EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 2, 2024, 1129-1145
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Parapseudo-complementation on Paradistributive Latticoids

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Abstract. In this paper, we introduce the concept of a parapseudo-complementation in a paradistributive latticoid(PDL) and investigate its elementary properties. We demonstrate the independence of the axioms related to its definition, highlighting the flexibility of this concept. Additionally, we establish necessary conditions for a PDL with a minimal element to be parapseudo-complemented and explore the properties required for parapseudo-complementation to be equationally definable. Moreover, we establish a one-to-one correspondence between the set of all minimal elements and the set of all parapseudo-complementations.

2020 Mathematics Subject Classifications: 06D99

Key Words and Phrases: Parapseudo-complementation, Paradistributive Latticoid(PDL), Minimal element, Filter, Boolean algebra.

1. Introduction

In the realm of algebraic structures, a variety of algebras, including lattices and Boolean algebras, provide generalizations of the concept of complement. Within this context, the notion of pseudo-complementation has been extended to encompass a wide range of semigroups, referred to as pseudo-complemented semilattices. The study of pseudo-complements in distributive lattices was first introduced and extensively researched by G. Birkhoff[2] and Orrin Frink[5]. I. Chajda et al.[3, 4] introduced the so-called sectionally

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i2.5042

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pseudocomplemented lattices and posets and demonstrated their roles in algebraic structures. They defined congruences and filters in their structures, derived mutual relationship between them and described basic properties of congruences in strongly sectionally pseudocomplemented posets. Later, the concept of a relative pseudocomplemented lattice was introduced by R. P. Dilworth(Dilworth 1939) where he interpreted relative pseudocomplement as logical connective implication. M. Mandelker[6] expanded on this concept by introducing and investigating the notions of relative annihilators in lattices and relatively pseudo-complemented lattices. Mandelker proposed the annihilator (a, b) of element a relative to b as a natural generalization of the pseudo-complement $a * b$. It represents the set of elements x satisfying $a \cap x \leq b$. The greatest element of (a, b), if it exists, is defined as the relative pseudo-complement $a * b$. Thus, a lattice is considered relatively pseudo-complemented if each annihilator has a greatest element, making it a principal ideal. A dual weakly complemented lattice was introduced by Wille[15] and Kwuida[13]. Their contributions connected to the notion of annihilators of distributive dual weakly complemented lattice with a certain type of ideals called as closed ideals and later proved that closed ideals depend on the dual weak complementation operation on the lattice of all ideals $I(L)$ of L. Eman Ghareeb Rezk[9] introduced the concept of closed ideals and annihilators over the class of distributive dual weakly complemented lattices. The connection between closed ideals and annihilators in this class was obtained. M. S. Rao[7] introduced the concept of δ-ideals in pseudo-complemented distributive lattices and then Stone lattices are characterized in terms of δ-ideals. Further the properties of normal ideals of pseudo-complemented distributive lattices and the characterization of disjunctive lattices with the help of normal ideals was studied by M. S. Rao et al.[8]

The theory of pseudo-complements for posets was developed by P. V. Venkatanarasimhan [14], who introduced the concepts of ideals and semi-ideals and derived several results that paved the way for research on pseudo-complements in distributive lattices. These findings revealed that if every element in a pseudo-complemented semilattice or dual semilattice is normal, the algebra can be classified as a Boolean algebra. This conclusion led to new proofs for well-known theorems, such as the existence of maximal ideals in posets and the product of all maximal dual ideals being the dual ideal of dense components in a poset with a zero element.

U. M. Swamy and G. C. Rao[11] introduced the concept of an Almost Distributive Lattice (ADL) as a unifying abstraction for various lattice-theoretic generalizations of Boolean algebras and Boolean rings. Furthermore, in collaboration with G. N. Rao[12], they extended the concept of pseudo-complementation to almost distributive lattices and demonstrated that the class of pseudo-complemented ADLs is equationally definable. They also explored the relationship between annihilator ideals and pseudo-complementations in an ADL and established a one-to-one correspondence between pseudo-complementations and maximal elements in an ADL assuming the existence of one pseudo-complementation.

Recently, R. K. Bandaru et al.[1] introduced the concept of a Paradistributive Latticoid(PDL) as a generalization of distributive lattice and investigated its properties. They introduced the notions of an ideal and a filter in a PDL and studied their properties. They proved a subdirect representation theorem for associative PDLs which simplifies many results in PDLs.

The main objective of this paper is to introduce the concept of parapseudo-complementation in a Paradistributive Latticoid (PDL) and investigate its properties. We provide examples to illustrate the independence of the axioms defined for parapseudo-complementation. Specifically, we prove that a PDL V is parapseudo-complemented if and only if the annihilator filter $[\rho]^{\bullet}$ is a principal filter for any $\rho \in V$. Additionally, we establish a one-to-one correspondence between the set of all minimal elements and the set of all parapseudo-complementations in V. Finally, we demonstrate that the corresponding Boolean algebras $V^{\blacklozenge}$ and $V^{\lozenge}$ are isomorphic.

The remainder of this paper is structured as follows: section 1 provides a brief introduction to the concept of pseudo-complementation, followed by preliminaries in section 2. Section 3 presents the definition of parapseudo-complementation on a PDL, highlighting the independence of the axioms through illustrative examples. In section 4, we delve into the heart of our investigation by proving the necessary and sufficient conditions for a Paradistributive Latticoid (PDL) with a minimal element to be parapseudo-complemented. Additionally, we establish that the class of parapseudo-complemented PDLs is equationally definable, providing a solid foundation for further exploration of this concept.

Moving forward to section 5, we focus on the independence of the parapseudo-complementation $\blacklozenge$ within the corresponding Boolean algebra $V^{\blacklozenge}$. By presenting a rigorous proof, we demonstrate that the structure and properties of $V^{\blacklozenge}$ are not affected by the specific choice of parapseudo-complementation. This insight enhances our understanding of the relationship between parapseudo-complementation and the underlying Boolean algebra.

In summary, through our research, we establish the necessary and sufficient conditions for parapseudo-complementation in PDLs, highlight the equationally definable nature of parapseudo-complemented PDLs, and demonstrate the independence of the parapseudo-complementation within the associated Boolean algebra. This study contributes to a deeper understanding of parapseudo-complementation and its implications in the context of Paradistributive Latticoids.

2. Preliminaries

First we recall the necessary definitions and results from [1].

Definition 1. *An algebra $(V, \vee, \wedge, 1)$ of type (2,2,0) is called a Paradistributive Latticoid, abbreviated as PDL, if it assures the subsequent axioms:*
$(LD\vee)$ $\kappa_1 \vee (\kappa_2 \wedge \kappa_3) = (\kappa_1 \vee \kappa_2) \wedge (\kappa_1 \vee \kappa_3)$.
$(RD\vee)$ $(\kappa_1 \wedge \kappa_2) \vee \kappa_3 = (\kappa_1 \vee \kappa_3) \wedge (\kappa_2 \vee \kappa_3)$.
(L_1) $(\kappa_1 \vee \kappa_2) \wedge \kappa_2 = \kappa_2$.
(L_2) $(\kappa_1 \vee \kappa_2) \wedge \kappa_1 = \kappa_1$.
(L_3) $\kappa_1 \vee (\kappa_1 \wedge \kappa_2) = \kappa_1$.
(I_1) $\kappa_1 \vee 1 = 1$.
for any $\kappa_1, \kappa_2, \kappa_3 \in V$.

For any $\kappa_1, \kappa_2 \in V$, we say that κ_1 is less than or equal to κ_2 and write $\kappa_1 \leq \kappa_2$ if

$\kappa_1 \wedge \kappa_2 = \kappa_1$ or equivalently $\kappa_1 \vee \kappa_2 = \kappa_2$ and it can be easily observed that $\leq$ is a partial order on V. The element 1, in Definition 1, is called the greatest element.

Example 1. *Let V be a non-empty set. Fix some element $\varrho_0 \in V$. Then, for any $\rho, \varrho \in V$ define $\vee$ and $\wedge$ on V by*

$$\rho \vee \varrho = \begin{cases} \rho & \varrho \neq \varrho_0 \\ \varrho_0 & \varrho = \varrho_0 \end{cases}$$

and

$$\rho \wedge \varrho = \begin{cases} \varrho & \varrho \neq \varrho_0 \\ \rho & \varrho = \varrho_0 \end{cases}$$

Then $(V, \vee, \wedge, \varrho_0)$ is a disconnected PDL with ϱ_0 as its greatest element.

