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On Quasi Generalized Exchange Algebras

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Abstract. A new type of algebraic structure, called a quasi generalized exchange algebra(qGE-algebra), with the GE-algebra conditions is introduced and its properties are investigated. The concepts of qGE-subalgebra, qGE-filter, closed qGE-filter and strong qGE-filter of a quasi GE-algebra are introduced and their relationships are discussed. The conditions for a subset of a quasi GE-algebra to be a qGE-filter are given.

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1. Introduction

L. Henkin and T. Skolem made significant contributions to the field of intuitionistic and non-classical logics during the 1950s by introducing Hilbert algebras. An interesting development came from A. Diego, who established the local finiteness of Hilbert algebras, as demonstrated in [3]. In an effort to extend the concept of dual BCK-algebras, H. S. Kim and Y. H. Kim introduced the notion of BE-algebras, as discussed in [4]. Drawing connections between Hilbert algebras and BE-algebras, A. Rezaei et al. explored their interrelations, as presented in [5]. The process of generalization is pivotal in the study of algebraic structures, leading to the introduction of GE-algebras by R. K. Bandaru et al., elaborated in [1]. An integral facet of GE-algebras' advancement lies in filter theory, which was leveraged by R. K. Bandaru et al. in the establishment of belligerent GE-filters within GE-algebras. Their properties were thoroughly investigated, as documented in [2].

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In this paper, we introduce a new type of algebraic structure, called a quasi GE-algebra (briefly, qGE-algebra), with the conditions of GE-algebra and investigate its properties. We show that GE-algebra and qGE-algebra are independent of each other through examples. We introduce the substructure of quasi GE-algebra called qGE-subalgebra, qGE-filter, strong qGE-filter, and closed qGE-filter, and further explore the relevant properties and interrelationship. We provide several conditions for a subset of a qGE-algebra to be a qGE-filter.

2. Preliminaries

We display the basic notions on GE-algebras.

A *GE-algebra* (see [[1]]) is a non-empty set X with a constant 1 and a binary operation "$*$" satisfying the following axioms:

(GE1) $\varpi * \varpi = 1$,

(GE2) $1 * \varpi = \varpi$,

(GE3) $\varpi * (\pi * \eta) = \varpi * (\pi * (\varpi * \eta))$

for all $\varpi, \pi, \eta \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall \varpi, \pi \in X)\,(\varpi \leq \pi \;\Leftrightarrow\; \varpi * \pi = 1)\,. \tag{1}$$

Every GE-algebra X satisfies the following items (see [[1]]).

$$(\forall \varpi \in X)\,(\varpi * 1 = 1)\,. \tag{2}$$

$$(\forall \varpi, \pi \in X)\,(\varpi * (\varpi * \pi) = \varpi * \pi)\,. \tag{3}$$

$$(\forall \varpi, \pi \in X)\,(\varpi \leq \pi * \varpi)\,. \tag{4}$$

$$(\forall \varpi, \pi, \eta \in X)\,(\varpi * (\pi * \eta) \leq \pi * (\varpi * \eta))\,. \tag{5}$$

$$(\forall \varpi \in X)\,(1 \leq \varpi \;\Rightarrow\; \varpi = 1)\,. \tag{6}$$

$$(\forall \varpi, \pi \in X)\,(\varpi \leq (\pi * \varpi) * \varpi)\,. \tag{7}$$

$$(\forall \varpi, \pi \in X)\,(\varpi \leq (\varpi * \pi) * \pi)\,. \tag{8}$$

$$(\forall \varpi, \pi, \eta \in X)\,(\varpi \leq \pi * \eta \;\Leftrightarrow\; \pi \leq \varpi * \eta)\,. \tag{9}$$

3. Quasi GE-algebras

In a GE-algebra X, we consider the following equality:

$$(\forall \kappa, \delta, \varsigma \in X)(\kappa * \delta = (\varsigma * \kappa) * (\varsigma * \delta)). \tag{10}$$

The following example shows that a GE-algebra may not satisfy the condition (10).

