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Inverse Domination in X-Trees and Sibling Trees

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Abstract. A set D of vertices in a graph G is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set in G is called the domination number and is denoted by $\gamma(G)$. Let D be a minimum dominating set of G. If $V-D$ contains a dominating set say $D^{'}$ of G, then $D^{'}$ is called an inverse dominating set with respect to D. The inverse domination number $\gamma^{'}(G)$ is the cardinality of a minimum inverse dominating set of G. A dominating set D is called a connected dominating set or an independent dominating set of G according as the induced subgraph $\langle D\rangle$ is connected or independent in G. The minimum of the cardinalities of the connected dominating sets of G or the independent dominating sets of G is called the connected domination number $\gamma_c(G)$ or the independent domination number $\gamma_i(G)$ respectively. In this paper, we determine the inverse domination numbers in X-Trees and Sibling Trees. We have also determined the independent domination numbers of both the trees and the connected domination number of Sibling Trees. A result on inverse domination number of some classes of Hypertrees is also included.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Domination, Inverse domination, Connected Domination, Hypertrees, Sibling trees

1. Introduction

Domination problems are studied to find sets of representatives to monitor communication or electrical networks and in land surveying where it is necessary to minimize the number of places a surveyor must stand to take height measurements for an entire region [24]. It also plays a vital role in parallel processing and supercomputing, which continue to exert great influence on the development of modern science and engineering. In any network, dominating sets are central sets and hence they play a key role in routing problems associated with parallel computing [22]. A non-empty subset $D \subseteq V(G)$ is a dominating set if each vertex in $V(G)-D$ is adjacent to at least one vertex in D. Such a set with minimum cardinality yields the domination number of a graph G and is denoted by $\gamma(G)$

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[13]. A dominating set D is said to be a connected dominating set or an independent dominating set if the induced subgraph $\langle D \rangle$ is connected or independent in G. The minimum of the cardinalities of the connected dominating sets of G or independent dominating sets of G is called the connected domination number of G denoted by $\gamma_c(G)$ or the independent domination number denoted by $\gamma_i(G)$[18]. The study of connected domination has extensive application in the study of routing problems and virtual backbone based routing in wireless networks[4, 15, 25]. Determining if an arbitrary graph has a dominating set of a given size is a well-known NP-complete problem [10]. Finding optimum dominating sets in networks has always been challenging.

Let D be a minimum dominating set of G. If $V - D$ contains a dominating set say $D^{'}$ of G, then $D^{'}$ is called an inverse dominating set with respect to D. The inverse domination number $\gamma^{'}(G)$ is the order of a smallest inverse dominating set in G [14]. Inverse domination in graphs introduced by Kulli and Sigarkanti [14] in 1991 plays a major role in reliable communication and electrical networks. Suppose D is a minimum dominating set in a graph G and some nodes of D fail, the inverse dominating set plays the role of D. Domke, Dunbar, and Markus (Ars Combin. 72 (2004), 149–160)[6] conjectured that the inverse domination number of G is at most the independence number of G. The above conjecture has been proved for special families of graphs, including claw-free graphs, bipartite graphs, split graphs, very well covered graphs, chordal graphs and cactus graphs in[8]. There are some graphs with equal domination and inverse domination numbers are identified by T.TamizhChelvam [2]. Inverse domination number of circulant graph proved by V.Cynthiya in [3]. Also we identified domination and inverse domination numbers for Wrapped butterfly network, Lollipop graph, Fly graph and Jellyfish graph in [19–21].

In this paper, we determine the inverse domination, independent domination and connected domination numbers in sibling tree networks and also find the domination, independent domination and inverse domination numbers for the X-tree networks. The inverse domination number of some classes of hypertrees is included.

2. Domination and Inverse Domination in X-Trees

Efficient inter-processor communication is one of the crucial issues in multiprocessor systems [1, 7, 9, 12, 23]. Multiple processors are interconnected in a tightly coupled, hierarchical, tree-structured network. An X-tree [5] is a complete binary tree with additional edges to connect consecutive nodes on the same level of the tree so that the vertices on each level induce a path. Edges on such paths are called horizontal edges. Horizontal edges are of two types: sibling edges and cousin edges. A sibling edge denotes a horizontal edge that connects two vertices with the same parent and a cousin edge denotes any of the remaining horizontal edges. Two sibling edges are said to be adjacent if there is exactly one cousin edge between them. The tree edges are addressed as vertical edges. The structural characteristic of X-tree was identified in [5]. The root of $X(k)$ is considered to be at Level 0. Vertices at level k are called leaf vertices. The vertices of $X(k)$ other than the root and the leaf vertices are called internal vertices. A k-level X-tree or a 2^k-leaf X-tree will be denoted by $X(k)$. A k-level X-tree has $2^{k+1} - 1$ vertices and $2^{k+2} - k - 4$ edges