Lemma 1. *Let $(V, \vee, \wedge, 1)$ be a PDL. Then for any $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in V$, we have the following:*
(1) $1 \wedge \kappa_1 = \kappa_1$.
(2) $\kappa_1 \wedge 1 = \kappa_1$.
(3) $1 \vee \kappa_1 = 1$.
(4) $(\kappa_1 \vee \kappa_2) \wedge \kappa_3 = (\kappa_1 \wedge \kappa_3) \vee (\kappa_2 \wedge \kappa_3)$.
(5) $\kappa_1 \vee (\kappa_2 \wedge \kappa_3) = \kappa_1 \vee (\kappa_3 \wedge \kappa_2)$.
(6) *The operation $\vee$ is associative in V i.e.,* $\kappa_1 \vee (\kappa_2 \vee \kappa_3) = (\kappa_1 \vee \kappa_2) \vee \kappa_3$.
(7) *The set $V_{\mu_1} = \{\kappa_1 \in V \mid \mu_1 \leq \kappa_1\} = \{\mu_1 \vee \kappa_1 \mid \kappa_1 \in V\}$ is a distributive lattice under induced operations $\vee$ and $\wedge$ with μ_1 as its least element.*
(8) $\kappa_4 \vee \{\kappa_1 \wedge (\kappa_2 \wedge \kappa_3)\} = \kappa_4 \vee \{(\kappa_1 \wedge \kappa_2) \wedge \kappa_3\}$.
(9) $\kappa_1 \vee (\kappa_2 \vee \kappa_3) = \kappa_1 \vee (\kappa_3 \vee \kappa_2)$.
(10) $\kappa_1 \vee \kappa_2 = 1$ *if and only if* $\kappa_2 \vee \kappa_1 = 1$.
(11) $\kappa_1 \wedge \kappa_2 = \kappa_2 \wedge \kappa_1$ *whenever* $\kappa_1 \vee \kappa_2 = 1$.

Theorem 1. *An algebra $(V, \vee, \wedge, 1)$ of type $(2, 2, 0)$ is a PDL if and only if it satisfies the following:*
$(LD\vee)$ $\kappa_1 \vee (\kappa_2 \wedge \kappa_3) = (\kappa_1 \vee \kappa_2) \wedge (\kappa_1 \vee \kappa_3)$
$(RD\vee)$ $(\kappa_1 \wedge \kappa_2) \vee \kappa_3 = (\kappa_1 \vee \kappa_3) \wedge (\kappa_2 \vee \kappa_3)$
$(RD\wedge)$ $(\kappa_1 \vee \kappa_2) \wedge \kappa_3 = (\kappa_1 \wedge \kappa_3) \vee (\kappa_2 \wedge \kappa_3)$
(L_1) $(\kappa_1 \vee \kappa_2) \wedge \kappa_2 = \kappa_2$
(L_3) $\kappa_1 \vee (\kappa_1 \wedge \kappa_2) = \kappa_1$
(I_1) $\kappa_1 \vee 1 = 1$
(I_2) $1 \wedge \kappa_1 = \kappa_1$.
for all $\kappa_1, \kappa_2, \kappa_3 \in V$.

Definition 2. *A Paradistributive Latticoid $(V, \vee, \wedge, 1)$ is said to be associative if it satisfies the following condition*

$$\kappa_1 \wedge (\kappa_2 \wedge \kappa_3) = (\kappa_1 \wedge \kappa_2) \wedge \kappa_3$$

for all $\kappa_1, \kappa_2, \kappa_3 \in V$.

Let V be a PDL. Then, an element $\mu_1 \in V$ is said to be a minimal element if for any $u \in V$, $u \leq \mu_1 \Rightarrow u = \mu_1$.

Lemma 2. *Let V be a PDL. Then, for any $\mu_1 \in V$, the following are equivalent:*
(1). *μ_1 is minimal*
(2). *$\kappa_1 \wedge \mu_1 = \mu_1$ for all $\kappa_1 \in V$*
(3). *$\kappa_1 \vee \mu_1 = \kappa_1$ for all $\kappa_1 \in V$.*

Definition 3. *A non-empty subset F of a PDL V is said to be a filter if it satisfies the following:*

$$\kappa_1, \kappa_2 \in F \Rightarrow \kappa_1 \wedge \kappa_2 \in F.$$
$$\kappa_1 \in F, \mu_1 \in V \Rightarrow \mu_1 \vee \kappa_1 \in F.$$

Theorem 2. *Let S be a non-empty subset of V. Then*

$$[S) \;=\; \{\kappa_1 \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, \kappa_1 \in V, 1 \leq i \leq n \text{ and } n \text{ is a positive integer } \}$$

is the smallest filter of V containing S.

Lemma 3. *Let V be a PDL and F be a filter of V. Then for any $\kappa_1, \kappa_2 \in V$, we have the following:*
(1) *$[\kappa_1) = \{\rho \vee \kappa_1 \mid \rho \in V\}$.*
(2) *$\kappa_1 \in [\kappa_2)$ if and only if $\kappa_1 = \kappa_1 \vee \kappa_2$ for all $\kappa_1, \kappa_2 \in V$.*
(3) *$\kappa_1 \vee \kappa_2 \in F$ if and only if $\kappa_2 \vee \kappa_1 \in F$.*
(4) *$[\kappa_1 \vee \kappa_2) = [\kappa_2 \vee \kappa_1)$.*
(5) *$[\kappa_1 \wedge \kappa_2) = [\kappa_2 \wedge \kappa_1) = [\kappa_1) \vee [\kappa_2)$.*

Theorem 3. *The collection $F(L)$ of all filters of a PDL V forms a distributive lattice under set inclusion, in which, the glb and lub of any F and G are given respectively by $F \wedge G = F \cap G$ and $F \vee G = \{\kappa_1 \wedge \kappa_2 \mid \kappa_1 \in F \text{ and } \kappa_2 \in G\}$.*

Definition 4. *By a homomorphism of a PDL $(V, \vee, \wedge, 1)$ into a PDL $(V', \vee', \wedge', 1')$, we mean, a mapping $f : V \to V'$ satisfying the following:*
(1) $f(\mu_1 \vee \mu_2) = f(\mu_1) \vee' f(\mu_2)$
(2) $f(\mu_1 \wedge \mu_2) = f(\mu_1) \wedge' f(\mu_2)$
(3) $f(1) = f(1')$.

3. Parapseudo-Complementation on Paradistributive Latticoids

In this section, we define a parapseudo-complementation on a PDL and present some fundamental findings which helps in verification of the axioms independency.

Definition 5. *Let $(V, \vee, \wedge, 1)$ be a Paradistributive Latticoid (PDL) and consider a unary operation denoted as $\rho \mapsto \rho^{\blacklozenge}$ on V. This operation is called a parapseudo-complementation on V if it satisfies the following conditions:*

(PPC_1) *If $\rho \vee \varrho = 1$, then $\rho \vee \varrho^{\blacklozenge} = \rho$.*
(PPC_2) $\rho \vee \rho^{\blacklozenge} = 1$.
(PPC_3) $(\rho \wedge \varrho)^{\blacklozenge} = \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}$.

If there is no ambiguity about the parapseudo-complementation on a PDL V, we can say that V is a parapseudo-complemented PDL (PPDL). In the case of a distributive lattice with one, PPC_3 becomes a consequence of PPC_1 and PPC_2. However, in the case of PDLs, PPC_1, PPC_2, and PPC_3 are independent. Now, we provide examples to demonstrate the independence of these axioms.

Example 2. *Consider a PDL V with at least two elements. Let's define the unary operation $\rho^{\blacklozenge} = 1$ for all $\rho \in V$. We will show that V satisfies (PPC_2) and (PPC_3) but fails to satisfy (PPC_1).*

(PPC_2): For any $\rho \in V$, we have $\rho \vee \rho^{\blacklozenge} = \rho \vee 1 = 1$. Hence, (PPC_2) is satisfied.

(PPC_3): Let $\rho, \varrho \in V$. We have $(\rho \wedge \varrho)^{\blacklozenge} = 1$ and $\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} = 1 \vee 1 = 1$. Therefore, (PPC_3) is satisfied.

Now, we examine (PPC_1). Suppose there exists $\varrho \in V$ such that $\varrho \neq 1$. We have $\varrho \vee 1 = 1$. However, $\varrho \vee 1^{\blacklozenge} = \varrho \vee 1 = 1 \neq \varrho$. Therefore, (PPC_1) is not satisfied when ϱ is not equal to 1.

In conclusion, the PDL V with the unary operation $\rho^{\blacklozenge} = 1$ satisfies (PPC_2) and (PPC_3) but fails to satisfy (PPC_1) when V has at least two elements.

Example 3. *Let V be a bounded distributive lattice with bounds $0 \neq 1$. Define $\rho^{\blacklozenge} = 0$ for all $\rho \in V$. We will show that V satisfies (PPC_1) and (PPC_3) but fails to satisfy (PPC_2).*

(PPC_1): Suppose $\rho \vee \varrho = 1$, where $\rho, \varrho \in V$. We have $\rho \vee \varrho^{\blacklozenge} = \rho \vee 0 = \rho$. Therefore, (PPC_1) is satisfied.

(PPC_3): For any $\rho, \varrho \in V$, we have $(\rho \wedge \varrho)^{\blacklozenge} = 0$ and $\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} = 0 \vee 0 = 0$. Thus, (PPC_3) is satisfied.

Now we examine (PPC_2). Suppose $0 \in V$. We have $0 \vee 0^{\blacklozenge} = 0 \vee 0 = 0$, but we require $0 \vee 0^{\blacklozenge} = 1$. Therefore, (PPC_2) is not satisfied in this case.