Example 1. *Let $X = \{1, a, b, c, d, e, f\}$ be a set with the binary operation "$*$" in the following Cayley Table.*

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	c	e	e	1
b	1	a	1	d	d	d	f
c	1	1	b	1	1	1	1
d	1	a	1	1	1	1	f
e	1	a	b	1	1	1	1
f	1	a	b	e	d	e	1

Then X is a GE-algebra and we have

$$(c * a) * (c * d) = 1 * 1 = 1 \neq e = a * d.$$

We would like to introduce a new type of algebra using (10) instead of (GE3) under the three conditions of GE-agebras.

Definition 1. *A quasi GE-algebra (briefly, qGE-algebra) is defined to be a set X with a special element "1" called the unit and a binary operation "$*$" that satisfies three conditions (GE1), (GE2) and* (10).

Example 2. *Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation " $*$ " given in the following table:*

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	b	e	d
b	d	c	1	e	b	a
c	c	d	e	1	a	b
d	b	e	d	a	1	c
e	e	b	a	d	c	1

It is routine to verify that $(X, *, 1)$ is a qGE-algebra.

Example 3. *Let $X = \{1, a, b\}$ be a set with the binary operation "$*$" in the following Cayley Table.*

$*$	1	a	b
1	1	a	b
a	b	1	a
b	a	b	1

Then X is a qGE-algebra.

Example 4. *Let X be the set of all integers or all real numbers. Define a binary operation "$*$" on X as follows:*

$$* : X \times X \to X, \ (\kappa, \delta) \mapsto \delta - \kappa.$$

It is routine to verify that $(X, *, 0)$ is a qGE-algebra.

Remark 1. *Example 1 explains that a GE-algebra may not be a qGE-algebra.*

The following example shows that a qGE-algebra may not be a GE-algebra.

Example 5. *The qGE-algebra X given in Example 2 is not a GE-algebra because of*

$$a * (b * a) = a * c = b \neq e = a * d = a * (b * 1) = a * (b * (a * a)).$$

By Remark 1 and Example 5, we can see that the two concepts GE-algebra and qGE-algebra are independent of each other.

In a qGE-algebra X, a binary relation "$\leq$" is also defined by (1). If X is a GE-algebra, then $(X, \leq)$ may not be a poset as shown in the following example.

Example 6. *Let $X = \{1, a, b, c, d\}$ be a set with a binary operation "$*$" given in the following table:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	a	1	d	d
c	1	a	1	1	1
d	1	a	1	1	1

Then $(X, *, 1)$ is a GE-algebra. We can observe that $c \leq d$ and $d \leq c$ but $c \neq d$. Hence $(X, \leq)$ is not be a poset.

But, if X is a qGE-algebra, then $(X, \leq)$ is a poset. In fact, it is reflexive by (GE1). Let $\kappa, \delta \in X$ be such that $\kappa \leq \delta$. Then $\kappa * \delta = 1$, and so $\delta * \kappa = (\kappa * \delta) * (\kappa * \kappa) = 1 * 1 = 1$, i.e., $\delta \leq \kappa$. Hence $\leq$ is symmetric. Let $\kappa, \delta, \varsigma \in X$ be such that $\kappa \leq \delta$ and $\delta \leq \varsigma$. Then $\kappa * \delta = 1$ and $\delta * \varsigma = 1$. Hence

$$\kappa * \varsigma = 1 * (\kappa * \varsigma) = (\kappa * \delta) * (\kappa * \varsigma) = \delta * \varsigma = 1,$$

i.e, $\kappa \leq \varsigma$. Thus $\leq$ is transitive. Therefore $(X, \leq)$ is a poset.

The relation $\leq$ is also antisymmetric. In fact, let $\kappa, \delta \in X$ be such that $\kappa \leq \delta$ and $\delta \leq \kappa$. Then $\kappa * \delta = 1$ and $\delta * \kappa = 1$. Hence

$$\delta = 1 * \delta = (\delta * 1) * (\delta * \delta) = (\delta * 1) * (\delta * \kappa) = 1 * \kappa = \kappa,$$

and therefore $\leq$ is antisymmetric.

In general, a GE-algebra has no left cancellation property as shown in the following example.

