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On the construction of a groupoid from an ample Hausdorff groupoid with twisted Steinberg algebra not isomorphic to its non-twisted Steinberg algebra

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Abstract. This study introduces an ample Hausdorff groupoid $\hat{A} \rtimes \mathcal{R}$ extracted from an ample Hausdorff groupoid $\mathcal{G}$ and a unital commutative ring R; a Hausdorff groupoid D which is the discrete twist over $\hat{A} \rtimes \mathcal{R}$. In the groupoid C*-algebra perspective, when $R = \mathbb{C}$ there is an isomorphism between the non-twisted groupoid C*-algebra ($C^*(\mathcal{G})$) and the twisted groupoid C*-algebra ($C^*(\hat{A} \rtimes \mathcal{R}; D)$). However, in this paper, in a purely algebraic setting, the non-twisted Steinberg algebra ($A_R(\mathcal{G})$) and the twisted Steinberg algebra ($A_R(D; \hat{A} \rtimes \mathcal{R})$) are non-isomorphic.

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1. Introduction

The study of groupoids was initiated by Brandt in 1926 in [2]. Brandt utilizes the notion of groupoid in [4] and other researchers produced more studies related to groupoids. In [12], groupoid is defined as a small category in which every morphism is invertible. Groupoid was used in various areas like the fibre bundle theory, in differential theory, in foliation theory and in differential topology.

In 1980s, Renault was motivated by the works of Feldman and Moore [5, 6] for von Neumann algebras and initiated the study of C*-algebras associated to groupoids in his PhD thesis [10]. This study proved itself useful as it caters many problem in C*-algebras.

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Significant works on characterization of Lei-type maps with C*-algebra are in [13] and [8]. Renault then introduced the twisted groupoid C*-algebras where the twist is done by incorporating a $\mathbb{T}$-valued 2-cocycle to its multiplication and involution. This study yields extreme importance in the structures of large classes of C*-algebras as seen in the works of Renault[11], Tu[14] and Barlak and Li[4]. In [15], Williams, Renault and Muhly proved that the groupoid C*-algebra ($C^*(\mathcal{G})$) and the twisted groupoid C*-algebra ($C^*(\hat{A}\rtimes\mathcal{R};D)$) are isomorphic when $R=\mathbb{C}$ with the additional conditions for G to be second countable locally compact groupoid with a Haar system and abelian isotropy.

Last 2010, thirty years after the introduction of twisted groupoid C*-algebras, Steinberg algebra was introduced independently in [3, 9]. It is an algebraic analogue of groupoid C*-algebra. In 2021, Becky Armstrong, Lisa Clark, et al. introduced the twisted Steinberg algebra in [1]. It is a purely algebraic analogue of Renault's twisted groupoid C*-algebra. This study is a generalisation of Steinberg algebra by twisting the convolution and involution in two ways: a locally constant 2-cocycle σ and a discrete twist Σ over a Hausdorff étale groupoid $\mathcal{G}$.

In this paper, we consider a purely algebraic perspective, that is, in the notion of Steinberg algebra. Without the analysis requirements for our groupoid, our goal is to show that the non-twisted Steinberg algebra and the twisted Steinberg algebra are non-isomorphic. Our first task is to construct an ample Hausdorff groupoid $\hat{A}\rtimes\mathcal{R}$ from an ample Hausdorff groupoid $\mathcal{G}$ and a unital commutative ring R with $R^\times$ as the set of units of R. From the unit space of $\hat{A}\rtimes\mathcal{R}$, we then construct a sequence $\hat{A}\times T \overset{i}{\hookrightarrow} D \overset{q}{\hookrightarrow} \hat{A}\rtimes\mathcal{R}$ where D is a Hausdorff groupoid, $T\leq R^\times$ and (D,i,q) is our desired twist over $\hat{A}\rtimes\mathcal{R}$. We then investigate properties of the Steinberg algebra of $\mathcal{G}$ over R or $A_R(\mathcal{G})$ and the twisted Steinberg algebra associated to the pair $(\hat{A}\rtimes\mathcal{R},D)$ or $A_R(D;\hat{A}\rtimes\mathcal{R})$ and look at when isomorphism between the two fails to hold.

2. Preliminaries

In this section, important concepts and notations on topological groupoids, Steinberg algebra and twisted Steinberg algebra arising from a discrete twist are presented.