In conclusion, the bounded distributive lattice V with the unary operation $\rho^{\blacklozenge} = 0$ satisfies (PPC_1) and (PPC_3) but fails to satisfy (PPC_2) when V contains 0 as an element.

Example 4. *Let V be a disconnected PDL with atleast two elements other than 1. Then $(V^3, \vee, \wedge, 1)$ is a PDL, where $\vee, \wedge$ are defined co-ordinate wise. Now, for any $\rho \in V^3$, we write $|\rho|$ for the number of non-units in ρ. Define $\blacklozenge$ on V^3 as follows:*
For any $\rho \in V^3$, define $\rho^{\blacklozenge} = (\rho_1^{\blacklozenge}, \rho_2^{\blacklozenge}, \rho_3^{\blacklozenge})$ where, for $i = 1, 2, 3$

$$\rho_i^{\blacklozenge} = \begin{cases} 1 & \rho_i \neq 1 \\ 0 & \rho_i = 1 \quad |\rho| = 2 \\ 2 & \rho_i = 1 \quad |\rho| = 1, |\rho| > 2 \end{cases}$$

and $1^{\blacklozenge} = (2, 2, 2)$.
Then $(V^3, \vee, \wedge, 1)$ is a PPDL which satisfies (PPC_1) and (PPC_2) but fails to satisfy (PPC_3). For, if $\rho = (0, 1, 1)$ and $\varrho = (1, 0, 1)$, then $\rho^{\blacklozenge} = (1, 2, 2)$ and $\varrho^{\blacklozenge} = (2, 1, 2)$ and $\rho \wedge \varrho = (0, 0, 1)$. Hence $(\rho \wedge \varrho)^{\blacklozenge} = (1, 1, 0)$ and $\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} = (1, 1, 2)$. Therefore, $(\rho \wedge \varrho)^{\blacklozenge} \neq \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}$.

Lemma 4. *Let* $(V, +, \cdot, 0, 1)$ *be a commutative regular ring with unity and let* ρ_0 *be the unique idempotent element in* V *such that* $\rho V = \rho_0 V$. *Now, for any* $\rho, \varrho \in V$, *define*
(1) $\rho \vee \varrho = \varrho_0 \rho$
(2) $\rho \wedge \varrho = \rho + \varrho - \varrho_0 \rho$
(3) $\rho^{\blacklozenge} = 1 - \rho_0$.
Then $(V, \vee, \wedge, 0)$ *is a PDL in which* 1 *is a minimal element and* $\blacklozenge$ *is a parapseudo-complementation on* V.

Proof. It is clear that $(V, \vee, \wedge, 0)$ is a PDL. Note that, for any $\rho, \varrho \in V$, $(\rho\varrho)_0 = \rho_0 \varrho_0$ and $(\rho + \varrho - \rho_0 \varrho)_0 = \rho_0 + \varrho_0 - \rho_0 \varrho_0$. Also, $0_0 = 0$ and $1_0 = 1$. Now, we prove that $\blacklozenge$ is a parapseudo-complementation on V. Let $\rho, \varrho \in V$ and $\rho \vee \varrho = 0$. Then $\varrho_0 \rho = 0$ and

$$\begin{aligned} \rho \vee \varrho^{\blacklozenge} &= (\varrho^{\blacklozenge})_0 \rho \\ &= \varrho^{\blacklozenge} \rho \\ &= (1 - \varrho_0)\rho \\ &= \rho \end{aligned}$$

Also, $\rho \vee \rho^{\blacklozenge} = \rho \vee (1 - \rho_0) = (1 - \rho_0)_0 \rho = (1 - \rho_0)\rho = \rho - \rho = 0$.
Let $\rho, \varrho \in V$. Then,

$$\begin{aligned} \rho^{\blacklozenge} \vee \varrho^{\blacklozenge} &= (1 - \rho_0) \vee (1 - \varrho_0) \\ &= (1 - \varrho_0)_0 (1 - \rho_0) \\ &= (1_0 - \varrho_0)(1 - \rho_0) \\ &= (1 - \varrho_0)(1 - \rho_0) \\ &= 1 - \rho_0 - \varrho_0 + \rho_0 \varrho_0 \\ &= 1 - (\rho_0 + \varrho_0 - \rho_0 \varrho_0) \\ &= 1 - (\rho + \varrho - \varrho_0 \rho)_0 \\ &= 1 - (\rho \wedge \varrho)_0 \\ &= (\rho \wedge \varrho)^{\blacklozenge} \end{aligned}$$

Therefore, $\blacklozenge$ is a parapseudo-complementation on V.

Example 5. *Let* $(V, \vee, \wedge, 1)$ *be a disconnected PDL. Fix* $\rho_1 \neq 1 \in V$ *and define* $\blacklozenge$ *on* V *as follows:*

$$\mu_1^{\blacklozenge} = \begin{cases} 1 & \mu_1 \neq 1 \\ \rho_1 & \mu_1 = 1 \end{cases}$$

Then $\blacklozenge$ *is a parapseudo-complementation on* V.

In the case of a distributive lattice the dual pseudo-complementation, if exists, is unique. But, in a PDL there can be several parapseudo-complementations. For, in Example 5, we get one parapseudo-complementation on V corresponding to each $\rho_1 (\neq 1) \in V$.

Theorem 4. *Every finite PDL is parapseudo-complemented.*

Proof. Let V be a finite PDL. Then V has a minimal element, say m. Now, we prove that V is parapseudo-complemented. For this, define $\lozenge$ on V by $\rho^{\lozenge} = (m \vee \rho)^{\blacklozenge}$, where $(m \vee \rho)^{\blacklozenge}$ is the dual pseudo-complement of $m \vee \rho$ in the finite distributive lattice $[m, 1]$ and $\rho \in V$. We prove $\lozenge$ is parapseudo-complementation on V. Let $\rho, \varrho \in V$. Then $\rho^{\lozenge} \vee \rho = \rho^{\lozenge} \vee (\rho \vee m) = (m \vee \rho)^{\blacklozenge} \vee (m \vee \rho) = 1$. Suppose $\varrho \vee \rho = 1$. Then $(m \vee \rho) \vee (m \vee \varrho) = 1$ and $m \vee \varrho \in [m, 1]$. Hence $(m \vee \rho)^{\blacklozenge} \leq (m \vee \varrho)$. Thus $\rho^{\lozenge} \leq (m \vee \varrho)$. Now, $\varrho \leq \varrho \vee \rho^{\lozenge} \leq \varrho \vee m \vee \varrho = \varrho$. Therefore, $\varrho \vee \rho^{\lozenge} = \varrho$. Let $\rho, \varrho \in V$. Then $(\rho \wedge \varrho)^{\lozenge} = (m \vee (\rho \wedge \varrho))^{\blacklozenge} = ((m \vee \rho) \wedge (m \vee \varrho))^{\blacklozenge} = (m \vee \rho)^{\blacklozenge} \vee (m \vee \varrho)^{\blacklozenge} = \rho^{\lozenge} \vee \varrho^{\lozenge}$.

Let $(V, \vee, \wedge, 1)$ be a PDL. By an interval in V, we mean the set $[\rho, \varrho] = \{\mu_1 \in V | \rho \leq \mu_1 \leq \varrho\}$ for some $\rho, \varrho \in V$ such that $\rho \leq \varrho$. Clearly, $[\rho, \varrho]$ is closed under $\vee, \wedge$. Since $[\rho, \varrho]$ is a PDL with ρ as its zero element and ϱ as its greatest element, every interval $[\rho, \varrho]$ is a bounded distributive lattice.

Definition 6. *A PDL $(V, \vee, \wedge, 1)$ is said to be relatively complemented if every interval $[\rho, \varrho]$, $\rho \leq \varrho$ in V is a complemented lattice.*