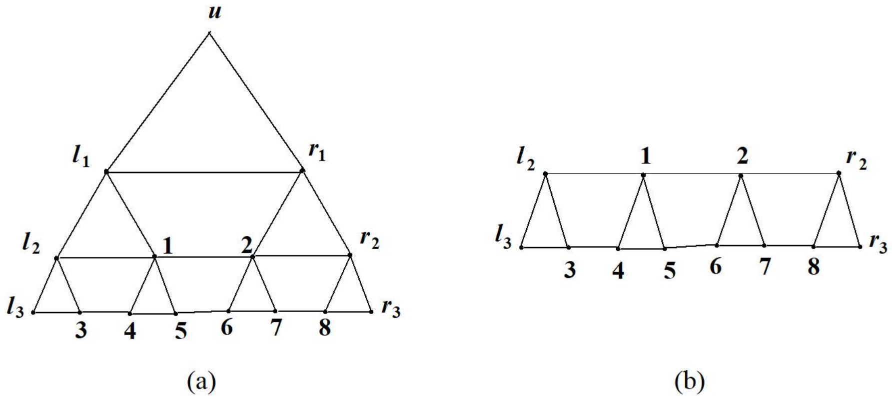

Figure 1: (a) $X(3)$ with labels. (b) Subgraph H in Lemma 2.

[16]. X-trees are fault tolerant variants of the basic tree network and have been the focus of more recent implementation in massively parallel systems.

In this section, we determine the domination number of $X(k),\ k \geq 1$. For convenience, we label the vertices of $X(3)$ as in Figure 1(a).

Lemma 1. *The domination number of* $X(3)$ *is given by* $\gamma(X(3)) = 4$.

Proof. Let D be a dominating set of $X(3)$. Three distinct vertices are necessary in D to dominate the degree 2 vertices u, l_3 and r_3. Refer Figure 1(a). To optimize the cardinality of the neighbourhoods of vertices adjacent to u, l_3 and r_2, we choose l_1, r_2 (or 8) and l_2 (or 3) in D. Another possibility is r_1, l_2 (or 3) and r_2 (or 8). In either case, the number of dominated vertices is 7. To dominate the subgraph induced by the remaining 5 vertices, we require at least one more vertex in D. Thus $|D| \geq 4$. It is easy to verify that $\{l_1, r_2, 3, 6\}$ is a dominating set of $X(3)$. Hence $\gamma(X(3)) = 4$.

Remark 1. $\{r_1, l_2, 5, 8\}$ *is also a minimum dominating set of* $X(3)$.

Lemma 2. *A minimal dominating set of* $X(3)$ *that contains the root of* $X(3)$ *is of cardinality 5.*

Proof. Let D be a dominating set of $X(3)$ that contains u. Then u dominates l_1 and r_1. Consider the subgraph H induced by Level 2 and Level 3 vertices of $X(3)$ which are yet to be dominated. See Figure 1(b). To dominate the degree 2 vertices l_3 and r_3, it is necessary to include 2 vertices of H in D. The possibilities that do not include l_3 or r_3 are $\{l_2, r_2\}, \{l_2, 8\}, \{r_2, 3\}$ and $\{3, 8\}$. In all cases, the remaining vertices to be dominated induce a path of length 3. As there are two pendant vertices, two more vertices from the path are to be included in D. On the otherhand, the possibilities that include l_3 or r_3 are $\{l_3, r_3\}, \{l_3, r_2\}$, and $\{l_2, r_3\}$. In all these cases, the remaining vertices to be dominated are 1,2,4,5,6 and 7. The subgraph induced by these vertices require 2 vertices to be included in D. Thus $|D| = 5$.

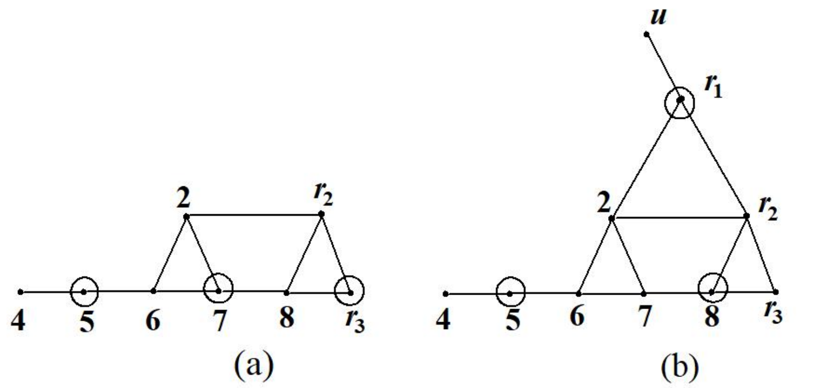

Figure 2: (a) Subgraph H_1 dominated by circled vertices. (b)Subgraph H_2 dominated by circled vertices.