Definition 1. [7] Let G be a set and $G^{(2)}$ be a subset of $G\times G$ such that there is a (composition) map $(\gamma,\alpha)\mapsto\gamma\alpha$ from $G^{(2)}$ to G. Suppose that there is an inverse map $\gamma\mapsto\gamma^{-1}$ on G such that $(\gamma^{-1})^{-1}=\gamma$. Then we say that G is a *groupoid* if the following are satisfied:

(G1) if $(\gamma,\alpha),(\alpha,\beta)\in G^{(2)}$, then $(\gamma\alpha,\beta),(\gamma,\alpha\beta)\in G^{(2)}$ and the following equation is satisfied: $(\gamma\alpha)\beta=\gamma(\alpha\beta)$;

(G2) for all $\gamma\in G$, $(\gamma^{-1},\gamma)\in G^{(2)}$;

(G3) if $(\gamma,\alpha)\in G^{(2)}$, then $(\gamma^{-1}\gamma)\alpha=\alpha$ and $\gamma(\alpha\alpha^{-1})=\gamma$.

We call $G^{(2)}$ as the set of all *composable pairs*. We write $G^{(3)}$ for the set of composable triples in G, that is, $G^{(3)}=\{(\alpha,\beta,\gamma):(\alpha,\beta),(\beta,\gamma)\in G^{(2)}\}$.

Lemma 1. [12] Let G be a groupoid and $\gamma, \beta \in G$. We say that $(\gamma, \beta) \in G^{(2)}$ if and only if $s(\gamma) = r(\beta)$.

Definition 2. [12] Define the functions s and r from G to itself by $s(\alpha) = \alpha^{-1}\alpha$ called the *source* of $\alpha \in G$ and $r(\alpha) = \alpha\alpha^{-1}$ called the *range* of $\alpha \in G$, respectively. The common image of r and s is the *unit space* of G and is denoted by $G^{(0)}$, that is, $G^{(0)} := s(G) = r(G)$.

Example 1. [12]

(i) Let G be a group with identity e. Then G is a groupoid with $G^{(2)} = G \times G; s(\gamma) = \gamma^{-1}\gamma = e; r(\gamma) = \gamma\gamma^{-1} = e$; and $G^{(0)} = \{e\}$.

(ii) If $\{G_i | i \in I\}$ is a family of groups with identities $\{\epsilon_1 | i \in I\}$, then the disjoint union $\bigcup_{i \in I} G_i$ has a groupoid structure with $d(g) = c(g) = \epsilon_i$ for every $g \in G_i$. The composition, defined only for pairs $(g, h) \in \bigcup_{i \in I} G_i \times G_i$, is just the relevant group law. This is known as a *group bundle*.

Lemma 2. [7] Let G be a groupoid. We have

(i) $(\alpha, \gamma), (\gamma, \beta) \in G^{(2)}$ and $\alpha\gamma = \beta\gamma$ imply $\alpha = \beta$. Similarly, if $(\gamma, \alpha), (\gamma, \beta) \in G^{(2)}$ and $\gamma\alpha = \gamma\beta$, then $\alpha = \beta$.

(ii) $r(\alpha\beta) = r(\beta)$ and $s(\alpha\beta) = s(\beta)$ for all $\alpha, \beta \in G^{(2)}$.

(iii) $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ for all $\alpha, \beta \in G^{(2)}$.

(iv) $r(x) = x = s(x)$ for all $x \in G^{(0)}$.

If $\gamma \in G$, then $(r(\gamma), \gamma)$ and $(\gamma, s(\gamma))$ belong to $G^{(2)}$, and $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$.

Definition 3. [12] Let ${}^xG = r^{-1}(x)$, $G^x = s^{-1}(x)$, and ${}^xG^y = {}^xG \cap G^y$. The *isotropy* of a groupoid G is the set $\text{Iso}(G) := \{\gamma \in G : r(\gamma) = s(\gamma)\} = \bigcup\limits_{x \in G^{(0)}} {}^xG^x$. We say that G is *principal* if $\text{Iso}(G) = G^{(0)}$.

Remark 1. [12] The isotropy of any groupoid is a group bundle.

Lemma 3. [7] A groupoid G is principal if $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective.