Theorem 5. *Let $(V, \vee, \wedge, 1)$ be a PDL with 1. Then the following are equivalent:*
(1). V is relatively complemented.
(2). V is sectionally complemented, i.e the interval $[\rho, 1]$, $\rho \in V$ is a complemented lattice.
(3). Given $\rho, \varrho \in V$, there exists a unique $\mu_1 \in V$ such that $\mu_1 \vee \rho = 1$ and $\mu_1 \wedge \rho = \varrho \wedge \rho$.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3) : Assume (2) and let $\rho, \varrho \in V$. So that the interval $[\varrho \wedge \rho, 1]$ is complemented and $\rho \in [\varrho \wedge \rho, 1]$. If μ_1 is the complement of ρ in $[\varrho \wedge \rho, 1]$, then $\mu_1 \wedge \rho = \varrho \wedge \rho$ and $\mu_1 \vee \rho = 1$. Since any $\mu_2 \in V$ satisfies $\mu_2 \wedge \rho = \varrho \wedge \rho$ and $\mu_2 \vee \rho = 1$ belongs to $[\varrho \wedge \rho, 1]$. Therefore, $[\varrho \wedge \rho, 1]$ is a boolean algebra and hence the uniqueness of μ_1 follows.
(3) $\Rightarrow$ (1) : Assume (3). Let $\rho, \varrho \in V$ such that $\varrho \leq \rho$ and let $\mu_1 \in [\varrho, \rho]$. Then by (3), there exists $\mu_2 \in V$ such that $\mu_1 \vee \mu_2 = 1$, $\mu_2 \wedge \mu_1 = \varrho \wedge \mu_1 = \varrho$. It is clear that $\varrho = \mu_2 \wedge \mu_1 = \mu_1 \wedge \mu_2 \leq \mu_2$. Now, we prove that the element $\mu_2 \wedge \rho \in [\varrho, \rho]$ and $\mu_2 \wedge \rho$ is complement of μ_1 in $[\varrho, \rho]$.
Clearly $\mu_2 \wedge \rho \leq \rho$. Now, $\varrho \vee (\mu_2 \wedge \rho) = (\varrho \vee \mu_2) \wedge (\varrho \vee \rho) = \mu_2 \wedge \rho$. Hence $\mu_2 \wedge \rho \in [\varrho, \rho]$.
Now, $\mu_1 \vee (\mu_2 \wedge \rho) = (\mu_1 \vee \mu_2) \wedge (\mu_1 \vee \rho) = 1 \wedge (\mu_1 \vee \rho) = \rho$ and

$$
\begin{aligned}
\varrho &= \mu_2 \wedge \mu_1 \\
&= [\mu_2 \vee (\mu_2 \wedge \rho)] \wedge \mu_1. \\
&= (\mu_2 \wedge \mu_1) \vee [(\mu_2 \wedge \rho) \wedge \mu_1]. \\
&= (\mu_2 \wedge \mu_1) \vee [\mu_1 \wedge (\mu_2 \wedge \rho)]. \\
&= [(\mu_2 \wedge \mu_1) \vee \mu_1] \wedge [(\mu_2 \wedge \mu_1) \vee (\mu_2 \wedge \rho)]. \\
&= \mu_1 \wedge [\varrho \vee (\mu_2 \wedge \rho)]. \\
&= \mu_1 \wedge [(\varrho \vee \mu_2) \wedge (\varrho \vee \rho)]. \\
&= \mu_1 \wedge (\mu_2 \wedge \rho).
\end{aligned}
$$

Therefore, $\mu_2 \wedge \rho$ is the complement of μ_1 in $[\varrho, \rho]$. Hence V is relatively complemented.

Note that every relatively complemented PDL is an associative PDL.

Theorem 6. *Let V be a relatively complemented PDL with a minimal element m_1. Then V is parapseudo-complemented PDL.*

Proof. Let V be a relatively complemented PDL with a minimal element m_1. For any $\rho \in V$, let $\rho^{\blacklozenge}$ be the complement of $\rho \in [m_1 \wedge \rho, 1]$. Now, we prove that $\blacklozenge$ is parapseudo-complementation on V. Clearly, for any $\rho \in V$, we have $\rho \vee \rho^{\blacklozenge} = 1$. Let $\varrho \in V$ be such that $\varrho \vee \rho = 1$. Then, $\varrho \vee \rho^{\blacklozenge} = \varrho \vee (\rho^{\blacklozenge} \wedge \rho) = \varrho \vee (m_1 \wedge \rho) = \varrho \vee m_1 = \varrho$. Now, we prove that, for any $\rho, \varrho \in V$, $(\rho \wedge \varrho)^{\blacklozenge} = \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}$. Now, $(\rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \vee (\rho \wedge \varrho) = (\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} \vee \rho) \wedge (\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} \vee \varrho) = (\rho^{\blacklozenge} \vee \rho \vee \varrho^{\blacklozenge}) \wedge 1 = 1 \wedge 1 = 1$. Then, $(\rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \wedge (\rho \wedge \varrho) = (\rho^{\blacklozenge} \wedge \rho \wedge \varrho) \vee (\varrho^{\blacklozenge} \wedge \rho \wedge \varrho) = (m_1 \wedge \rho \wedge \varrho) \vee (\varrho^{\blacklozenge} \wedge \varrho \wedge \rho) = (m_1 \wedge \rho \wedge \varrho) \vee (m_1 \wedge \varrho \wedge \rho) = (m_1 \wedge \rho \wedge \varrho) \vee (m_1 \wedge \rho \wedge \varrho) = (m_1 \wedge \rho \wedge \varrho)$.

4. Properties

We present here some elementary properties of parapseudo-complemented PDL and further prove some essential conditions for a PDL with a minimal element to be parapseudo-complemented. The following lemma can be proved easily.

Lemma 5. *Let V be a parapseudo-complemented PDL. Then, for any $\rho, \varrho \in V$, we have the following:*
(1). $1^{\blacklozenge}$ *is a minimal element.*
(2). *If ρ is a minimal element, then $\rho^{\blacklozenge} = 1$.*
(3). $1^{\blacklozenge\blacklozenge} = 1$.
(4). $\rho^{\blacklozenge} \vee \rho = 1$.
(5). $\rho \vee \rho^{\blacklozenge\blacklozenge} = \rho$.
(6). $\rho^{\blacklozenge} = \rho^{\blacklozenge\blacklozenge\blacklozenge}$.
(7). $\rho^{\blacklozenge} = 1 \Leftrightarrow \rho^{\blacklozenge\blacklozenge}$ *is minimal element.*
(8). $1^{\blacklozenge} \leq \rho^{\blacklozenge}$.
(9). $\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} = \varrho^{\blacklozenge} \vee \rho^{\blacklozenge}$.
(10). $\rho \leq \varrho \Rightarrow \varrho^{\blacklozenge} \leq \rho^{\blacklozenge}$.
(11). $(\rho \vee \varrho)^{\blacklozenge} \leq \varrho^{\blacklozenge}$, $(\rho \vee \varrho)^{\blacklozenge} \leq \rho^{\blacklozenge}$.
(12). $\rho^{\blacklozenge} \leq \varrho^{\blacklozenge} \Leftrightarrow \varrho^{\blacklozenge\blacklozenge} \leq \rho^{\blacklozenge\blacklozenge}$.
(13). $\rho = 1 \Leftrightarrow \rho^{\blacklozenge\blacklozenge} = 1$.

Lemma 6. *Let V be a PDL with two minimal elements m_1 and m_2. Then the bounded distributive lattices $[m_1, 1]$ and $[m_2, 1]$ are isomorphic.*

Proof. Let V be a PDL with two minimal elements, m_1 and m_2. Define $f : [m_1, 1] \to [m_2, 1]$ by $f(\rho) = m_2 \vee \rho$. Now, we prove that f is an isomorphism. Clearly f is well defined. Let $\rho, \varrho \in [m_1, 1]$ and $f(\rho) = f(\varrho)$. Then $m_2 \vee \rho = m_2 \vee \varrho$. Now $\rho = m_1 \vee \rho = m_1 \vee m_2 \vee \rho = m_1 \vee m_2 \vee \varrho = m_1 \vee \varrho = \varrho$. Therefore, f is one-one. Let $t \in [m_2, 1]$. Then $m_1 \vee t \in [m_1, 1]$ and $f(m_1 \vee t) = m_2 \vee m_1 \vee t = m_2 \vee t = t$. Hence, f is onto. Let $\rho, \varrho \in [m_1, 1]$. Then $f(\rho \vee \varrho) = m_2 \vee \rho \vee \varrho = (m_2 \vee \rho) \vee (m_2 \vee \varrho) = f(\rho) \vee f(\varrho)$ and $f(\rho \wedge \varrho) = m_2 \vee (\rho \wedge \varrho) = (m_2 \vee \rho) \wedge (m_2 \vee \varrho) = f(\rho) \wedge f(\varrho)$, which implies f satisfies homomorphism property. Also, $f(1) = m_2 \vee 1 = 1$. Therefore, f is an isomorphism.

Theorem 7. *Let V be a PDL with a minimal element, m. Then the following are equivalent:*
(1). *V is a parapseudo-complemented PDL.*
(2). *$[m,1]$ is a dual pseudo-complemented lattice.*
(3). *$[m_1,1]$ is a dual pseudo-complemented lattice for all minimal elements m_1 in V.*

