radical* or *H-radical* if:
(i) $t(\mathcal{R}(N)) \subseteq \mathcal{R}(t(N))$ for all homomorphisms t of N in $\mathcal{N}$;
(ii) $\mathcal{R}(N/\mathcal{R}(N)) = \{0\}$ for all N in $\mathcal{N}$.
For a class of near-rings $\mathcal{M} \subseteq \mathcal{N}$ and for N in $\mathcal{N}$, we have
$(N)\mathcal{M} := \cap(J \mid J$ is an ideal of N and $N/J \in \mathcal{M})$.
Corresponding to any class of near-rings $\mathcal{M} \subseteq \mathcal{N}$, we have an ideal-mapping $\mathcal{R}$ defined by $\mathcal{R}(N) := (N)\mathcal{M}$ and it is well known that this ideal-mapping $\mathcal{R}$ is a H-radical.

Definition 3. *An ideal P of a near-ring N is a* right prime ideal of type 2 *if $P = (H:0)$ for a prime N-group H of type 2.*

It can be observed that a right prime ideal of N of type 2 is a prime ideal of N [4]. If N is a ring then a right prime ideal of N of type 2 is a prime ideal of the ring N.

Definition 4. *A near-ring is a* right prime near-ring of type 2 *if the zero ideal of N is a right prime ideal of type 2.*

Definition 5. The right prime radical of N of type 2 *is the intersection of all right prime ideals of N of type 2 and it will be denoted by $\mathcal{P}_{2(r)}(N)$.*

Now it will be proved that $\mathcal{P}_{2(r)}$ is a H-radical.

Theorem 1. *$\mathcal{P}_{2(r)}$ is a H-radical.*

Proof. Let $\mathcal{A}$ be the H-radical determined by the class of right prime near-ring of type 2. Let Q be a right prime ideal of N of type 2. We get a prime N-group H of type 2 such that $Q = (H:0)$. By Proposition 2, there is a prime N/Q-group B of type 2 such that $(B:0)_{N/Q} = (H:0)_N/Q = Q/Q = \{0\}$. So N/Q is right prime near-ring of type 2. Therefore $\mathcal{P}_{2(r)}(N) \supseteq \mathcal{A}(N)$. On the other hand suppose that P is an ideal of N and N/P is right prime near-ring of type 2. We get a prime N/P-group A of type

2 such that $(A:0)_{N/P} = \{0\}$. By Proposition 3, A is a prime N-group of type 2 and $(A:0)_N/P = (A:0)_{N/P} = \{0\}$. So $(A:0)_N = P$. Therefore $\mathcal{P}_{2(r)}(N) \subseteq \mathcal{A}(N)$. Hence $\mathcal{P}_{2(r)}(N) = \mathcal{A}(N)$. Since $\mathcal{A}$ is a H-radical, $\mathcal{P}_{2(r)}$ is also a H-radical.

Theorem 2. *Let H be a prime N-group of type 2. If K is an ideal of N and $K \nsubseteq (H:0)$ then H is a prime K-group of type 2 and $(H:0)_K \supseteq K \cap (H:0)_N$.*

Proof. Suppose that H is a prime N-group of type 2 and K is an ideal of N and $HK \neq \{0\}$. We claim that H is a prime K-group of type 2. Under restriction, clearly H is a K-group. Let $0 \neq h \in H$. Assume that $hK = \{0\}$. Now $(hR)K = h(RK) \subseteq hK = \{0\}$ and hence $AK = \{0\}$, where A is the N-subgroup of H generated by hK. So $HK = \{0\}$, a contradiction. Therefore $hK \neq \{0\}$. Let $0 \neq hk \in hK, k \in K$. We have that $(hk)N$ contains a non-zero distributive element $(hk)x$, $x \in N$. Clearly $(hk)x = h(kx) \in hK$ as required. Finally we prove that $\mathrm{An}_K(hK) = \mathrm{An}_K(H)$. As seen above, we have $0 \neq hk \in hK, k \in K$. Note that $\mathrm{An}_N((hk)N) = \mathrm{An}_N(H)$. Since $(hk)N = h(kN) \subseteq hK \subseteq H$ we have $\mathrm{An}_N(H) \subseteq \mathrm{An}_N(hK) \subseteq \mathrm{An}_N((hk)N)$. Therefore, $\mathrm{An}_N(H) = \mathrm{An}_N(hK) = \mathrm{An}_N((hk)N)$. So $\mathrm{An}_K(H) = K\cap \mathrm{An}_N(H) = K\cap \mathrm{An}_N(hK) = \mathrm{An}_K(hK)$. Now it is also clear that $K \cap (H:0)_N \subseteq (H:0)_K$.

A H-radical $\mathcal{R}$ is *complete* if $K \subseteq \mathcal{R}(N)$ for all ideals K of N for which $\mathcal{R}(K) = K$.

Theorem 3. *The H-radical $\mathcal{P}_{2(r)}$ is complete.*

Proof. Let K be an ideal N and $\mathcal{P}_{2(r)}(K) = K$. Suppose that $K \nsubseteq \mathcal{P}_{2(r)}(N)$. So there is a prime N-group H of type 2 such that $K \nsubseteq (H:0)_N$. By Theorem 2, H is a prime K-group of type 2. This contradicts $\mathcal{P}_{2(r)}(K) = K$. Therefore $K \subseteq \mathcal{P}_{2(r)}(N)$. Hence $\mathcal{P}_{2(r)}$ is complete.