Example 7. *The GE-algebra X in Example 6 doesn't have the left cancellation property since $a * a = 1 = a * 1$, but $a \neq 1$.*

Theorem 1. *A qGE-algebra X has the left cancellation property.*

Proof. Let $\kappa, \delta, \varsigma \in X$ be such that $\kappa * \delta = \kappa * \varsigma$. Then

$$\delta = 1 * \delta = (\kappa * 1) * (\kappa * \delta) = (\kappa * 1) * (\kappa * \varsigma) = 1 * \varsigma = \varsigma$$

by (GE2) and (10). Hence κ is left-cancellative. Since κ is arbitrary, X has the left cancellation property.

Proposition 1. *Every qGE-algebra X satisfies:*

$$(\forall \kappa, \delta \in X)(\kappa \leq \delta \ \Leftrightarrow \ \kappa = \delta). \tag{11}$$

Proof. It is clear that if $\kappa = \delta$, then $\kappa \leq \delta$. Let $\kappa, \delta \in X$ be such that $\kappa \leq \delta$. Then $\kappa * \delta = 1 = \kappa * \kappa$ by (GE1). It follows from Theorem 1 that $\kappa = \delta$.

Remark 2. *By Proposition 1, we know that the binary relation $\leq$ is only the set*

$$\leq = \{(\kappa, \kappa) \in X \times X \mid \kappa \in X\}.$$

Proposition 2. *Every qGE-algebra X satisfies:*

$$(\forall \kappa, \delta \in X)(\kappa * \delta = (\delta * \kappa) * 1), \tag{12}$$

$$(\forall \kappa, \delta \in X)((\kappa * 1) * (\kappa * \delta) = \delta), \tag{13}$$

$$(\forall \kappa, \delta \in X)(\kappa * ((\kappa * 1) * \delta) = \delta). \tag{14}$$

$$(\forall \kappa, \delta, \varsigma \in X)(\kappa \leq \delta \ \Rightarrow \ \varsigma * \kappa \leq \varsigma * \delta). \tag{15}$$

Proof. The combination of (GE1) and (10) induces (12), and the combination of (GE2) and (10) induces (13). If we take $\kappa = 1$ in (12) and use (GE2), then $\delta = (\delta * 1) * 1$ for all $\delta \in X$. It follows from (13) that

$$\kappa * ((\kappa * 1) * \delta) = ((\kappa * 1) * 1) * ((\kappa * 1) * \delta) = \delta$$

for all $\kappa, \delta \in X$. (15) is clear by (10).

Corollary 1. *Every qGE-algebra X satisfies:*

$$(\forall \kappa, \delta, \varsigma \in X)((\varsigma * \kappa) * (\varsigma * \delta) = (\delta * \kappa) * 1), \tag{16}$$

$$(\forall \kappa, \delta, \varsigma \in X)(\kappa * \varsigma = \delta * \varsigma \ \Rightarrow \ \kappa = \delta). \tag{17}$$

Proof. The combination of (10) and (12) induces (16). Let $\kappa, \delta, \varsigma \in X$ be such that $\kappa * \varsigma = \delta * \varsigma$. Using (12), we have

$$\varsigma * \kappa = (\kappa * \varsigma) * 1 = (\delta * \varsigma) * 1 = \varsigma * \delta.$$

It follows from Theorem 1 that $\kappa = \delta$.

Remark 3. *In Proposition 2,* (12) *shows that* X *consists of elements* κ *that satisfy* $(\kappa * 1) * 1 = \kappa$. *That is,* $X = \{\kappa \in X \mid (\kappa * 1) * 1 = \kappa\}$.

Theorem 2. *Let* $(X, *_X, 1_X)$ *and* $(Y, *_Y, 1_Y)$ *be qGE-algebras. Let* $Z = X \times Y$ *be the Cartesian product of* X *and* Y. *Define a binary operation "$*$" on* Z *as follows:*

$$* : Z \times Z \to Z,\ ((\kappa, \varpi), (\delta, \pi)) \mapsto (\kappa *_X \delta, \varpi *_Y \pi). \tag{18}$$

Then $(Z, *, 1)$ *is a qGE-algebra where* $1 = (1_X, 1_Y)$. *We call it the product qGE-algebra of* $(X, *_X, 1_X)$ *and* $(Y, *_Y, 1_Y)$.

Proof. It is straightforward.

An example to explain Theorem 2 is presented as follows.