Proof. (1) $\Rightarrow$ (2) : Let $\Diamond$ be a parapseudo-complementation on V. We know that $[m,1]$ is a bounded distributive lattice. Now define $\blacklozenge$ on $[m,1]$ by $\rho^{\blacklozenge} = m \vee \rho^{\Diamond}$ for all $\rho \in [m,1]$. Then $\rho^{\blacklozenge} \in [m,1]$ and $\rho \vee \rho^{\blacklozenge} = \rho \vee (m \vee \rho^{\Diamond}) = \rho \vee \rho^{\Diamond} \vee m = 1 \vee m = 1$. Let $\varrho \in [m,1]$ and $\rho \vee \varrho = 1$. Then $\varrho \vee \rho = 1$ and hence $\varrho \vee \rho^{\Diamond} = \varrho$. Now $\rho^{\blacklozenge} \vee \varrho = m \vee \rho^{\Diamond} \vee \varrho = m \vee \varrho \vee \rho^{\Diamond} = m \vee \varrho = \varrho$. Therefore, $\rho^{\blacklozenge} \leq \varrho$. Hence, $^{\blacklozenge}$ is dual pseudo-complementation on $[m,1]$.
(2) $\Rightarrow$ (3) : Suppose $[m,1]$ is a dual pseudo-complemented lattice. Then $[m_1,1]$ is a dual pseudo-complemented lattice for all minimal elements m_1 in V.
(3) $\Rightarrow$ (1) : Suppose $[m_1,1]$ is a dual pseudo-complemented lattice for all minimal elements m_1 in V. For any $\rho \in V$, we have $m \wedge \rho$ is a minimal element in V and $[m \wedge \rho, 1]$ is a dual pseudo-complemented lattice. Let $\rho^{\blacklozenge}$ be the dual pseudo-complement of ρ in $[m \wedge \rho, 1]$. We prove that $\rho \rightarrowtail \rho^{\blacklozenge}$ is a parapseudo-complementation on V. Clearly, $\rho \vee \rho^{\blacklozenge} = 1$. Let $\varrho \in V$ and $\varrho \vee \rho = 1$. Put $\tau = (m \wedge \rho) \vee \varrho = (m \vee \varrho) \wedge (\rho \vee \varrho) = (m \vee \varrho) \wedge 1 = m \vee \varrho$. Then $\tau \in [m \wedge \rho, 1]$ and $\rho \vee \tau = \rho \vee m \vee \varrho = \rho \vee \varrho = 1$. So that $\rho^{\blacklozenge} \leq \tau$. Now, $\tau = \rho^{\blacklozenge} \vee \tau$ implies that $m \vee \varrho = \rho^{\blacklozenge} \vee m \vee \varrho$ and hence $\varrho \vee m \vee \varrho = \varrho \vee \rho^{\blacklozenge} \vee \varrho$. Thus, we get $\varrho = \varrho \vee \rho^{\blacklozenge}$.
Finally, let $\rho, \varrho \in V$. We have $\rho \in [m \wedge \rho, 1]$ and $\varrho \in [m \wedge \varrho, 1]$.
Now,

$$\begin{aligned}
(m \wedge (\rho \wedge \varrho)) \vee (\rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) &= (m \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \wedge ((\rho \wedge \varrho) \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}). \\
&= (m \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \wedge (\rho \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \wedge (\varrho \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \\
&= (m \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}). \\
&= \rho^{\blacklozenge} \vee \varrho^{\blacklozenge} \quad (\text{ since } \rho^{\blacklozenge} \in [m \wedge \rho, 1])
\end{aligned}$$

Therefore, $\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} \in [m \wedge (\rho \wedge \varrho), 1]$.
Now, $(\rho \wedge \varrho) \vee (\rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) = (\rho \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) \wedge (\varrho \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) = 1$. Let $\tau \in [m \wedge (\rho \wedge \varrho), 1]$ and $(\rho \wedge \varrho) \vee \tau = 1$. Then, $(\rho \vee \tau) \wedge (\varrho \vee \tau) = 1$ which implies that $\rho \vee \tau = 1$ and $\varrho \vee \tau = 1$. Also, $(m \wedge \rho) \vee \tau \in [m \wedge \rho, 1]$ and $\rho \vee ((m \wedge \rho) \vee \tau) = \rho \vee (\tau \vee (m \wedge \rho)) = 1$. Hence, we get $\rho^{\blacklozenge} \leq (m \wedge \rho) \vee \tau$. So that, $(m \wedge \rho) \vee \tau = \rho^{\blacklozenge} \vee (m \wedge \rho) \vee \tau = [(\rho^{\blacklozenge} \vee m) \wedge (\rho^{\blacklozenge} \vee \rho)] \vee \tau = \rho^{\blacklozenge} \vee \tau$. Therefore, $m \vee \tau = \rho^{\blacklozenge} \vee \tau$. Now, $[m \wedge (\rho \wedge \varrho)] \leq \tau$ implies $\tau = [m \wedge (\rho \wedge \varrho)] \vee \tau = (m \vee \tau) \wedge [(\rho \vee \tau) \wedge (\varrho \vee \tau)] = m \vee \tau$.
Therefore, $\rho^{\blacklozenge} \vee \tau = \tau$ implies $\rho^{\blacklozenge} \leq \tau$. Similarly, we get $\varrho^{\blacklozenge} \leq \tau$. Hence $\rho^{\blacklozenge} \vee \varrho^{\blacklozenge} \leq \tau$. Thus, we get that $(\rho \wedge \varrho)^{\blacklozenge} = \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}$. Therefore, $\blacklozenge$ is a parapseudo-complementation on V.

Lemma 7. *Let V be a PDL and $A \subseteq V$. Then the set*

$$A^{\bullet} = \{t \in V \mid t \vee \rho = 1 \textit{ for all } \rho \in A\}$$

is a filter of V.

Proof. Let $t_1, t_2 \in A^\bullet$. Then $t_1 \vee \rho = 1$, $t_2 \vee \rho = 1$ for all $\rho \in V$. Hence $(t_1 \wedge t_2) \vee \rho = (t_1 \vee \rho) \wedge (t_2 \vee \rho) = 1 \wedge 1 = 1$. Therefore, $t_1 \wedge t_2 \in A^\bullet$. Now, let $t_1 \in A^\bullet$ and $\kappa_1 \in V$. Then $\kappa_1 \vee t_1 \vee \rho = \kappa_1 \vee 1 = 1$. Hence, we get $\kappa_1 \vee t_1 \in A^\bullet$. Thus, $A^\bullet$ is a filter of V.

The filter $A^\bullet$ is called the annihilator filter corresponding to A. If $A = \{\rho\}$, we write $A^\bullet = [\rho]^\bullet$.

Lemma 8. *Let V be a PDL. Then for any $\rho, \varrho \in V$, $[\rho \wedge \varrho]^\bullet = [\rho]^\bullet \cap [\varrho]^\bullet$.*

Proof. Let $\kappa_1 \in V$. Then, $\kappa_1 \in [\rho \wedge \varrho]^\bullet \Leftrightarrow \kappa_1 \vee (\rho \wedge \varrho) = 1 \Leftrightarrow (\kappa_1 \vee \rho) \wedge (\kappa_1 \vee \varrho) = 1 \Leftrightarrow \kappa_1 \vee \rho = 1$ and $\kappa_1 \vee \varrho = 1 \Leftrightarrow \kappa_1 \in [\rho]^\bullet$ and $\kappa_1 \in [\varrho]^\bullet \Leftrightarrow \kappa_1 \in [\rho]^\bullet \cap [\varrho]^\bullet$.

Lemma 9. *Let V be a PDL and $\rho \in V$. Then $[\rho) = V$ if and only if ρ is a minimal element.*

Proof. Suppose $[\rho) = V$. Then, for any $\kappa_1 \in V$, we have $\kappa_1 \in [\rho)$ and hence $\kappa_1 \vee \rho = \kappa_1$. Therefore, ρ is a minimal element. Conversely, suppose that ρ is a minimal element. We have $[\rho) \subseteq V$. Let $\kappa_1 \in V$. Then $\kappa_1 \vee \rho = \kappa_1$. Therefore $\kappa_1 \in [\rho)$. Hence $[\rho) = V$.

Now, we prove the following theorem which characterize parapseudo-complementation on PDL.

Theorem 8. *Let V be a PDL. Then V is a parapseudo-complemented PDL if and only if for any $\rho \in V$, the annihilator filter $[\rho]^\bullet$ is a principal filter.*

Proof. Let $\rho \in V$ be such that $[\rho]^\bullet = [\kappa_1)$ for some $\kappa_1 \in V$. Since $1 \in V$, we have $V = [1]^\bullet = [m)$ for some $m \in V$. Hence by Lemma 9, m is a minimal element in V. Define $\rho^\Diamond = m \vee \kappa_1$. Now, we prove that $\Diamond$ is a parapseudo-complementation on V. Let $\rho \in V$ and suppose $[\rho]^\bullet = [\kappa_1) = [\kappa_2)$ for some $\kappa_1, \kappa_2 \in V$. Then $\kappa_1 = \kappa_1 \vee \kappa_2$ and $\kappa_2 = \kappa_2 \vee \kappa_1$. Therefore, $m \vee \kappa_1 = m \vee \kappa_1 \vee \kappa_2 = m \vee \kappa_2 \vee \kappa_1 = m \vee \kappa_2$ which implies $\Diamond$ is well-defined. Let $\rho \in V$. Then $\rho \vee \rho^\Diamond = \rho \vee m \vee \kappa_1 = \rho \vee \kappa_1 = 1$. Let $\varrho \in V$ and $\varrho \vee \rho = 1$. Then $\varrho \in [\rho]^\bullet = [\kappa_1)$. Therefore, $\varrho = \varrho \vee \kappa_1 = \varrho \vee m \vee \kappa_1 = \varrho \vee \rho^\Diamond$. Finally, let $\rho, \varrho \in V$ and $[\rho]^\bullet = [\kappa_1)$, $[\varrho]^\bullet = [\kappa_2)$ for some $\kappa_1, \kappa_2 \in V$. Then, $[\rho \wedge \varrho]^\bullet = [\rho]^\bullet \cap [\varrho]^\bullet = [\kappa_1) \cap [\kappa_2) = [\kappa_1 \vee \kappa_2)$. Therefore $(\rho \wedge \varrho)^\Diamond = m \vee (\kappa_1 \vee \kappa_2) = (m \vee \kappa_1) \vee (m \vee \kappa_2) = \rho^\Diamond \vee \varrho^\Diamond$. Thus $\Diamond$ is parapseudo-complementation on V.
Conversely, if V is parapseudo-complemented PDL, then for any $\rho \in V$, we have $[\rho]^\bullet = [\rho^\blacklozenge)$. Hence, every annihilator filter is a principal filter.

