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Certified Hop Independence: Properties and Connections with Other Variants of Independence

Sharmia H. Kaida[1], Kaimar Jay S. Maharajul[1], Javier A. Hassan[1,*], Ladznar S. Laja[1], Abdurajan B. Lintasan[1], Aljon A. Pablo[2]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU-Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*
[2] *Banaran Main Junior High School, Secondary Education Department, MSU-Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*

Abstract. Let G be a graph. Then $B \subseteq V(G)$ is called a certified hop independent set of G if for every $a, b \in B$, $d_G(a,b) \neq 2$ and for every $x \in B$ has either zero or at least two neighbors in $V(G) \setminus B$. The maximum cardinality among all certified hop independent sets in G, denoted by $\alpha_{ch}(G)$, is called the certified hop independence number of G. In this paper, we initiate the study of certified hop independence in graphs and we establish some of its properties. We give realization results involving hop independence and certified hop independence parameters, and we show that the difference between these two parameters can be made arbitrarily large. We characterize certified hop independent sets in some graphs and we use these results to obtain the exact values or bounds of the parameter. Moreover, we show that the certified hop independence and independence parameters are incomparable.

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Key Words and Phrases: Hop independence, certified independent set, certified hop independence number

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1. Introduction

In this paper, we introduce a new variant of hop independence called certified hop independence. Indeed, while a hop independent set of a graph requires that no two distinct vertices in the set are at distance two from each other, the concept that we will be dealing with here imposes additional condition that each vertex in the set must have either zero

*Corresponding author.
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Email addresses: sharmiakaida@msutawi-tawi.edu.ph (S. Kaida)
kaimarjaymaharajul@msutawi-tawi.edu.ph (K. Maharajul)
javierhassan@msutawi-tawi.edu.ph (J. Hassan)
ladznarlaja@msutawi-tawi.edu.ph (L. Laja)
abdurajanlintasan@msutawi-tawi.edu.ph (A. Lintasan)
aljonpablo@msutawi-tawi.edu.ph (A. Pablo)

or at least two neighbors outside the set. Some related studies on hop independence can be found in [1–4].

The motivation of introducing the concept is the ever increasing number of studies on independence and some of its variations. We show that the certified hop independence and the standard independence parameters of a graph are incomparable. Moreover, we give realization results involving certified hop independence and hop independence parameters, and we show that the latter is always at most equal to the hop independence parameter.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A *path graph* is non-empty graph with vertex-set $\{x_1, x_2, ..., x_n\}$ and edge-set $\{x_1x_2, x_2x_3, ..., x_{n-1}x_n\}$, where the $x_i's$ are all distinct. The path of order n is denoted by P_n. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is called $u - v$ path. The *cycle graph* C_n is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$.

A graph is *complete* if every pair of distinct vertices are adjacent. A complete graph of order n is denoted by K_n. A subset C of a vertex-set $V(G)$ of G is a clique if the graph $\langle C \rangle$ induced by C is complete. The maximum cardinality of a clique of G, denoted by $\omega(G)$, is called the clique number of G.

A graph G is *connected* if every pair of its vertices can be joined by a path. Otherwise, G is *disconnected*. A maximal connected subgraph (not a subgraph of any connected subgraph) of G is called a *component* of G.

Let G and H be two graphs. The *join* $G + H$ of G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{ab : a \in V(G), b \in V(H)\}.$$

The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

A subset I of $V(G)$ is called an *independent* if for every pair of distinct vertices $x, y \in I$, $d_G(x, y) \neq 1$. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence* number of G. Any independent set I with cardinality equal to $\alpha(G)$ is called an α-set of G.

A subset S of $V(G)$ is called a *hop independent* set of G if any two distinct vertices in S are not at distance two from each other, that is, $d_G(v, w) \neq 2$ for any two distinct vertices $v, w \in S$. The *hop independence number* of G, denoted by $\alpha_h(G)$, is the maximum cardinality of a hop independent set of G.