Example 8. *Let* $(X, *_X, 1_X)$ *be a qGE-algebra and consider the qGE-algebra* $(\mathbb{Z}, -, 0)$ *which is given in Example 4. Let* $Y = X \times \mathbb{Z}$ *and the binary operation "$*$" on* Y *is given as follows:*

$$(\kappa, \varpi) * (\delta, \pi) = (\kappa *_X \delta, \pi - \varpi)$$

for all $(\kappa, \varpi), (\delta, \pi) \in Y$. *Then* $(Y, *, 1)$ *is the product qGE-algebra of* $(X, *_X, 1_X)$ *and* $(\mathbb{Z}, -, 0)$ *where* $1 = (1_X, 0)$.

4. qGE-subalgebras

In what follows, let X be a qGE-algebra unless otherwise specified.

Definition 2. *A non-empty subset* E *of* X *is called a qGE-subalgebra of* X *if it satisfies:*

$$(\forall \kappa, \delta \in X)(\kappa, \delta \in E \ \Rightarrow\ \kappa * \delta \in E). \tag{19}$$

It is obvious that the singleton $\{1\}$ is a qGE-subalgebra of X.

Example 9. *Consider the qGE-algebra* X *given in Example 2. It is routine to verify that the set* $E = \{1, b, d\}$ *is a qGE-subalgebra of* X.

Example 10. *Let* $X := \mathbb{R} \setminus \{0\}$ *where* $\mathbb{R}$ *is the set of all real numbers. Define binary operations "$*_+$" and "$*_-$" on* X *as follows:*

$$*_+ : X \times X \to X,\ (\kappa, \delta) \mapsto \tfrac{\delta}{\kappa}, \tag{20}$$

$$*_- : X \times X \to X,\ (\kappa, \delta) \mapsto -\tfrac{\delta}{\kappa}, \tag{21}$$

respectively. It can be easily confirmed that $(X, *_+, 1)$ *and* $(X, *_-, -1)$ *are qGE-algebras. Let* $E := \mathbb{R}^+$ *and* $D := \mathbb{R}^-$ *be the set of all positive real numbers and the set of all positive real numbers, respectively. Then* E *is a qGE-subalgebra of* $(X, *_+, 1)$, *but* D *is not a qGE-subalgebra of* $(X, *_+, 1)$. *Also* D *is a qGE-subalgebra of* $(X, *_-, -1)$, *but* E *is not a qGE-subalgebra of* $(X, *_-, -1)$.

Proposition 3. *Every qGE-subalgebra of X contains the unit 1.*

Proof. It is straightforward by (GE1).

Theorem 3. *Let $(Z, *, 1)$ be the product qGE-algebra of qGE-algebras $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$. If D and E are qGE-subalgebras of X and Y, respectively, then their product $D \times E$ is a qGE-subalgebra of Z.*

Proof. Let $(\kappa, \delta), (\varpi, \pi) \in D \times E$. Then $\kappa, \varpi \in D$ and $\delta, \pi \in E$, and thus $\kappa *_X \varpi \in D$ and $\delta *_Y \pi \in E$. It follows that

$$(\kappa, \delta) * (\varpi, \pi) = (\kappa *_X \delta, \varpi *_Y \pi) \in D \times E.$$

Hence $D \times E$ is a qGE-subalgebra of Z.

The following example illustrates Theorem 3.

Example 11. *Consider the qGE-algebra $(X, *_X, 1_X)$ given in Example 2 and the qGE-algebra $(\mathbb{R}, -, 0)$ which is given in Example 4. Then $(X \times \mathbb{R}, *, 1)$ is the product qGE-algebra of $(X, *_X, 1_X)$ and $(\mathbb{R}, -, 0)$ where $*$ is defined by*

$$(\forall (\kappa, \delta), (r, s) \in X \times \mathbb{R})((\kappa, \delta) * (r, s) = (\kappa *_X r, s - \delta)).$$

Let $D = \{1, a\}$ and $E = \mathbb{Z}$. Then D and E are qGE-subalgebras of X and $\mathbb{R}$, respectively. Let $(\kappa, \delta), (u, v) \in D \times E$. Then $\kappa, u \in D$ and $\delta, v \in E$, and thus $\kappa *_X u \in D$ and $v - \delta \in E$. It follows that

$$(\kappa, \delta) * (u, v) = (\kappa *_X u, v - \delta) \in D \times E.$$

Hence $D \times E$ is a qGE-subalgebra of $X \times \mathbb{R}$.