Lemma 3. *A minimal dominating set of* $X(3)$ *containing* (*i*) $\{l_1, l_2\}, (ii)\{l_2, l_3\}$, *or* (*iii*) $\{l_3, l_1\}$ *is of cardinality 5.*

Proof. Let D be a minimal dominating set of $X(3)$. Suppose D contains $\{l_1, l_2\}$ or $\{l_3, l_1\}$. Now if $N[S]$ denotes the closed neighbourhood of a set S of vertices, then $N[\{l_1, l_2\}] = \{u, l_1, l_2, l_3, r_1, 1, 3\} = N[\{l_3, l_1\}]$. The subgraph H_1 induced by the remaining vertices require 3 vertices to be included in D to dominate all vertices of the subgraph. See Figure 2(a). Again, suppose D contains $\{l_2, l_3\}$. We have $N[\{l_2, l_3\}] = \{l_1, l_2, l_3, 1, 3\}$. The subgraph H_2 induced by the remaining vertices require 3 vertices to be included in D to dominate all vertices of the subgraph. See Figure 2(b). In either cases, we have $|D|= 5$.

We now proceed to determine the domination number of $X(k)$, $k \geq 3$. Let H be the subgraph induced by the vertices in Levels k, $k-1$, $k-2$ and $k-3$ of $X(k)$, $k \geq 3$. H has the following properties:
(i) $V(H) = \bigcup_{i=1}^{2^{k-3}} V_i$ such that the subgraph induced by V_i is isomorphic to $X(3)$, $1 \leq i \leq 2^{k-3}$. Let these copies of $X(3)$ be named H_1, H_2,..., $H_{2^{k-3}}$ from left to right as shown in Figure 3. Let the roots of H_i, $1 \leq i \leq 2^{k-3}$ be labeled u_1, u_2,..., $u_{2^{k-3}}$ respectively.

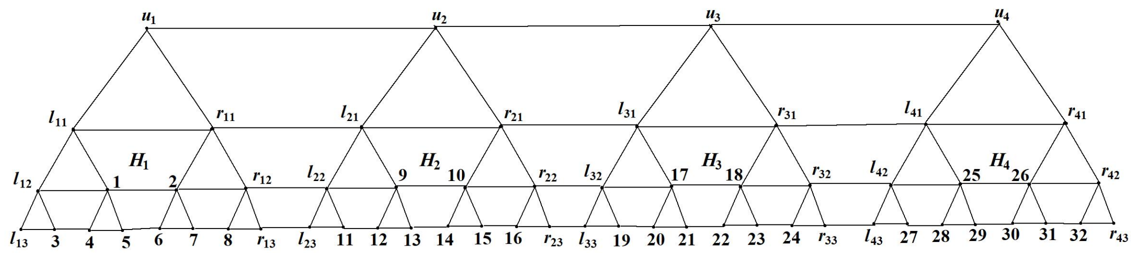

Figure 3: Subgraph induced by vertices in Levels 2,3,4,5 of $X(5)$

Let the leftmost descendants of u_i be labeled l_{i1}, l_{i2} and l_{i3}; similarly let the rightmost descendants of u_i be labeled r_{i1}, r_{i2} and r_{i3}, $1 \leq i \leq 2^{k-3}$. See Figure 3. Let the 2 unlabeled vertices in Level $k-1$ in each H_i be labeled as $(i-1)8+1$ and $(i-1)8+2$ and the 6 unlabeled vertices in Level k of each H_i as $(i-1)8+3$, $(i-1)8+4$,..., $(i-1)8+8$

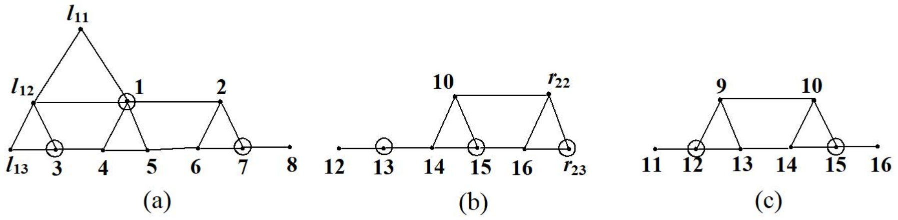

Figure 4: (a) Subgraph H_1 in Case 1. (b) Subgraph H_2 in Case 1.(c) Subgraph H_2 in Case 2.

from left to right, $1 \leq i \leq 2^{k-3}$.