3. Results

We begin this section by introducing the concept of certified hop independence in a graph.

Definition 1. Let G be a graph. Then $B \subseteq V(G)$ is called a *certified hop independent set* of G if for every $a, b \in B$, $d_G(a,b) \neq 2$ and for every $x \in B$ has either zero or at least two neighbors in $V(G) \setminus B$. The maximum cardinality among all certified hop independent sets in G, denoted by $\alpha_{ch}(G)$, is called the *certified hop independence number* of G. Any certified hop independent set B with $|B| = \alpha_{ch}(G)$ is called the maximum certified hop independent set of G or an α_{ch}-set of G.

Example 1. Consider the graph G in Figure 1.

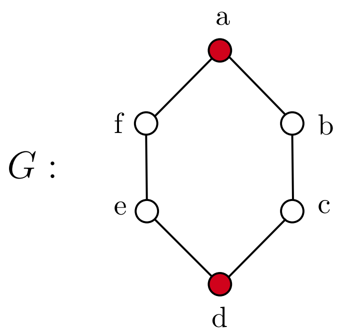

Figure 1: Graph G with $\alpha_{ch}(G) = 2$

Let $B = \{a, d\}$. Then $d_G(a,d) = 3$. Thus, B is a hop independent set of B. Now, observe that $b, f \in N_G(a)$ and $c, e \in N_G(d)$, where $b, c, f, e \in V(G) \setminus B$. It follows that B is a certified hop independent set of G. Moreover, it can be verified that $\alpha_{ch}(G) = 2$.

Theorem 1. *Let G be a graph. Then*

(i) $\alpha_{ch}(G) \leq \alpha_h(G)$; *and*

(ii) $1 \leq \alpha_{ch}(G) \leq |V(G)|$.

Proof. (i) Let $B \subseteq V(G)$ be a maximum certified hop independent set of G. Then $\alpha_{ch}(G) = |B|$ and B is a hop independent set of G. Since $\alpha_h(G)$ is the maximum cardinality among all hop independent sets in G, it follows that $\alpha_h(G) \geq |B| = \alpha_{ch}(G)$.

(ii) Since an empty set cannot be a certified hop independent, we have $\alpha_{ch}(G) \geq 1$. Since $\alpha_h(G) \leq |V(G)|$, we have $\alpha_{ch}(G) \leq |V(G)|$ by (i). Consequently,

$$1 \leq \alpha_{ch}(G) \leq |V(G)|. \quad \square$$

Theorem 2. *Let G be a graph. Then $\alpha_{ch}(G) = |V(G)|$ if and only if $diam(H) \leq 1$ for each component H of G.*

Proof. Suppose that $\alpha_{ch}(G) = |V(G)|$. Then $V(G)$ is the maximum certified hop independent set of G. Assume that G is connected. Suppose further that $diam(G) \geq 2$. There exist $a, b \in V(G)$ such that $d_G(a, b) = 2$. This means that a and b cannot be both elements of any certified hop independent set of G. Thus, $\alpha_{ch}(G) \leq |V(G)| - 1$, a contradiction. Next, suppose that G is disconnected. Suppose further that $diam(H) \geq 2$ for some component H of G. Then there exist $x, y, \in V(H)$ such that

$$d_H(x, y) = 2 = d_G(x, y).$$

Then either x or y cannot be an element of a certified hop independent set of G. Thus $\alpha_{ch}(G) \leq |V(G)| - 1$, a contradiction. Therefore, $diam(H) \leq 1$ for each component H of G.