Theorem 4. *The intersection of two qGE-subalgebras is a qGE-subalgebra.*

The union of two qGE-subalgebras may not be a qGE-subalgebra as shown in the following example.

Example 12. *Consider the qGE-algebra X given in Example 2. It is routine to verify that the set $E_1 = \{1, a\}$ and $E_2 = \{1, c\}$ are qGE-subalgebras of X. But $E_1 \cup E_2 = \{1, a, c\}$ is not a qGE-subalgebra of X since $a, c \in E_1 \cup E_2$ but $a * c = b \notin E_1 \cup E_2$.*

5. qGE-filters

In this section, we introduce the qGE-filter in a qGE-algebra in the same way as the GE-filter in a GE-algebra as follows.

Definition 3. *A subset F of X is called a qGE-filter of X if it satisfies:*

$$1 \in F, \tag{22}$$

$$(\forall \kappa, \delta \in X)(\kappa * \delta \in F, \kappa \in F \Rightarrow \delta \in F). \tag{23}$$

Example 13. *Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation "$*$" given in the following table:*

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	b	e	d
b	b	d	1	e	a	c
c	d	b	e	1	c	a
d	c	e	a	d	1	b
e	e	c	d	a	b	1

Then $(X, *, 1)$ is a qGE-algebra, and it is routine to check that the set $F = \{1, c, d\}$ is a qGE-filter of X.

Example 14. *Consider the qGE-algebra $(Y, *, 1)$ which is given in Example 8. Consider a subset $K := X \times \mathbb{N}^0$ of Y where $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of all natural numbers. It is clear that $1 = (1_X, 0) \in K$. Let $(\kappa_1, \varpi_1), (\kappa_2, \varpi_2) \in Y$ be such that $(\kappa_1, \varpi_1) * (\kappa_2, \varpi_2) \in K$ and $(\kappa_1, \varpi_1) \in K$. Then*

$$(\kappa_1, \varpi_1) * (\kappa_2, \varpi_2) = (\kappa_1 *_X \kappa_2, \varpi_2 - \varpi_1) \in K,$$

and so $\varpi_1 \in \mathbb{N}^0$ and $\varpi_2 - \varpi_1 \in \mathbb{N}^0$. Hence $\varpi_2 \in \mathbb{N}^0$, and thus $(\kappa_2, \varpi_2) \in K$. Therefore K is a qGE-filter of Y.

Theorem 5. *Every qGE-filter F of X satisfies:*

$$(\forall \varpi, \pi \in F)(Q(\varpi, \pi) := \{\kappa \in X \mid \varpi * \kappa = \pi\} \subseteq F), \tag{24}$$

Proof. Assume that F is a qGE-filter of X and let $\kappa \in Q(\varpi, \pi)$ for $\varpi, \pi \in F$. Then $\varpi * \kappa = \pi \in F$ and so $\kappa \in F$. Hence $Q(\varpi, \pi) \subseteq F$.

Proposition 4. *If F is a subset of X that satisfies the condition (24), then F satisfies the condition (23).*

Proof. Let F be a subset of X that satisfies the condition (24). Let $\kappa, \delta \in X$ be such that $\kappa \in F$ and $\kappa * \delta \in F$. Then the equality $\kappa * \delta = \kappa * \delta$ induces $\delta \in Q(\kappa, \kappa * \delta) \subseteq F$, and so F satisfies the condition (23).

We present the following open question.

Question 4. *If F is a subset of X that satisfies the condition (24), then does F include the unit 1?*

If we can get the positive answer to the Question 4, then we know that every subset F of X which satisfies the condition (24) is a qGE-filter of X.

If F is a subset of X that satisfies the condition (24) for all $\varpi, \pi \in X$ with $\varpi \neq \pi$, then F may not be a qGE-filter of X as shown in the following example.