Theorem 9. *Let V be a PDL with a minimal element m. Then V is parapseudo-complemented PDL if and only if the set PF(V) of all principal filters of V is a pseudo-complemented lattice.*

Proof. Suppose V is a parapseudo-complemented PDL. Then the set PF(V) forms a distributive lattice. Let $[\rho) \in PF(\mathrm{V})$. Define $[\rho)^\Diamond = [\rho^\blacklozenge)$ where $\rho^\blacklozenge$ is the parapseudo-complement of $\rho \in V$. We prove that $\Diamond$ is a pseudo-complementation on PF(V). Now,

$[\rho) \cap [\rho)^{\lozenge} = [\rho) \cap [\rho^{\blacklozenge}) = [\rho \vee \rho^{\blacklozenge}) = [1)$. Let $\tau \in V$ and $[\rho) \cap [\tau) = [1)$. Then $[\rho \vee \tau) = [1)$ implies $\rho \vee \tau = 1$. Hence $\tau \vee \rho^{\blacklozenge} = \tau$. Therefore, $[\tau) \cap [\rho^{\blacklozenge}) = [\tau \vee \rho^{\blacklozenge}) = [\tau)$ so that $[\tau) \subseteq [\rho^{\blacklozenge}) = [\rho)^{\lozenge}$. Thus $[\rho)^{\lozenge}$ is the pseudo-complement of $[\rho)$ in PF(V).
Conversely, suppose PF(V) is a pseudo-complemented lattice. Let $\rho \in V$. Then $[\rho) \in PF(\mathrm{V})$. Write $[\rho)^{\blacklozenge} = [\rho_1)$ the pseudo-complement of $[\rho) \in PF(\mathrm{V})$. Now, define $\rho^{\lozenge} = m \vee \rho_1$. Then we prove that $\lozenge$ is a parapseudo-complementation on V. First we observe that $\lozenge$ is well defined. Suppose $[\rho)^{\blacklozenge} = [\rho_1) = [\rho_2)$. Then $m \vee \rho_1 = m \vee \rho_1 \vee \rho_2 = m \vee \rho_2 \vee \rho_1 = m \vee \rho_2$. Hence $\lozenge$ is well defined. Now, $\rho \vee \rho^{\lozenge} = \rho \vee m \vee \rho_1 = \rho \vee \rho_1 = 1$, since $[\rho \vee \rho_1) = [\rho) \cap [\rho_1) = [\rho) \cap [\rho)^{\blacklozenge} = [1)$.
Let $\varrho \in V$ and $\varrho \vee \rho = 1$. Then, $[\rho) \cap [\varrho) = [1)$ and hence $[\varrho) \cap [\rho)^{\blacklozenge} = [\varrho)$. Therefore $[\varrho) \cap [\rho_1) = [\varrho)$ which implies that $[\varrho) \subseteq [\rho_1)$. Hence $\varrho = \varrho \vee \rho_1 = \varrho \vee \rho_1 \vee m = \varrho \vee \rho^{\lozenge}$.
Finally, let $\rho, \varrho \in V$ and suppose $[\rho)^{\blacklozenge} = [\rho_1)$, $[\varrho)^{\blacklozenge} = [\varrho_1)$ for some $\rho_1, \varrho_1 \in V$. Now, $[\rho \wedge \varrho)^{\blacklozenge} = [\rho)^{\blacklozenge} \cap [\varrho)^{\blacklozenge} = [\rho_1) \cap [\varrho_1) = [\rho_1 \vee \varrho_1)$. Hence, by definition, $(\rho \wedge \varrho)^{\lozenge} = m \vee \rho_1 \vee \varrho_1 = m \vee \rho_1 \vee m \vee \varrho_1 = \rho^{\lozenge} \vee \varrho^{\lozenge}$. Thus $\lozenge$ is a parapseudo-complementation on V.

In 1949, P.Ribenboim[10] had first observed that the class of pseudo-complemented distributive lattices is equational. Now, we prove that the parapseudo-complementation on paradistributive latticoids is also equationally definable. We give certain equivalent sets of identities which characterize the parapseudo-complementation on V. For this, first we need the following lemmas. As there are no hidden difficulties to prove the following two lemmas, we omit their proofs.

Lemma 10. *Let V be a parapseudo-complemented PDL. Then for any $\rho, \varrho \in V$, the following are equivalent:*
(1). $\rho \vee \varrho = 1$.
(2). $\rho \vee \varrho^{\blacklozenge\blacklozenge} = 1$.
(3). $\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} = 1$.
(4). $\rho \vee \varrho^{\blacklozenge\blacklozenge} = 1$.

Lemma 11. *Let V be a parapseudo-complemented PDL. Then for any $\rho, \varrho \in V$, the following hold:*
(1). $(\rho \vee \varrho)^{\blacklozenge\blacklozenge} = \rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge}$.
(2). $(\rho \vee \varrho)^{\blacklozenge} = (\varrho \vee \rho)^{\blacklozenge}$.
(3). $(\rho \wedge \varrho)^{\blacklozenge} = (\varrho \wedge \rho)^{\blacklozenge}$.

Now, we prove that the parapseudo-complementation PDL is equationally definable.

Lemma 12. *A unary operation $\blacklozenge$ on PDL V is a parapseudo-complementation on V if and only if it satisfies the following equations:*
(1). $\rho \vee \rho^{\blacklozenge} = 1$.
(2). $\rho \wedge \rho^{\blacklozenge\blacklozenge} = \rho^{\blacklozenge\blacklozenge}$.
(3). $(\rho \wedge \varrho)^{\blacklozenge} = \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}$.
(4). $(\rho \vee \varrho)^{\blacklozenge\blacklozenge} = \rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge}$.
(5). $\rho \vee 1^{\blacklozenge} = \rho$.

Proof. Let V be a PDL and $\blacklozenge$ be a unary operation on V satisfying the given conditions. We prove that $\blacklozenge$ is a parapseudo-complementation on V. Let $\rho, \varrho \in V$ and $\varrho \vee \rho = 1$. Then,

$$\begin{array}{rcll}
\varrho & = & \varrho \vee \varrho^{\blacklozenge\blacklozenge} & (\text{ by } (2)) \\
 & = & \varrho \vee 1^{\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} & (\text{ by } (5)) \\
 & = & \varrho \vee (\rho^{\blacklozenge} \vee \rho^{\blacklozenge\blacklozenge})^{\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} & (\text{ by } (1)) \\
 & = & \varrho \vee (\rho \wedge \rho^{\blacklozenge})^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} & (\text{ by } (3)) \\
 & = & \varrho \vee ((\rho \wedge \rho^{\blacklozenge}) \vee \varrho)^{\blacklozenge\blacklozenge} & (\text{ by } (4)) \\
 & = & \varrho \vee ((\rho \vee \varrho) \wedge (\rho^{\blacklozenge} \vee \varrho))^{\blacklozenge\blacklozenge} & \\
 & = & \varrho \vee (\rho^{\blacklozenge} \vee \varrho)^{\blacklozenge\blacklozenge} & \\
 & = & \varrho \vee \rho^{\blacklozenge\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} & (\text{ by } (4)) \\
 & = & \varrho \vee \rho^{\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} & (\text{ by } (2)) \\
 & = & \varrho \vee \varrho^{\blacklozenge\blacklozenge} \vee \rho^{\blacklozenge} & (\text{ by } (3)) \\
 & = & \varrho \vee \rho^{\blacklozenge} & (\text{ by } (2))
\end{array}$$

Therefore, it follows from (1) and (3) that $\blacklozenge$ is a parapseudo-complementation on V. Converse follows from Lemma 10 and Lemma 11.

Lemma 13. *A unary operation $\blacklozenge$ on PDL V is a parapseudo-complementation on V if and only if it satisfies the following equations:*
(1). $\varrho \vee \rho^{\blacklozenge} = \varrho \vee (\rho \vee \varrho)^{\blacklozenge}$.
(2). $\rho \vee 1^{\blacklozenge} = \rho$.
(3). $1^{\blacklozenge\blacklozenge} = 1$.
(4). $(\rho \wedge \varrho)^{\blacklozenge} = \rho^{\blacklozenge} \vee \varrho^{\blacklozenge}$.