Conversely, suppose that $diam(H) \leq 1$ for each component H of G. Let $u, v \in V(G)$. If u, v are vertices of one component K of G, then $d_K(u, v) = 1 = d_G(u, v)$, and we are done. Suppose that $u \in V(Q)$ and $v \in V(T)$, where Q and T are components of G. Then $d_G(u, v) \neq 2$. Since u and v are arbitrary, it follows that $V(G)$ is a hop independent set of G. Since each vertex in G has zero neighbor outside $V(G)$, $V(G)$ is a certified hop independent set of G. Consequently, $\alpha_{ch}(G) = |V(G)|$. □

Corollary 1. *Let m be a positive integer. Then $\alpha_{ch}(K_m) = m = \alpha_{ch}(\overline{K}_m)$ for all $m \geq 1$.*

Theorem 3. *If S is a certified hop independent set of a path graph P_n, then $d_{P_n}(x, y) \geq 3$ for all $x, y \in S$, where $x \neq y$.*

Proof. let $S \subseteq V(P_n)$ be a certified hop independent set of P_n, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Let $x, y \in S$. Suppose that $d_{p_n}(x, y) = 1$. If $x = v_1$, then $y = v_2$ and y have only one neighbor v_3 outside S, a contradiction. Similarly, when $y = v_1$, $x = v_n$ or $y = v_n$. Suppose that $x = v_i$ and $y = v_j$, where $i, j = \{2, \ldots, n-1\}$. Since $|N_{P_n}(v_k)| = 2$ for all $k \in \{2, \ldots, n-1\}$, x and y have only one neighbor outside S, which is a contradiction. Now, since any certified hop independent set is a hop independent, it follows that $d_{P_n}(x, y) \neq 2$. Therefore, $d_{P_n}(x, y) \geq 3$ for all $x, y \in S$, where $x \neq y$. □

The following result follows from Theorem 3.

Corollary 2. *Let n be a positive integer. Then*

$$\alpha_{ch}(P_n) = \begin{cases} n \text{ if } n = 1, 2 \\ \lfloor \frac{n}{3} \rfloor \text{ if } n \geq 3. \end{cases}$$

Theorem 4. *If S' is a certified hop independent set of a cycle graph C_n, then $d_{C_n}(u, v) \geq 3$ for all $u, v \in S'$, where $u \neq v$.*

Proof. Let $S' \subseteq V(C_n)$ be a certified hop independent set of C_n. Let $u, v \in S'$. If $d_{C_n}(u, v) = 1$, then both u and v have only one neighbor in $V(C_n) \setminus S'$, a contradiction. Now, since S' is a hop independent set of C_n, $d_{C_n}(u, v) \neq 2$ for all $u, v \in S'$. Therefore, $d_{C_n}(u, v) \geq 3$ for all $u, v \in S'$, where $u \neq v$. □

The following result follows from Theorem 4.

Corollary 3. *Let n be a positive integer. Then*

$$\alpha_{ch}(C_n) = \begin{cases} n \text{ if } n = 3 \\ \lfloor \frac{n}{3} \rfloor \text{ if } n \geq 4. \end{cases}$$

The following is a realization results involving certified hop independence and hop independence parameters.

Theorem 5. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\alpha_{ch}(G) = a$ and $\alpha_h(G) = b$.*

Proof. Consider the following two cases:

Case 1 : $a = b$

Subcase 1 : $a \geq 5$ is odd

Consider the graph G in Figure 2.

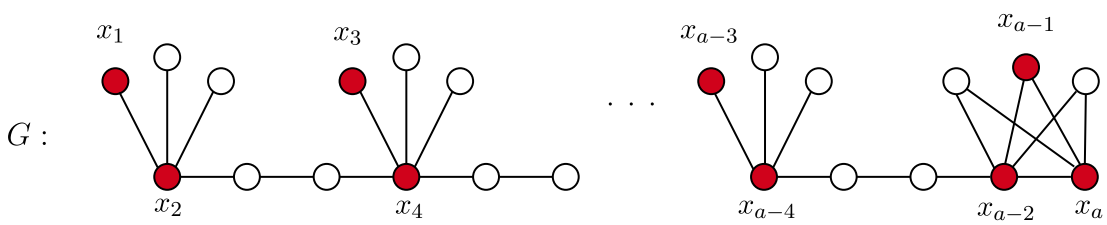

Figure 2: Graph G with $\alpha_{ch}(G) = a = \alpha_h(G)$.