Lemma 4. *Let H be the subgraph of $X(k)$ induced by Levels k, $k-1$, $k-2$ and $k-3$ of $X(k)$, $k \geq 3$. Then $\gamma(H) = 2^{k-1}$.*

Proof. Due to symmetricity and the fact that vertices of H_{i-1} and H_{i+1} can dominate vertices in H_i, it is enough to consider the domination parameter in the subgroup $H_1 \cup H_2 \cup H_3$.

Case 1. Consider $H_1 \cup H_2$. Suppose u_1, r_{11}, r_{12} and r_{13} are dominated by vertices that are not in H_1. Then the subgraph of H_1 induced by the remaining vertices of H_1 require 3 vertices of H_1 in any dominating set D of H. See Figure 4(a). In this case necessarily $\{u_2, l_{21}, l_{22}, l_{23}\} \subseteq D$. To dominate the remaining vertices in H_2, 3 more vertices in H_2 are to be included in D. See Figure 4(b). Thus to dominate $H_1 \cup H_2$ at least 9 vertices of $H_1 \cup H_2$ are to be included in D.

Case 2: Consider $H_1 \cup H_2 \cup H_3$. Suppose $u_2, l_{21}, l_{22}, l_{23}, r_{21}, r_{22}$ and r_{23} are already dominated by vertices that are not in H_2. Then the subgraph of H_2 induced by the remaining vertices of H_2 require 2 vertices of H_2 in D. See Figure 4(c). In this case necessarily $\{r_{11}, r_{12}, r_{13}, l_{31}, l_{32}, l_{33}\} \subseteq D$. Three more vertices in each of H_1 and H_3 are to be included in D to dominate all the vertices in $H_1 \cup H_2 \cup H_3$. Thus to dominate $H_1 \cup H_2 \cup H_3$ at least $6 + 2 + 6 = 14$ vertices of $H_1 \cup H_2 \cup H_3$ are to be included in D.

By virtue of Lemmas 2 and 3 and arguments similar to the above cases, we claim that selecting 4 vertices in each H_i, $1 \leq i \leq 2^{k-3}$ as in Lemma 1 yields a minimum dominating set D of $X(k)$.

By Lemma 1, D is a dominating set of H. To prove that D is a minimum dominating set, we need to consider only $H_1 \cup H_2 \cup H_3$ and consider the dominating sets $D_1 = \{l_{11}, r_{12}, 3, 6\}$, $D_1' = \{r_{11}, l_{12}, 5, 8\}$ of H_1 and $D_3 = \{l_{31}, r_{32}, 19, 22\}$, $D_3' = \{r_{31}, l_{32}, 21, 24\}$ of H_3. If D_1 and D_3' are in D, then we observe that r_{12} dominates l_{22} and l_{32} dominates r_{22}. The subgraph of H_2 induced by the remaining vertices requires more than 4 vertices to dominate the subgraph. See Figure 5(a). On the other hand if D_1 and D_3 are in D then r_{12} dominates l_{22} and l_{31} dominates r_{21}. In this case, the subgraph of H_2 induced by the remaining vertices require 4 vertices to dominate the subgraph . See Figure 5(b). In either case, the number of vertices dominating H_2, considering the already dominated vertices of H_2, is not less than 4.

Thus 4 vertices from each H_i, $1 \leq i \leq 2^{k-3}$ is a minimum count in D. Hence D is a minimum dominating set and $|D| = 4 \times 2^{k-3} = 2^{k-1}$.

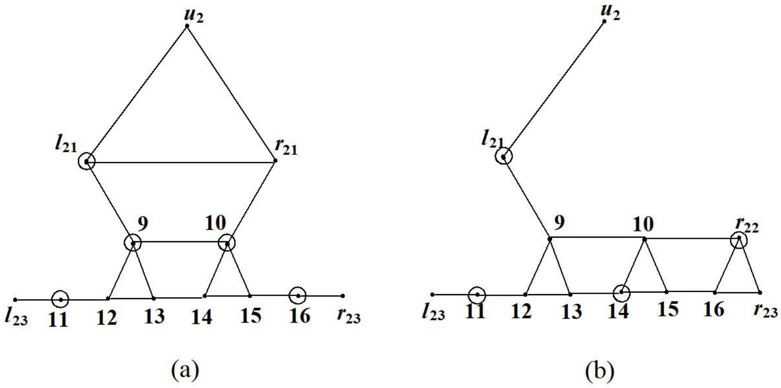

Figure 5: (a) l_{22} and r_{22} of H_2 are already dominated. (b) l_{22} and r_{21} of H_2 are already dominated.