Example 15. *Consider the qGE-algebra $(X, *, 1)$ given in Example 2. Let $F = \{1, d\}$. Then we can observe that $Q(1, d) = Q(d, 1) = \{d\} \subseteq F$ for all $1, d \in F$. But F is not a qGE-filter of X since $d \in F$ and $d * b = d \in F$ but $b \notin F$.*

Question 5. *Does any qGE-filter F of X satisfy the condition below?*

$$(\forall \kappa, \delta, \varsigma \in X)(\varsigma * (\delta * \kappa) \in F, \varsigma * \delta \in F \;\Rightarrow\; \varsigma * \kappa \in F). \tag{25}$$

The example below shows that the answer to Question 5 is negative.

Example 16. *Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation "$*$" given in the following table:*

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	d	e	b	c
b	c	d	1	b	e	a
c	b	e	c	1	a	d
d	d	c	e	a	1	b
e	e	b	a	d	c	1

Then $(X, *, 1)$ is a qGE-algebra, and it is routine to verify that the set $F := \{1, b, c\}$ is a qGE-filter of X. But it does not satisfy (25) since $a * (a * 1) = a * a = 1 \in F$ and $a * a = 1 \in F$, but $a * 1 = a \notin F$.

We use two conditions (22) and (25) to make a qGE-filter from a subset.

Theorem 6. *Let F be a subset of X that satisfies (22). If F satisfies the condition (25), then it is a qGE-filter of X.*

Proof. Assume that a subset F of X satisfies two conditions (22) and (25). Let $\kappa, \delta \in X$ be such that $\delta * \kappa \in F$ and $\delta \in F$. If we take $\varsigma := 1$ in (25) and use (GE2), then $1 * (\delta * \kappa) = \delta * \kappa \in F$ and $1 * \delta = \delta \in F$. It follows from (25) and use (GE2) that $\kappa = 1 * \kappa \in F$. Thus F is a qGE-filter of X.

For a subset F of X, consider the condition below.

$$(\forall \kappa, \delta, \varsigma \in X)(\kappa * (\delta * \varsigma) \in F \;\Rightarrow \delta * \varsigma \in F). \tag{26}$$

The following example shows that a qGE-filter F of X may not satisfy the condition (26).

Example 17. *Let $(X, *, 1)$ be a qGE-algebra and $F = \{1, b, c\}$ a qGE-filter of X given in Example 16. Then F does not satisfy (26) since $d*(c*e) = d*d = 1 \in F$ but $c*e = d \notin F$.*

We explore the conditions for a qGE-filter to satisfy the condition (25).

Theorem 7. *Let F be a qGE-filter of X. If F satisfies (26), then it satisfies the condition (25).*

Proof. Let F be a qGE-filter of X that satisfies (26). Let $\kappa, \delta, \varsigma \in X$ be such that $\varsigma * (\delta * \kappa) \in F$ and $\varsigma * \delta \in F$. Then $\delta * \kappa \in F$ and $\varsigma * \delta \in F$. It follows from (10) that $(\varsigma * \delta) * (\varsigma * \kappa) = \delta * \kappa \in F$ and $\varsigma * \delta \in F$. Hence $\varsigma * \kappa \in F$ by (23), and therefore the condition (25) is valid.

We explore the conditions for a subset F of X to be a qGE-filter of X.

Theorem 8. *Let F be a subset of X which includes the unit 1. If F satisfies the condition* (26)*, then F is a qGE-filter of X.*

Proof. Assume that a subset F of X includes the unit 1 and satisfies the condition (26). Let $\kappa, \delta \in X$ be such that $\kappa * \delta \in F$ and $\kappa \in F$. Then $\kappa * (1 * \delta) = \kappa * \delta \in F$ by (GE2). It follows from (GE2) and (26) that $\delta = 1 * \delta \in F$. Hence F is a qGE-filter of X.

Theorem 9. *Let F be a subset of X with the unit 1. If it satisfies:*

$$(\forall \kappa, \delta, \varsigma \in X)(\kappa * (\delta * \varsigma) \in F,\ \delta \in F \ \Rightarrow\ \kappa * \varsigma \in F), \tag{27}$$

then it is a qGE-filter of X.

Proof. Let $\kappa, \delta \in X$ be such that $\kappa * \delta \in F$ and $\kappa \in F$. Using (GE2), we have $1 * (\kappa * \delta) = \kappa * \delta \in F$, and so $\delta = 1 * \delta \in F$ by (GE2) and (27). Hence F is a qGE-filter of X.