Let $B_1 = \{x_1, x_2, ..., x_{a-1}, x_a\}$. Then B_1 is both a maximum certified hop independent and a maximum hop independent set of G. Thus,

$$\alpha_{ch}(G) = a = \alpha_h(G).$$

Next, for $a = 3 = b$, consider K_3. Then $\alpha_{ch}(K_3) = a = \alpha_h(K_3)$.

Subcase 2 : a is even

Consider the graph G' below.

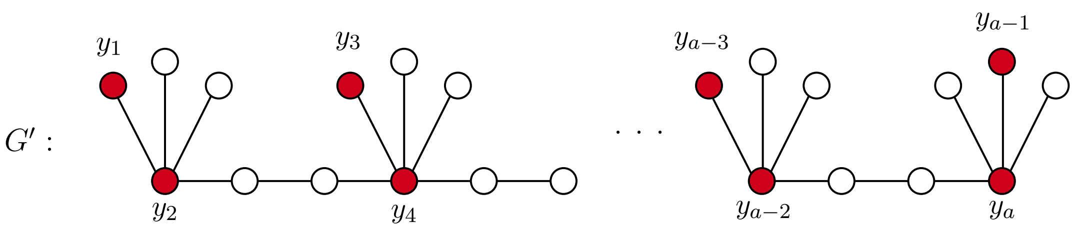

Figure 3: Graph G' with $\alpha_h(G') = a = \alpha_{ch}(G')$

Let $B_2 = \{y_1, y_2, \ldots, y_a\}$. Then B_2 is both a maximum hop independent and a maximum certified hop independent set of G. Therefore, $\alpha_h(G') = a = \alpha_{ch}(G')$.

Case 2 : $a < b$

Let $m = b - a$ and consider the following cases.

Subcase 1 : a is odd.

Consider the graph H below.

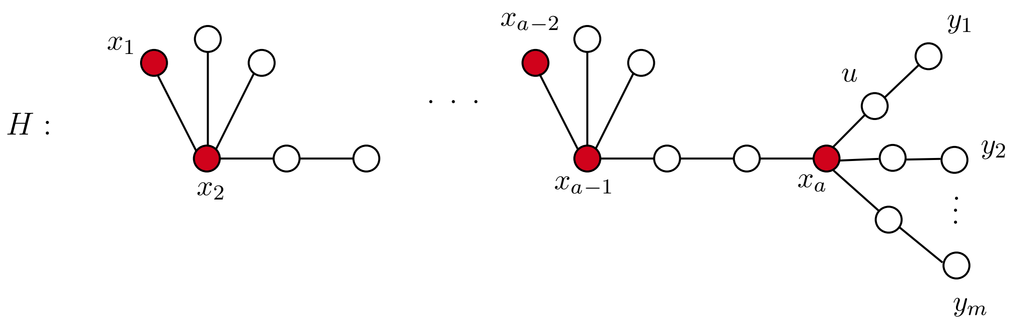

Figure 4: Graph H with $\alpha_{ch}(H) < \alpha_h(H)$

Let $B' = \{x_1, x_2, ..., x_a\}$ and $B'' = \{x_1, x_2, ..., x_{a-1}, u, y_1, y_2, ..., y_m\}$. Then B' and B'' are maximum certified hop independent and maximum hop independent set of H, respectively. Hence, $\alpha_{ch}(H) = a$ and $\alpha_h(H) = a + m = b$.

Case 2 : a is even

Consider the graph H' below.