Definition 4. [12] A groupoid G is *effective* if the interior of the isotropy group of G is equal to its unit space.

Definition 5. [12] Given groupoid G and H, we call a map $\phi : G \to H$ a *groupoid homomorphism* if $(\phi \times \phi)(G^{(2)}) \subseteq H^{(2)}$ and $\phi(\alpha)\phi(\beta) = \phi(\alpha\beta)$ for all $(\alpha, \beta) \in G^{(2)}$.

The following concepts is taken from [12].

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure such that the composition and involution are continuous and $G^{(2)}$ has the induced topology from the product topology. Every groupoid is a topological groupoid with the discrete topology. An open set $B \subseteq G$ is an *open bisection* if $r|_B$ and $s|_B$ are homeomorphisms onto an open subset of G. A topological groupoid is *étale* if r (or equivalently s) is a local homeomorphism. An étale groupoid is *ample* if the topology of G has a basis of compact open bisections. Discrete group, discrete groupoids, and discrete space are some examples of an ample groupoid with the discrete topology.

Definition 6. [12] Given a topological space X and a topological ring R, the *open support* of a function $f : X \to R$ is the set $\text{supp}(f) := \{x \in X : f(x) \neq 0\} = f^{-1}(R\backslash\{0\})$. We say that f is *compactly supported* if $\text{supp}(f)$ is contained in a compact set.

We use the following notion for the characteristic function of a subset U of G: $1_U : G \to R$ defined by

$$1_U(g) = \begin{cases} 1 & \text{if}\, g \in U \\ 0 & \text{if}\, g \notin U \end{cases}$$

Let R^G be the set of all functions $f : G \to R$. Canonically R^G has the structure of an R-module with operations defined pointwise.

Definition 7. [12] Let $A_R(G)$ be the R-submodule of R^G generated by the set $\{1_U | U$ is a Hausdorff compact open subset of $G\}$, that is, $A_R(G) = \{f : G \to R | f$ is continuous and supp(f) is compact$\}$. The *convolution* of $f, g \in A_R(G)$ is defined as

$$(f * g)(x) := \sum_{\substack{y \in G, \\ s(y)=s(x)}} f(xy^{-1})g(y) = \sum_{\substack{(z,y)\in G^{(2)}, \\ zy=x}} f(z)g(y)$$

for all $x \in G$. The R-module $A_R(G)$ with the convolution, is called the *Steinberg algebra* of G over R.

The following example is a Steinberg algebra of $\mathbb{R}$ over $\mathbb{Z}$.

Example 2. [12] Consider the set of $G = \mathbb{R}\backslash\{0\}$ which is a group under multiplication. By Example 1, G is a groupoid with the following structures: $G^{(2)} = G \times G$, $s(\gamma) = \gamma^{-1}\gamma = 1$, $r(\gamma) = \gamma\gamma^{-1} = 1$ and $G^{(0)} = \{1\}$. We note that G is an ample Hausdorff groupoid with respect to the discrete topology. Its base is composed of singletons $\{r\}$ where $r \in G$. We let G as our unital commutative ring and $1_{\{r\}} : G \to \mathbb{Z}$ is the characteristic function of a subset $\{r\}$ of G. Then

$$\begin{aligned} A_{\mathbb{Z}}(G) &:= \text{span}_{\mathbb{Z}}\{1_{\{r\}} : \{r\} \text{is a compact open bisection}\} \\ &:= \{f : G \to \mathbb{Z}\}. \end{aligned} \tag{1}$$

Hence, $A_{\mathbb{Z}}(G)$ together with the convolution

$$(f * g)(x) := \sum_{\substack{y \in G, \\ s(y)=s(x)}} f(xy^{-1})g(y) = \sum_{\substack{(z,y)\in G^{(2)}, \\ zy=x}} f(z)g(y)$$

is a Steinberg algebra of $\mathbb{Z}$ over G.

The following discussions is taken from [1] where the Steinberg algebra is twisted via the discrete twist.