In the following example, we can find a qGE-filter of X which does not satisfy the condition (27).

Example 18. *Consider the qGE-algebra $(X, *, 1)$ given in Example 13. It is routine to verify that the set $F := \{1, e\}$ is a qGE-filter of X. But F does not satisfy* (27) *since*

$$a * (e * b) = a * d = e \in F \text{ and } e \in F \text{ but } a * b = c \notin F.$$

Definition 6. *If a subset F of X satisfies* (22) *and* (27)*, we say that F is a strong qGE-filter of X.*

Example 19. *Let $X = \{1, a, b, c\}$ be a set with a binary operation "$*$" given in the following table:*

$*$	1	a	b	c
1	1	a	b	c
a	b	1	c	a
b	a	c	1	b
c	c	b	a	1

Then $(X, *, 1)$ is a qGE-algebra, and the set $F := \{1, c\}$ is a strong qGE-filter of X.

It is obvious that every strong qGE-filter is a qGE-filter (see Theorem 9). But a qGE-filter may not be a strong qGE-filter as seen in the following example.

Example 20. *Consider the qGE-algebra $(X, *, 1)$ given in Example 13. It is routine to verify that the set $F := \{1, e\}$ is a qGE-filter of X. But F is not a strong qGE-filter of X since $a * (e * b) = a * d = e \in F$ and $e \in F$ but $a * b = c \notin F$.*

The following example shows that a strong qGE-filter may not be a qGE-subalgebra.

Example 21. *Consider the qGE-algebra $(\mathbb{R} \setminus \{0\}, *_+, 1)$ given in Example 10. If we take $F_+ := \{\kappa \in \mathbb{R} \mid \kappa \geq 1\}$, then $1 \in F_+ \subseteq \mathbb{R} \setminus \{0\}$. Let $\kappa, \delta, \varsigma \in \mathbb{R} \setminus \{0\}$ be such that $\kappa * (\delta * \varsigma) \in F_+$ and $\delta \in F_+$. Then $\frac{\varsigma}{\kappa\delta} = \kappa * (\delta * \varsigma) \geq 1$ and $\delta \geq 1$. It follows that $\kappa * \varsigma = \frac{\varsigma}{\kappa} = \frac{\delta\varsigma}{\kappa\delta} \geq 1$, i.e., $\kappa * \varsigma \in F_+$. Hence F_+ is a strong qGE-filter of $\mathbb{R} \setminus \{0\}$. But F_+ is not a qGE-subalgebra of $\mathbb{R} \setminus \{0\}$ because of $3.5 * 2.5 = \frac{2.5}{3.5} < 1$ and so $3.5 * 2.5 \notin F_+$ for $2.5, 3.5 \in F_+$.*

In Example 10, the set $F_- := \{\kappa \in \mathbb{R} \mid \kappa \leq -1\}$ is neither a strong qGE-filter nor a qGE-subalgebra, as checked in the following example.

Example 22. *Consider the qGE-algebra $(\mathbb{R} \setminus \{0\}, *_+, 1)$ given in Example 10. Let $F_- := \{\kappa \in \mathbb{R} \mid \kappa \leq -1\}$. Then $-1 \in F_- \subseteq \mathbb{R}\setminus\{0\}$. Let $\kappa, \delta, \varsigma \in \mathbb{R}\setminus\{0\}$ be such that $\kappa*(\delta*\varsigma) \in F_-$ and $\delta \in F_-$. Then $\frac{\varsigma}{\kappa\delta} = \kappa * (\delta * \varsigma) \leq -1$ and $\delta \leq -1$. But $\kappa * \varsigma = \frac{\varsigma}{\kappa} = \frac{\delta\varsigma}{\kappa\delta} \geq 0$, i.e., $\kappa * \varsigma \notin F_-$. Thus F_- is not a strong qGE-filter of $\mathbb{R} \setminus \{0\}$. Also if $\kappa, \delta \in F_-$, then $\kappa \leq -1$ nd $\delta \leq -1$. Hence $\kappa * \delta = \frac{\delta}{\kappa} \geq 0$, that is, $\kappa * \delta \notin F_-$. Therefore F_- is not a strong qGE-subalgebra of $\mathbb{R} \setminus \{0\}$.*

The following example shows that a qGE-subalgebra may not be a strong qGE-filter.