Theorem 1. *Let G be the X-Tree of dimension k and it is denoted by $X(k)$, $k \geq 0$, Then*

$$\gamma(X(k)) = \begin{cases} \frac{2^{k+3}-4}{15}; & k+1 \equiv 0 \ (mod\ 4) \\ \frac{2^{k+3}+7}{15}; & k+1 \equiv 1 \ (mod\ 4) \\ \frac{2^{k+3}-1}{15}; & k+1 \equiv 2 \ (mod\ 4) \\ \frac{2^{k+3}-2}{15}; & k+1 \equiv 3 \ (mod\ 4) \end{cases}$$

Proof. By Lemma 2 and Lemma 4, it is clear that a count of 4 levels starting from the last Level k of $X(k)$ contributes 2^{k-1} vertices to any minimum dominating set D of $X(k)$. Deleting these 4 levels from $X(k)$ yields $X(k-4)$. Applying Lemma 4 and deleting the last 4 levels repeatedly we are left with $X(0)$, $X(1)$, $X(2)$ or $X(3)$ according as $k+1 \equiv 0, 1, 2$ or 3 $(mod\ 4)$ respectively. We have $\gamma(X(0)) = \gamma(X(1)) = 1$, $\gamma(X(2)) = 2$. Thus we compute the domination number of $X(k)$ as follows.
when $k+1 \equiv 0 \ (mod\ 4)$, $|D| = 2^{k-1} + 2^{k-5} + ... + 2^{10} + 2^6 + 2^2 = \frac{2^{k+3}-4}{15}$
when $k+1 \equiv 1 \ (mod\ 4)$, $|D| = (2^{k-1} + 2^{k-5} + ... + 2^7 + 2^5) + 1 = \frac{2^{k+3}+7}{15}$
when $k+1 \equiv 2 \ (mod\ 4)$, $|D| = (2^{k-1} + 2^{k-5} + ... + 2^8 + 2^4) + 1 = \frac{2^{k+3}-1}{15}$ and
when $k+1 \equiv 3 \ (mod\ 4)$, $|D| = (2^{k-1} + 2^{k-5} + ... + 2^{11} + 2^7) + 2 = \frac{2^{k+3}-2}{15}$

Notations: The sets of vertices $\{l_1, r_2, 3, 6\}$ and $\{r_1, l_2, 5, 8\}$ in a copy of $X(3)$ as in Figure 1(a) are disjoint dominating sets of $X(3)$. We refer to them as D-Twin sets.

Theorem 2. *Let $X(k)$ be the X-tree of dimension $k \geq 0$. Then $\gamma'(X(k)) = \gamma(X(k))$, $k \geq 0$.*

Proof. We construct two minimum dominating sets D and D' of $X(k)$, $k \geq 0$ as follows: Include one set of D-Twin vertices in D and the other set of D-Twin vertices in D' from each copy of $X(3)$ considered in Theorem 1. The vertices that are not covered by these copies of $X(3)$ induce $X(0)$, $X(1)$ or $X(2)$ according as $k+1 \equiv 1$, 2 or $3(mod\ 4)$. See Figure 6. When $k+1 \equiv 1$ or $2(mod\ 4)$, no D-Twin set includes l or r. Hence put l in D and r in D'. On the otherhand, when $k+1 \equiv 3(mod\ 4)$, none of the Level 2 vertices in

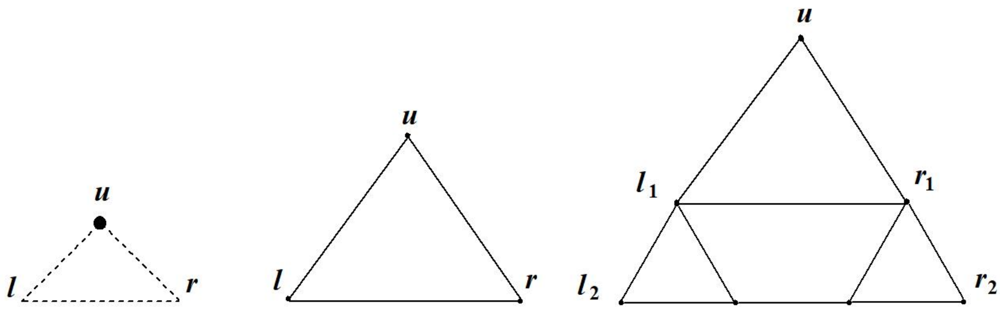

Figure 6: Subgraph induced by leftout vertices when $k+1 \equiv 1$, 2, 3 or $(mod\ 4)$