Proof. Let V be a PDL and $\blacklozenge$ be a unary operation on V satisfying the given equations. We prove $\blacklozenge$ is a parapseudo-complementation on V.
Let $\rho, \varrho \in V$ and $\varrho \vee \rho = 1$. Then $\varrho \vee \rho^{\blacklozenge} = \varrho \vee (\rho \vee \varrho)^{\blacklozenge} = \varrho \vee 1^{\blacklozenge} = \varrho$.
Now

$$\begin{array}{rcl}
\rho \vee \rho^{\blacklozenge} & = & \rho \vee (\rho \vee 1^{\blacklozenge})^{\blacklozenge} \\
 & = & \rho \vee (1^{\blacklozenge} \vee \rho)^{\blacklozenge} \\
 & = & \rho \vee 1^{\blacklozenge\blacklozenge} \\
 & = & \rho \vee 1 \\
 & = & 1
\end{array}$$

shows that $\blacklozenge$ is a parapseudo-complementation on V.
Conversely, assume that $\blacklozenge$ is a parapseudo-complementation on V. Then, by Lemma 5 and by Definition 5 we have $(2), (3), (4)$. So, it is enough if we prove (1). For this, let $\rho, \varrho \in V$. Then

$$\begin{array}{rcl}
\rho \vee \varrho \vee (\rho \vee \varrho)^{\blacklozenge} & = & 1 \\
\Rightarrow \varrho \vee (\rho \vee \varrho)^{\blacklozenge} \vee \rho^{\blacklozenge} & = & \varrho \vee (\rho \vee \varrho)^{\blacklozenge} \\
\Rightarrow \varrho \vee \rho^{\blacklozenge} \vee (\rho \vee \varrho)^{\blacklozenge} & = & \varrho \vee (\rho \vee \varrho)^{\blacklozenge} \\
\Rightarrow \varrho \vee \rho^{\blacklozenge} & = & \varrho \vee (\rho \vee \varrho)^{\blacklozenge}
\end{array}$$

5. One to One Correspondence

In this section, we prove that, if $^\blacklozenge$ is a parapseudo-complementation on V, then the set $V^\blacklozenge = \{\rho^\blacklozenge \mid \rho \in V\}$ is a Boolean algebra. Furthermore, there exists a one-to-one correspondence between the set of all minimal elements of V and the set of all parapseudo-complementations on V. Finally, it is worth noting that the Boolean algebra $V^\blacklozenge$ is independent of the specific choice of parapseudo-complementation $^\blacklozenge$.

Theorem 10. *Let V be a PDL with a parapseudo-complementation $\blacklozenge$. For any $\rho^\blacklozenge, \varrho^\blacklozenge \in V^\blacklozenge$, define $\rho^\blacklozenge \leq \varrho^\blacklozenge$ if and only if $\rho^\blacklozenge \wedge \varrho^\blacklozenge = \rho^\blacklozenge$. Then $(V^\blacklozenge, \leq)$ is a Boolean algebra.*

Proof. Let V be a PDL with a parapseudo-complementation $\blacklozenge$. Clearly, $\leq$ is reflexive and anti-symmetric. Now, for $\rho^\blacklozenge \leq \varrho^\blacklozenge$ and $\varrho^\blacklozenge \leq \tau^\blacklozenge$, we have $\rho^\blacklozenge \wedge \varrho^\blacklozenge = \rho^\blacklozenge$, $\varrho^\blacklozenge \wedge \tau^\blacklozenge = \varrho^\blacklozenge$. Therefore, $\rho^\blacklozenge \vee \tau^\blacklozenge = \rho^\blacklozenge \vee \varrho^\blacklozenge \vee \tau^\blacklozenge = \varrho^\blacklozenge \vee \tau^\blacklozenge = \tau^\blacklozenge$ which implies $\rho^\blacklozenge \wedge \tau^\blacklozenge = \rho^\blacklozenge \wedge (\rho^\blacklozenge \vee \tau^\blacklozenge) = \rho^\blacklozenge$, hence $\leq$ is transitive. Therefore, $\leq$ is a partial ordering on $V^\blacklozenge$. Let $\rho^\blacklozenge, \varrho^\blacklozenge \in V^\blacklozenge$. Then $(\rho \wedge \varrho)^\blacklozenge = \rho^\blacklozenge \vee \varrho^\blacklozenge$. Hence $\rho^\blacklozenge \vee \varrho^\blacklozenge \in V^\blacklozenge$ and we have $\rho^\blacklozenge \vee \varrho^\blacklozenge = \varrho^\blacklozenge \vee \rho^\blacklozenge$. So that, $\rho^\blacklozenge \vee \varrho^\blacklozenge$ is the least upper bound of $\rho^\blacklozenge, \varrho^\blacklozenge \in V^\blacklozenge$. We have $\rho^{\blacklozenge\blacklozenge} \leq \rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge}$ implies $(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge \leq \rho^\blacklozenge$. Similarly, we get $(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge \leq \varrho^\blacklozenge$. Therefore, $(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge$ is a lower bound of $\rho^\blacklozenge, \varrho^\blacklozenge \in V^\blacklozenge$. Let $\tau^\blacklozenge \in V^\blacklozenge$ and $\tau^\blacklozenge \leq \rho^\blacklozenge$, $\tau^\blacklozenge \leq \varrho^\blacklozenge$. Then $\rho^{\blacklozenge\blacklozenge} \leq \tau^{\blacklozenge\blacklozenge}$, $\varrho^{\blacklozenge\blacklozenge} \leq \tau^{\blacklozenge\blacklozenge}$. Hence $\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge} \leq \tau^{\blacklozenge\blacklozenge}$. Therefore, $\tau^\blacklozenge \leq (\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge$. Thus $(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge$ is the greatest lower bound of $\rho^\blacklozenge, \varrho^\blacklozenge \in V^\blacklozenge$. Hence $(V^\blacklozenge, \leq)$ is a lattice. From now, we represent $(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge = (\rho^\blacklozenge \overline{\wedge} \varrho^\blacklozenge)$.
Now, by Lemma 5(8), we have $1^\blacklozenge \leq \rho^\blacklozenge$ for all $\rho \in V$. So that, $1^\blacklozenge$ is the least element in $V^\blacklozenge$ and since $1^{\blacklozenge\blacklozenge} = 1$, $1 \in V^\blacklozenge$ it is the greatest element in $V^\blacklozenge$. Therefore, $(V^\blacklozenge, \leq)$ is a bounded lattice. Finally, we prove that $(V^\blacklozenge, \leq)$ has complement and satisfies distributive property.
Let $\rho^\blacklozenge \in V^\blacklozenge$. Then $\rho^{\blacklozenge\blacklozenge} \in V^\blacklozenge$ and $\rho^\blacklozenge \vee \rho^{\blacklozenge\blacklozenge} = 1$ and $\rho^\blacklozenge \overline{\wedge} \rho^{\blacklozenge\blacklozenge} = (\rho^{\blacklozenge\blacklozenge} \vee \rho^{\blacklozenge\blacklozenge\blacklozenge})^\blacklozenge = (\rho^{\blacklozenge\blacklozenge} \vee \rho^\blacklozenge)^\blacklozenge = 1^\blacklozenge$. Hence $\rho^{\blacklozenge\blacklozenge}$ is the complement of $\rho^\blacklozenge \in V^\blacklozenge$.
Now, let $\rho^\blacklozenge, \varrho^\blacklozenge, \tau^\blacklozenge \in V^\blacklozenge$. Then,

$$\begin{aligned}
(\rho^\blacklozenge \overline{\wedge} \varrho^\blacklozenge) \vee (\rho^\blacklozenge \overline{\wedge} \tau^\blacklozenge) &= (\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^\blacklozenge \vee (\rho^{\blacklozenge\blacklozenge} \vee \tau^{\blacklozenge\blacklozenge})^\blacklozenge. \\
&= [(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge}) \wedge (\rho^{\blacklozenge\blacklozenge} \vee \tau^{\blacklozenge\blacklozenge})]^\blacklozenge. \\
&= [\rho^{\blacklozenge\blacklozenge} \vee (\varrho^{\blacklozenge\blacklozenge} \wedge \tau^{\blacklozenge\blacklozenge})]^\blacklozenge. \\
&= [\rho^{\blacklozenge\blacklozenge} \vee (\varrho^{\blacklozenge\blacklozenge} \wedge \tau^{\blacklozenge\blacklozenge})]^{\blacklozenge\blacklozenge\blacklozenge}. \\
&= [\rho^{\blacklozenge\blacklozenge\blacklozenge\blacklozenge} \vee (\varrho^{\blacklozenge\blacklozenge} \wedge \tau^{\blacklozenge\blacklozenge})^{\blacklozenge\blacklozenge}]^\blacklozenge. \\
&= [\rho^{\blacklozenge\blacklozenge} \vee (\varrho^{\blacklozenge\blacklozenge\blacklozenge} \vee \tau^{\blacklozenge\blacklozenge\blacklozenge})^\blacklozenge]^\blacklozenge. \\
&= [\rho^{\blacklozenge\blacklozenge} \vee (\varrho^\blacklozenge \vee \tau^\blacklozenge)^\blacklozenge]^\blacklozenge. \\
&= \rho^\blacklozenge \overline{\wedge} (\varrho^\blacklozenge \vee \tau^\blacklozenge).
\end{aligned}$$

Thus $(V^\blacklozenge, \leq)$ is a Boolean algebra.