Example 23. *Consider the qGE-algebra $(X, *, 1)$ given in Example 13. It is routine to verify that the set $F := \{1, e\}$ is a qGE-subalgebra of X. But F is not a strong qGE-filter of X since $a * (e * b) = a * d = e \in F$ and $e \in F$ but $a * b = c \notin F$.*

By Examples 21 and 23, we can see that the two concepts qGE-subalgebra and strong qGE-filter are independent of each other.

We discuss relationship between a qGE-subalgebra and a qGE-filter.

Theorem 10. *Every qGE-subalgebra is a qGE-filter.*

Proof. Let E be a qGE-subalgebra of X. Proposition 3 shows that $1 \in E$. Let $\kappa, \delta \in X$ be such that $\kappa*\delta \in E$ and $\kappa \in E$. Then $\kappa*1 \in E$ by (19), and so $\delta = 1*\delta = (\kappa*1)*(\kappa*\delta) \in E$ by (GE2), (10) and (19). Therefore E is a qGE-filter of X.

In the following example, we know that the converse of Theorem 10 may not be true.

Example 24. *Consider the qGE-filter $K := X \times \mathbb{N}^0$ of Y which is described in Example 14. Since*

$$(\kappa_1, 7) * (\kappa_2, 3) = (\kappa_1 *_X \kappa_2, -4) \notin K$$

for all $\kappa_1, \kappa_2 \in X$, we know that K is not a qGE-subalgebra of Y.

Definition 7. *A qGE-filter F of X is said to be closed if F is closed under the binary operation "$*$" on X, i.e., F is a qGE-subalgebra of X.*

Example 25. *Consider the qGE-algebra X given in Example 16. It is routine to verify that the set $F = \{1, b, c\}$ is a closed qGE-filter of X.*

Example 26. *Consider the qGE-algebra $(\mathbb{R}, *, 0)$ in Example 4. It is routine to verify that $(\mathbb{Z}, *, 0)$ is a closed qGE-filter of $(\mathbb{R}, *, 0)$.*

Proposition 5. *Every closed qGE-filter F of X satisfies:*

$$(\forall \kappa \in X)(\kappa \in F \ \Rightarrow \ \kappa * 1 \in F). \tag{28}$$

Proof. It is clear.

Remark 4. *The Proposition 5 is not applicable when the qGE-filter F of X is not closed. In fact, the qGE-filter $K := X \times \mathbb{N}^0$ of Y which is described in Example 14 is not closed (see Example 24), and $(\kappa, 5) \in K$ for all $\kappa \in X$. But $(\kappa, 5) * 1 = (\kappa, 5) * (1_X, 0) = (\kappa *_X 1_X, 0 - 5) = (\kappa *_X 1_X, -5) \notin K$.*

We present the following open question.

Question 8. *If a qGE-filter F of X satisfies the condition (28), then is it closed?*

Theorem 11. *The intersection of two qGE-filters is a qGE-filter.*

Proof. This can be easily checked.

The union of two qGE-filters may not be a qGE-filter as shown in the following example.

Example 27. *Consider the qGE-algebra X given in Example 2. It is routine to verify that the set $E_1 = \{1, a\}$ and $E_2 = \{1, c\}$ are qGE-filters of X. But $E_1 \cup E_2 = \{1, a, c\}$ is not a qGE-filter of X since $a \in E_1 \cup E_2$ and $a * b = c \in E_1 \cup E_2$ but $b \notin E_1 \cup E_2$.*

6. Conclusions

We have introduced a new type of algebraic structure, called a quasi GE-algebra (briefly, qGE-algebra) and investigated its properties. We have introduced the concepts of qGE-subalgebra, qGE-filter, closed qGE-filter and strong qGE-filter of a qGE-algebra and discussed their relationships between them. We have provided conditions for a subset of qGE-algebra to be a qGE-filter. In our future work, we will introduce different types of qGE-filters of a qGE-algebra and investigate their properties.

Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Data and code Availability

No data were used to support this study

Supplementary information

Not Applicable

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

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