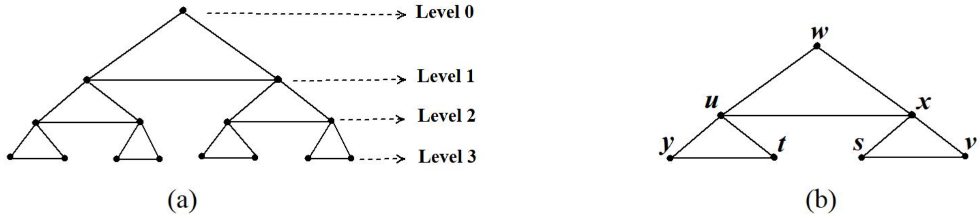

Figure 7: (a) Levels of $ST(3)$. (b) $ST(2)$ with labels.

$X(2)$ is included in any D-Twin set. Hence include $\{l_1, r_2\}$ in D and $\{r_1, l_2\}$ in D'.

Thus D and D' are dominating sets of the same cardinality as the one constructed in Theorem 1. Thus $\gamma'(X(k)) = \gamma(X(k))$, $k \geq 0$.

We note that the dominating set D of $X(k)$, $k \geq 0$ constructed in Theorem 1 is an independent dominating set. Thus we have the following result.

Theorem 3. *Let $X(k)$ be the X-Tree of dimension k. Then $\gamma(X(k)) = \gamma'(X(k)) = \gamma_i(X(k))$.*

3. Inverse and Connected Domination in Sibling Trees

A Sibling tree $ST(r)$ of dimension r is obtained from the complete binary tree $T(r)$ of height r by adding edges called sibling edges joining left and right children of the same parent node. The root node is at Level 0. Level i vertices are the children of vertices in Level $i-1$, $1 \leq i \leq r$. $ST(r)$ has $2^{r+1}-1$ vertices and $3(2^r-1)$ edges, See Figure 7(a).

Notation: Let H be a graph, isomorphic to $ST(2)$ as shown in the Figure 7(b). We call the pair of vertices u and v as irregular D-Twins as u and v together dominate all the vertices of H. Similarly, x and y form another pair of irregular D-Twins. Incidentally, the vertices u and x form a pair of regular of D-Twins. We refer to vertex w as the apex vertex of H.

Remark 2. *The domination number of $ST(r)$, $r \geq 0$, has been determined in [17] making use of regular D-Twins. In this section, we make use of irregular D-Twins, leading to the computation of inverse domination number of $ST(r)$.*

Lemma 5. *Let H be an induced subgraph from the last three levels of $ST(r)$, $r \geq 2$, isomorphic to $ST(2)$. Then $\gamma(H) = 2$.*

Proof. Even if the apex vertex w of H is already dominated by some vertex from $V(ST(r)) - V(H)$, there is no vertex of degree 5 in H that dominates the remaining vertices. Thus $\gamma(H) \geq 2$. Further, any pair of irregular D-Twin vertices in H dominates all vertices in H. Hence $\gamma(H) = 2$.

Theorem 4. *Let G be the sibling tree $ST(r)$ of dimension $r \geq 0$. Then*

$$\gamma(ST(r)) = \begin{cases} \frac{1}{7}(2^{r+2}+3) & ; r \equiv 0 \ (mod\ 3) \\ \frac{1}{7}(2^{r+2}-1) & ; r \equiv 1 \ (mod\ 3) \\ \frac{1}{7}(2^{r+2}-2) & ; r \equiv 2 \ (mod\ 3) \end{cases}$$

Proof. We partition the levels L_0, L_1, L_2, L_3,..., L_r of $ST(r)$ into maximum number of disjoint 3-levels, beginning from L_r. Clearly the left out levels will be L_0 when $r\ (mod\ 3) = 0$; L_0 and L_1 when $r\ (mod\ 3) = 1$. In otherwords, we have the partitioning set P of the levels of $ST(r)$ as follows:

(i) When $r\ (mod\ 3) = 0$, $P = \{L_0\} \cup \bigcup_{i=0}^{\lfloor \frac{r}{3} \rfloor - 1} \{L_{r-3i-2}, L_{r-3i-1}, L_{r-3i}\}$;

(ii) When $r\ (mod\ 3) = 1$, $P = \{L_0, L_1\} \cup \bigcup_{i=0}^{\lfloor \frac{r}{3} \rfloor - 1} \{L_{r-3i-2}, L_{r-3i-1}, L_{r-3i}\}$;

(iii) When $r\ (mod\ 3) = 2$, $P = \bigcup_{i=0}^{\lfloor \frac{r}{3} \rfloor} \{L_{r-3i-2}, L_{r-3i-1}, L_{r-3i}\}$.