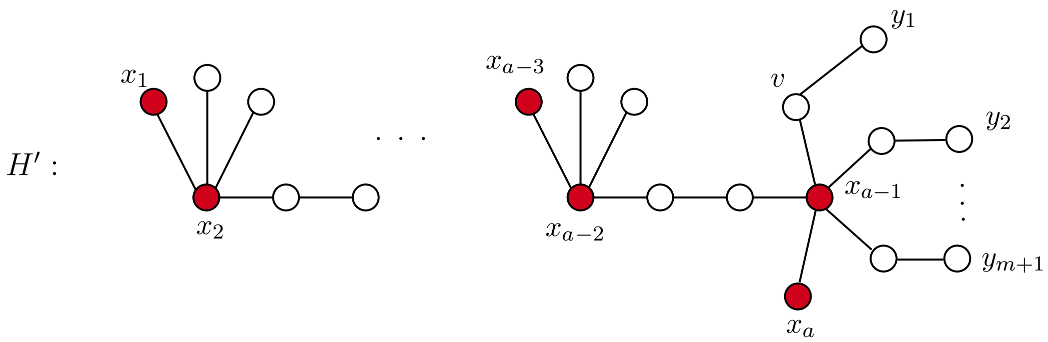

Figure 5: Graph H' with $\alpha_{ch}(H') < \alpha_h(H')$

Let $C_1 = \{x_1, x_2, ..., x_a\}$ and $C_2 = \{x_1, x_2, ..., x_{a-2}, v, y_1, y_2, ..., y_{m+1}\}$. Then C_1 and C_2 are maximum certified hop independent and maximum hop independent set of H', respectively. Thus, $\alpha_{ch}(H') = a$ and $\alpha_h(H') = a + m = b$. □

Remark 1. *The standard independence and certified hop independence parameters are incomparable.*

To see this, consider the graph G below.

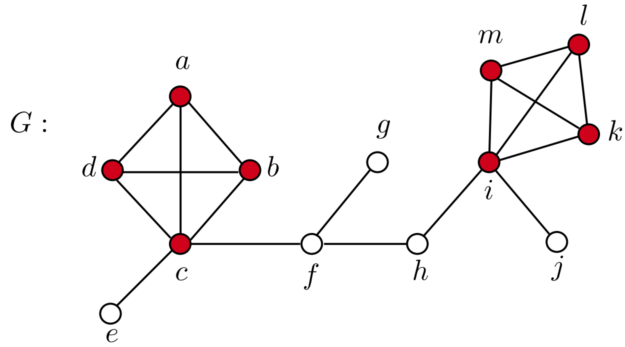

Figure 6: Graph G with $\alpha(G) < \alpha_{ch}(G)$

Let $C = \{a, b, c, d, i, k, l, m\}$. Then C is a maximum certified hop independent set of G. Thus, $\alpha_{ch}(G) = 8$. Next, let $C' = \{a, e, g, h, j, k\}$. Then C' is a maximum independent set of G. Hence, $\alpha(G) = 6$.

On the other hand, consider the graph H below.

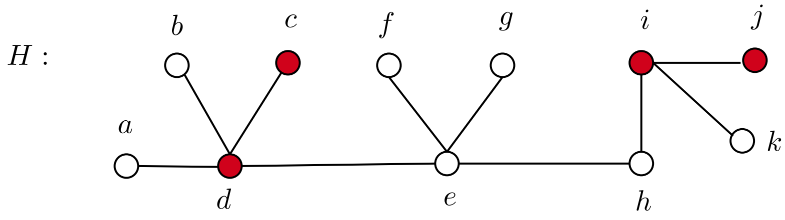

Figure 7: Graph H with $\alpha_{ch}(H) < \alpha(H)$

Let $O_1 = \{a, b, c, f, g, h, j, k\}$. Then O_1 is a maximum independent set of H. Thus, $\alpha(H) = 8$. Next, let $O_2 = \{c, d, i, j\}$. Then O_2 is a maximum certified hop independent set of H. Therefore, $\alpha_{ch}(H) = 4$. □

Theorem 6. *Let G and H be non-trivial graphs such that G and H have no complete subgraphs of order $|V(G)| - 1$ and $|V(H)| - 1$, respectively. Then $L \subseteq V(G + H)$ is a certified hop independent set of $G + H$ if and only if L satisfies one of the following conditions:*

(i) $L = L \cap V(G) = L_G$ *is clique in* G.