Corollary 7. *Let V be a parapseudo-complemented PDL with parapseudo-complemenation $\blacklozenge$. Then the map $f : V \to V^\blacklozenge$ defined by $f(\rho) = \rho^{\blacklozenge\blacklozenge}$ is an epimorphism.*

In the following theorem, we establish a one-to-one correspondence between the set of all minimal elements in V and the set of all parapseudo-complemenations on V. First we prove the following lemma.

Lemma 14. *Let V be a PDL with two parapseudo-complemenations $\blacklozenge$ and $\lozenge$. Then, for any $\rho, \varrho \in V$, we have the following:*
(1). $\rho^{\lozenge} \vee \rho^{\blacklozenge} = \rho^{\lozenge}$ *and* $\rho^{\lozenge} \wedge \rho^{\blacklozenge} = \rho^{\blacklozenge}$.
(2). $\rho^{\blacklozenge\lozenge} = \rho^{\lozenge\lozenge}$.
(3). $\rho^{\blacklozenge} = \varrho^{\blacklozenge} \Leftrightarrow \rho^{\lozenge} = \varrho^{\lozenge}$.
(4). $\rho^{\blacklozenge} = 1 \Leftrightarrow \rho^{\lozenge} = 1 \Leftrightarrow (\rho \vee \varrho = 1 \Rightarrow \varrho = 1)$.
(5). $\rho^{\lozenge} = 1^{\lozenge} \vee \rho^{\blacklozenge}$.
(6). $\rho^{\blacklozenge} \wedge \rho^{\blacklozenge\blacklozenge} = 1^{\blacklozenge} \Leftrightarrow \rho^{\lozenge} \wedge \rho^{\lozenge\lozenge} = 1^{\lozenge}$.

Proof. Let V be a PDL with two parapseudo-complemenations $\blacklozenge$ and $\lozenge$ and $\rho, \varrho \in V$.
(1). Since $\rho^{\lozenge} \vee \rho = 1$, we get that $\rho^{\lozenge} \vee \rho^{\blacklozenge} = \rho^{\lozenge}$. Hence $\rho^{\lozenge} \wedge \rho^{\blacklozenge} = \rho^{\blacklozenge}$.
(2). $\rho^{\blacklozenge\lozenge} = (\rho^{\lozenge} \wedge \rho^{\blacklozenge})^{\lozenge} = (\rho^{\blacklozenge} \wedge \rho^{\lozenge})^{\lozenge} = \rho^{\lozenge\lozenge}$.
(3). Let $\rho^{\blacklozenge} = \varrho^{\blacklozenge}$. Then $\rho^{\lozenge} = \rho^{\lozenge\lozenge\lozenge} = \rho^{\blacklozenge\lozenge\lozenge} = \varrho^{\blacklozenge\lozenge\lozenge} = \varrho^{\lozenge\lozenge\lozenge} = \varrho^{\lozenge}$.
(4). Let $\rho^{\blacklozenge} = 1$. Then, we have $\rho^{\lozenge} = \rho^{\lozenge} \vee \rho^{\blacklozenge} = \rho^{\lozenge} \vee 1 = 1$. Let $\rho^{\lozenge} = 1$ and $\rho \vee \varrho = 1$. Then $\varrho = \varrho \vee \rho^{\lozenge} = \varrho \vee 1 = 1$. Suppose $\varrho = 1$ whenever $\rho \vee \varrho = 1$. So that $\rho^{\blacklozenge} = 1$ since $\rho \vee \rho^{\blacklozenge} = 1$.
(5). We have $(1^{\lozenge} \vee \rho^{\blacklozenge}) \vee \rho^{\lozenge} = 1^{\lozenge} \vee \rho^{\lozenge} \vee \rho^{\blacklozenge} = \rho^{\lozenge} \vee \rho^{\blacklozenge} = \rho^{\lozenge}$.
(6). Let $\rho^{\blacklozenge} \wedge \rho^{\blacklozenge\blacklozenge} = 1^{\blacklozenge}$. Then $\rho^{\lozenge} \wedge \rho^{\lozenge\lozenge} = \rho^{\lozenge} \wedge \rho^{\blacklozenge\lozenge} = (1^{\lozenge} \vee \rho^{\blacklozenge}) \wedge (1^{\lozenge} \vee \rho^{\blacklozenge\blacklozenge}) = 1^{\lozenge} \vee (\rho^{\blacklozenge} \wedge \rho^{\blacklozenge\blacklozenge}) = 1^{\lozenge} \vee 1^{\blacklozenge} = 1^{\lozenge}$.

Let $(V, \vee, \wedge, 1)$ be a PDL with a parapseudo-complementation $\blacklozenge$, and m a minimal element in V. We define a new operation $\blacklozenge_m : V \to V$ as follows: for any $\rho \in V$, we have $\rho^{\blacklozenge_m} = m \vee \rho^{\blacklozenge}$. Then $\blacklozenge_m$ is also a parapseudo-complementation on V in which $1^{\blacklozenge_m} = m$.

Theorem 11. *Let V be a parapseudo-complemented PDL. Let M be the set of all minimal elements in V and PC(V) be the set of all parapseudo -complemenations on V. For any $m \in M$, define $\blacklozenge_m : V \to V$ by $\rho^{\blacklozenge_m} = m \vee \rho^{\blacklozenge}$ for all $\rho \in V$. Then $m \rightarrowtail \rho^{\blacklozenge_m}$ is a bijection of M onto PC(V).*

Proof. Let $m, n \in V$ be such that $\blacklozenge_m = \blacklozenge_n$. Then $1^{\blacklozenge_m} = 1^{\blacklozenge_n}$ so that $m \vee 1^{\blacklozenge} = n \vee 1^{\blacklozenge}$. Hence $m = n$. Also, for any $\lozenge \in PC(\mathrm{V})$, if $m = 1^{\lozenge}$, then $\rho^{\blacklozenge_m} = m \vee \rho^{\blacklozenge} = 1^{\lozenge} \vee \rho^{\blacklozenge} = \rho^{\lozenge}$ by Lemma 14(5). Then $\lozenge$ is same as $\blacklozenge_m$ and m is a minimal element. Thus $m \rightarrowtail \rho^{\blacklozenge_m}$ is a bijection of M onto PC(V).

Theorem 12. *If V is a PDL with two parapseudo-complementations $\blacklozenge$ and $\lozenge$, then the map $f : V^{\blacklozenge} \to V^{\lozenge}$ defined by $f(\rho^{\blacklozenge}) = \rho^{\lozenge}$ is an isomorphism of Boolean algebras.*

Proof. Let V be a PDL with two parapseudo-complementations $\blacklozenge$ and $\lozenge$. Clearly the map $f : V^{\blacklozenge} \to V^{\lozenge}$ defined by $f(\rho^{\blacklozenge}) = \rho^{\lozenge}$ is well -defined and one -one by Lemma 14. By definition, f is onto. Let $\rho^{\blacklozenge}, \varrho^{\blacklozenge} \in V$. Then, $f(\rho^{\blacklozenge} \overline{\wedge} \varrho^{\blacklozenge}) = f((\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^{\blacklozenge}) =$

$(\rho^{\blacklozenge\blacklozenge} \vee \varrho^{\blacklozenge\blacklozenge})^{\lozenge} = (\rho \vee \varrho)^{\blacklozenge\blacklozenge\lozenge} = (\rho \vee \varrho)^{\lozenge\lozenge\lozenge} = (\rho^{\lozenge\lozenge} \vee \varrho^{\lozenge\lozenge})^{\lozenge} = \rho^{\lozenge}\overline{\wedge}\varrho^{\lozenge} = f(\rho^{\blacklozenge})\overline{\wedge}f(\varrho^{\blacklozenge})$. Also, $f(\rho^{\blacklozenge} \vee \varrho^{\blacklozenge}) = f((\rho \wedge \varrho)^{\blacklozenge}) = (\rho \wedge \varrho)^{\lozenge} = \rho^{\lozenge} \vee \varrho^{\lozenge} = f(\rho^{\blacklozenge}) \vee f(\varrho^{\blacklozenge})$. Therefore f is an isomorphism.

6. Conclusions

In this paper, we have introduced the concept of a parapseudo-complementation on a paradistributive lattiocoid and examined its elementary properties. By establishing necessary conditions, we have provided insights into when a PDL with a minimal element can be parapseudo-complemented. Furthermore, our investigation has focused on the equationally definable nature of parapseudo-complementation, identifying the properties required for this concept to be equationally definable within a PDL. This contributes to a deeper understanding of the formalization and algebraic implications of parapseudo-complementation. Additionally, we have established a one-to-one correspondence between the set of all minimal elements and the set of all parapseudo-complementations in a PDL. This correspondence highlights the interplay between minimal elements and parapseudo-complementation, providing a valuable connection between the structural elements of a PDL and the concept under study. In future, our work will focus on $\blacklozenge$-PDL, Stone PDL and study their topological properties.

Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions.

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