We note that L_{r-3i-2} has 2^{r-3i-2} vertices for any i, $0 \leq i \leq \lfloor \frac{r}{3} \rfloor - 1$.

Hence L_{r-3i-2}, L_{r-3i-1}, L_{r-3i} together induce 2^{r-3i-2} disjoint copies of $ST(2)$, $0 \leq i \leq \lfloor \frac{r}{3} \rfloor - 1$. Thus the number α of vertex disjoint copies of $ST(2)$ in $ST(r), r \geq 3$, is

$$\alpha = \sum_{i=0}^{\lfloor \frac{r}{3} \rfloor - 1} 2^{r-3i-2} = \frac{2^{r+1}}{7}\Big(\frac{2^{3\lfloor \frac{r}{3} \rfloor} - 1}{2^{\lfloor \frac{r}{3} \rfloor}}\Big) = \frac{1}{7}(2^{r+1} - 2^{r(mod\ 3)+1})$$

By Lemma 5, each $ST(2)$ contributes 2 vertices to any minimum dominating set D of $ST(r)$. Hence $\gamma(ST(r)) = 2\alpha + 1$ for $r \equiv 0, 1(mod\ 3)$ and $\gamma(ST(r)) = 2\alpha + 2$ for $r \equiv 2(mod\ 3)$. Hence the result.

Theorem 5. *Let G be the sibling tree $ST(r)$ of dimension $r \geq 0$. Then $\gamma'(ST(r)) = \gamma(ST(r))$, $r \geq 0$.*

Proof. We construct two minimum dominating sets D and D' of $ST(r)$, $r \geq 0$ as follows. Include one pair of irregular D-Twin vertices in D and the other pair of D-Twin vertices in D' from each of the copies of $ST(2)$ considered in Theorem 4. The vertices of $ST(r)$ that are not covered by these copies of $ST(2)$ induce $ST(0)$ or $ST(1)$ according as $r \equiv 0$ or $1\ mod\ 3$. Since the children u and x of the root node w (Refer Figure 7(b)) of $ST(r)$ do not belong to any D-Twin set include u in D and x in D'. Thus D and D' are dominating sets of the same cardinality as the one constructed in Theorem 4. Thus $\gamma^{'}(ST(r)) = \gamma(ST(r))$, $r \geq 0$.

We note that the dominating set of $ST(r)$ constructed in Theorem 4 is an independent dominating set. Thus we have the following result.

Theorem 6. *Let G be the Sibling tree $ST(r)$ of dimension $r \geq 0$. Then $\gamma(ST(r)) = \gamma'(ST(r)) = \gamma_i(ST(r))$.*

Theorem 7. *Let G be the Sibling tree $ST(r)$, $r \geq 2$, then $\gamma_c(G) = 2^r - 2$.*

Proof. Let D be a connected dominating set of $ST(r)$, $r \geq 2$. Let v be an arbitrary vertex of $ST(r)$ in Level i, $1 \leq i \leq r-1$. Then v is a cut vertex of $ST(r)$. Let G_v and G'_v be the components of $ST(r) \setminus \{v\}$. Suppose $v \notin D$. Let $D(G_v) = V(G_v) \cap D$ and $D(G'_v) = V(G'_v) \cap D$. Then $D = D(G_v) \cup D(G'_v)$. Choose $x \in D(G_v)$ and $y \in D(G'_v)$. Since D is connected, there exists a path between x and y in D. This path has to necessarily pass through v. But $v \notin D$ and hence there is no path between x and y in D, a contradiction to the connectedness of D. This implies $v \in D$. Thus any internal vertex v in Level i, $1 \leq i \leq r-1$ is a member of D. Obviously, these vertices also dominate G. Hence $\gamma_c(G) = (2^{r+1} - 1) - 1 - 2^r = 2^r - 2$.

4. Inverse Domination in Hypertree Networks

The fundamental skeleton of a hypertree is a complete binary tree T_n of height n. Here the nodes of the tree are numbered as follows: The root node has label 1. The root is supposed to be at *level* 0. Labels of left and right children are formed by appending 0 and 1, respectively to the labels of the parent node. The decimal and binary labels of the hypertree are given in Figure 8. Here the children of the node x are labelled as $2x$ and $2x+1$. Additional links in a hypertree are horizontal and two nodes are joined in the same *level* i of the tree if their label difference is 2^{i-1}. We denote an *n-level* hypertree as $HT(n)$. It has $2^{n+1} - 1$ vertices and $3(2^n - 1)$edges. Hypertree is a multiprocessor interconnection topology which has a frequent data exchange in algorithms such as sorting and Fast and Fourier Transforms (FFT's) [11].