(ii) $L = L \cap V(H) = L_H$ *is clique in* H.

(iii) $L = L_G \cup L_H$, *where* L_G *and* L_H *are cliques in* G *and* H, *respectively.*

Proof. Let L be a certified hop independent set of $G + H$ and let $L_G = L \cap V(G)$ and $L_H = L \cap V(H)$. If $L_H = \varnothing$, then $L = L_G$. Let $x, y \in L_G$. Then

$$d_{G+H}(x, y) = d_G(x, y) \neq 2.$$

This means that $d_G(x, y) = 1$. Thus, L_G is a clique in G. Similarly, if $L_G = \varnothing$, then $L_H = L \cap V(H)$ is a clique in H. Suppose that $L_G \neq \varnothing$ and $L_H \neq \varnothing$. Since L is a hop independent, L_G and L_H are clique in G and H, respectively.

Conversely, suppose that (i) holds. Then $d_G(a, b) = 1 = d_{G+H}(a, b)$ for every $a, b \in L = L_G$. This means that L is a hop independent set of $G + H$. Since H is non-trivial, there exist at least two vertices $u, v \in V(H)$ such that $u, v \in N_{G+H}(a)$ and $u, v \in N_{G+H}(b)$. Hence, L is a certified hop independent set of $G + H$. Similarly, when (ii) holds, the assertion follows. Now, suppose that (iii) holds. Then L is a hop independent set of $G + H$. Since G and H have no complete subgraphs of order $|V(G)| - 1$ and $|V(H)| - 1$, respectively, clearly, L is a certified hop independent set of $G + H$. □

Corollary 4. *Let G and H be non-trivial graphs such that G and H have no complete subgraphs of order $|V(G)|-1$ and $|V(H)|-1$, respectively. Then*

$$\alpha_{ch}(G+H) = \omega(G) + \omega(H).$$

Proof. Let L be a maximum certified hop independent set of $G+H$. Then by Theoorem 6, $L = L_G \cup L_H$, where L_G and L_H are cliques in G and H, respectively. It follows that $|L_G| \leq \omega(G)$ and $|L_H| \leq \omega(H)$. Hence,

$$\alpha_{ch}(G+H) = |L| = |L_G| + |L_H| \leq \omega(G) + \omega(H).$$

On the other hand, suppose that $L = L_G \cup L_H$, where L_G and L_H are maximum cliques in G and H, respectively. Then by Theorem 6, L is a certified hop independent set of $G+H$. Thus, $\alpha_{ch}(G+H) \geq |L| = \omega(G) + \omega(H)$. Consequently,

$$\alpha_{ch}(G+H) = \omega(G) + \omega(H).$$

□

4. Conclusion

The concept of certified hop independence in graphs has been introduced and investigated in this study. Its relationship with hop independence parameter has been presented. Some bounds with respect to the order of a graph and exact values of parameters of some special graphs have been determined. Moreover, characterizations of certified hop independent sets in some graphs have been used to determine the exact values of parameters of some graphs. Some graphs that were not considered in this study could be an interesting topic to consider for further investigation of the concept. In addition, researchers may consider the complexity, algorithm, and real life application of the concept.

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References

[1] J. Hassan and S. Canoy Jr. Hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 15(2):1783–1796, 2022.

[2] J. Hassan, S. Canoy Jr., and A. Aradais. Hop independent sets in graphs. *Eur. J. Pure Appl. Math.*, 15(2):467–477, 2022.

[3] J. Hassan, A. Lintasan, and N.H. Mohammad. Some properties and realization problems involving connected outer-hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(3):1848–1861, 2023.

[4] J. Manditong, J. Hassan, LS Laja, AA. Laja, NHM. Mohammad, and SU. Kamdon. Conneted outer-hop independent dominating sets in graphs under some binary operations. *Eur. J. Pure Appl. Math.*, 16(3):1817–1829, 2023.