Theorem 4. *Let H be prime K-group of type 2 and K be an ideal of N. Then there is a K-subgroup C of H which is a prime N-group of type 2 and $(C:0)_N \cap K \subseteq (H:0)_K$.*

Proof. Suppose that K is an ideal of N and H is a prime K-group of type 2. We have a distributive element $h_0 \in H$. So $h_0(y+z) = h_0y + h_0z$ for all $y, z \in K$. Clearly $h_0K := \{h_0k \mid k \in K\}$ is a non-zero K-subgroup of H. The claim now is h_0K is an N-group. For this, define $(h_0k)x := h_0(kx)$ for all $h_0k \in K, x \in N$, where $k \in K$. To show that this operation is well defined, suppose that $h_0y = h_0z, y, z \in K$. Let $x \in N$. Now $(h_0(yx) - h_0(zx))k = (h_0(yx))k - (h_0(zx))k = h_0((yx)k) - h_0((zx)k) = h_0(y(xk)) - h_0(z(xk)) = (h_0y)(xk) = (h_0z)(xk) = ((h_0y) - (h_0z))(xk) = 0(xk) = 0$ for all $k \in K$. Therefore $h_0(yx) = h_0(zx)$ and that the operation is well defined. It can be easily verified that h_0K is an N-group. We see now that h_0K is a prime N-group of type 2. Let $0 \neq h_0t \in h_0K, t \in K$.

Since H is a prime K-group of type 2, we get a distributive element $0 \neq h_0y \in (h_0t)K$ $(\subseteq (h_0t)N), y \in K$. Also, for $a, b \in N$, $[(h_0y)(a+b) - ((h_0y)a + (h_0y)b)]k = (h_0y)(ak+bk) - ((h_0y)ak + (h_0y)bk) = ((h_0y)ak + (h_0y)bk) - ((h_0y))ak + (h_0y))bk) = 0$ for all $k \in K$. Therefore $(h_0y)(a+b) = (h_0y)a + (h_0y)b$ and that h_0y is distributive over N as required.

We see now that $\mathrm{An}_N((h_0t)N) = \mathrm{An}_N(h_0K)$. Obviously $\mathrm{An}_N(h_0K) \subseteq \mathrm{An}_N((h_0t)N)$ as $(h_0t)N \subseteq h_0K$. Since H is a prime K-group of type 2, $\mathrm{An}_K((h_0t)K) = \mathrm{An}_K(H)$. We have $((h_0t)K)K = (h_0t)KK \subseteq (h_0t)K \subseteq (h_0t)N$. Let $x \in \mathrm{An}_N((h_0t)N)$. Now $((h_0t)K)Kx = \{0\}$. So $Kx \subseteq \mathrm{An}_K((h_0t)K) = \mathrm{An}_K(H)$ and hence $h_0Kx = \{0\}$, that is $x \in \mathrm{An}_N(h_0K)$. Therefore $\mathrm{An}_N((h_0t)N) \subseteq \mathrm{An}_N(h_0K)$. This gives the required $\mathrm{An}_N((h_0t)N) = \mathrm{An}_N(h_0K)$. Hence $C := h_0K$ is a prime N-group of type 2.
Finally, let $T := (C : 0)_N$. Now $T \cap K$ is an ideal of K. Also $T \cap K \subseteq \mathrm{An}_K(h_0K) = \mathrm{An}_K(H)$. Therefore $T \cap K \subseteq (H : 0)_K$, that is, $(C : 0)_N \cap K \subseteq (H : 0)_K$.

A H-radical $\mathcal{R}$ is *idempotent* if $\mathcal{R}(N) = \mathcal{R}(\mathcal{R}(N))$ for all N.

Theorem 5. *The H-radical $\mathcal{P}_{2(r)}$ is idempotent.*

Proof. Let K be an ideal of N. We claim that $\mathcal{P}_{2(r)}(K) \supseteq K \cap \mathcal{P}_{2(r)}(N)$. Let P be a right prime ideal of K of type 2. There is K-group H of type 2 with $P = (H : 0)$. By Theorem 4, there is a K-subgroup C of the K-group H which is a prime N-group of type 2 and $(C : 0)_N \cap K \subseteq (H : 0)_K$. Moreover $Q := (C : 0)_N$ is a right prime ideal of N type 2 and $P \supseteq K \cap Q$. Therefore $\mathcal{P}_{2(r)}(K) \supseteq K \cap \mathcal{P}_{2(r)}(N)$. Now take $K = P_{2(r)}(N)$. This gives $P_{2(r)}(N) \cap P_{2(r)}(N) \subseteq P_{2(r)}(P_{2(r)}(N))$, that is, $P_{2(r)}(N) \subseteq P_{2(r)}(P_{2(r)}(N))$. The other inclusion is obvious and hence $P_{2(r)}(N) = P_{2(r)}(P_{2(r)}(N))$. So the H-radical $\mathcal{P}_{2(r)}$ is idempotent.

A H-radical $\mathcal{R}$ which is idempotent and complete is a *Kurosh-Amitsur radical* or *KA-radical.*
From Theorems 3 and 5 we have:

Theorem 6. *The H-radical $\mathcal{P}_{2(r)}$ is a KA-radical.*

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