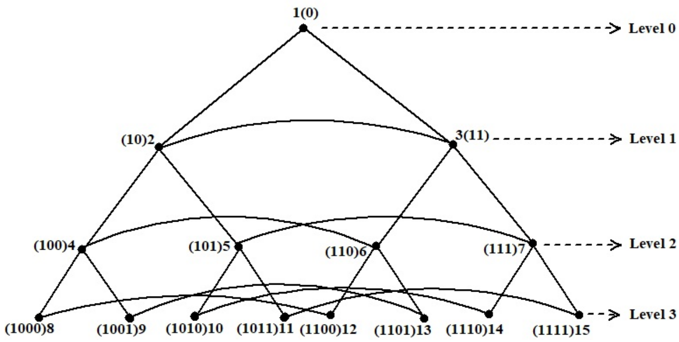

Figure 8: $HT(3)$ with decimal and binary labels.

Following the lines of Theorem 4, we have obtained $\gamma'(HT(n))$ for $n \equiv 0, 1(mod\ 3)$. An independent proof has been given in [17] to obtain $\gamma(HT(n))$.

Theorem 8. *Let G be the hypertree $HT(r)$ of dimension $r \geq 0$. Then*

$$\gamma'(HT(r)) = \gamma(HT(r)) = \begin{cases} \frac{1}{7}(2^{r+2}+3); & r \equiv 0 \ (mod\ 3) \\ \frac{1}{7}(2^{r+2}-1); & r \equiv 1 \ (mod\ 3) \end{cases}$$

Conjecture 8.1. *Let G be the hypertree $HT(r)$, $r \geq 0$. Then*
$\gamma'(HT(r)) = \gamma(HT(r)) + 1 = \frac{1}{7}(2^{r+2}-2)+1$, *for* $r \equiv 2 \ (mod\ 3)$.

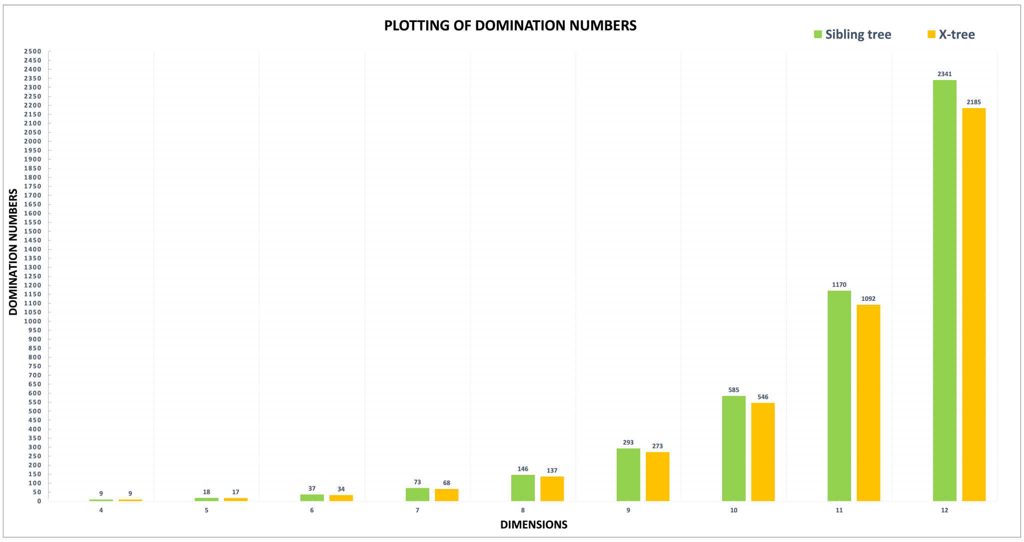

Figure 9: Plotting of Domination numbers.

5. Conclusion

In this paper, we have obtained domination, inverse domination and independent domination numbers of the X-Tree Networks. Similarly, we have obtained the domination, inverse domination, independent domination and connected domination numbers of Sibling Tree Networks. We have also obtained the inverse domination number of a few classes of Hypertree networks. All its tree networks have the same basic structure as the complete binary tree and the number of vertices in $X(k)$ and $ST(k)$ and $HT(k)$ $k \geq 0$, are equal. Hence, it is worth comparing the domination numbers of $X(k)$ and $ST(k)$ of the same dimension k. See Figure 9. We conclude that, as far as domination parameter is concerned, $X(k)$ is a better architecture than $ST(k)$, $k \geq 0$.

6. Future work

It is worth studying the domination and inverse domination numbers of architectures like Benes Networks and Hyper-Butterfly Networks. It would be an interesting line of research to explore domination parameters in tree-like architectures like Christmas Trees and Slim Trees.

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