Definition 8. Let G be a Hausdorff étale groupoid, and R be a commutative unital ring, and let $T < R^{\times}$. A *discrete twist* by T over G is a sequence $G^{(0)} \times T \overset{i}{\hookrightarrow} \Sigma \overset{q}{\hookrightarrow} G$, where the groupoid $G^{(0)} \times T$ is regarded as trivial group bundle with fibres T, Σ is a Hausdorff étale groupoid with $\Sigma^{(0)} = i(G^{(0)} \times \{1\})$, and i and q are continuous groupoid homomorphisms that restricts to homeomorphism of unit spaces, such that the following condition holds:

(1) The sequence is *exact*, in the sense that $i(\{x\} \times T) = q^{-1}(x)$ for every $x \in G^{(0)}$, i is injective, and q is a quotient map

(2) The groupoid Σ is a *locally trivial G-bundle*, in the sense that for each $\alpha \in G$, there is an open bisection B_α of G containing α, and a continuous map $P_\alpha : B_\alpha \to \Sigma$ such that

  (i) $q \circ P_\alpha = id_{B_\alpha}$

  (ii) the map $(\beta, z) \to i(r(\beta), z)P_\alpha(\beta)$ is a homeomorphism from $B_\alpha \times T$ to $q^{-1}(B_\alpha)$.

(3) The image of i is *central* in Σ, in the sense that $i(r(\epsilon), z)\epsilon = \epsilon i(s(\epsilon), z)$ for all $\epsilon \in \Sigma$ and $z \in T$.

We will denote a discrete twist over G by (Σ, i, q).

The following is an example of a discrete twist.

Example 3. Consider the the set of integers $\mathbb{Z}$ as our groupoid and unital commutative ring. Then our groupoid $\mathbb{Z}$ will have the following structures: $\mathbb{Z}^{\times} = \{-1, 1\}$, $s(x) = x^{-1} + x = \{0\}$, $r(x) = x + x^{-1} = \{0\}$, $\mathbb{Z}^{(0)} = \{0\}$ and $\mathbb{Z}^{(2)} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | s(x) = r(y)\}$. Note that $\mathbb{Z}$ is a Hausdorff étale groupoid having the discrete topology. Let the function $\sigma : \mathbb{Z}^{(2)} \to T \leq R^{\times}$ be a continuous 2-cocycle. Choose $T \leq \mathbb{Z}^{\times} = \{1\}$. Then $\mathbb{Z} \times T$ is a Hausdorff groupoid with respect to the product topology with multiplication given by

$$(\alpha, z)(\beta, w) := (\alpha\beta, \sigma(\alpha, \beta)zw),$$

and inversion given by

$$(\alpha, z)^{-1} := (\alpha^{-1}, \sigma(\alpha, \alpha^{-1})^{-1}z^{-1}) = (\alpha^{-1}, \sigma(\alpha^{-1}, \alpha)^{-1}z^{-1}),$$

for all $(\alpha, \beta) \in \mathbb{Z}^{(2)}$ and $z, w \in T$. Then, $(\mathbb{Z} \times T, i, q)$ is a discrete twist by T over $\mathbb{Z}$ with the sequence $\mathbb{Z}^{(0)} \times T \overset{i}{\hookrightarrow} \mathbb{Z} \times T \overset{q}{\hookrightarrow} \mathbb{Z}$ where $i(x, z) = (x, z)$ and $q(\gamma, z) = \gamma$ for all $x, \gamma \in \mathbb{Z}$ and $z \in T$.

Definition 9. A continuous map $P_\alpha : B_\alpha \to \Sigma$ is called a *continuous local section* if it satisfies Definition 8(2i). If $P(G^{(0)}) = \Sigma^{(0)} = i(G^{(0)} \times 1)$, then P_α is a *continuous global section.*

Definition 10. Let G be an ample Hausdorff groupoid and let (Σ, i, q) be a discrete twist by $T \leq R^{\times}$ over G. Denote $C(\Sigma, R)$ as the collection of continuous functions from Σ to R. We say that $f \in C(\Sigma, R)$ is *T-equivariant* if $f(z \cdot \epsilon) = zf(\epsilon)$ for all $z \in T$ and $\epsilon \in \Sigma$, and we define $A_R(G; \Sigma) := \{f \in C(\Sigma, R) : f \text{ is } T\text{-equivariant and } \overline{q(supp(f))} \text{ is compact}\}$.

Lemma 4. Let G be an ample Hausdorff groupoid, and let (Σ, i, q) be a discrete twist by $T \leq R^{\times}$ over G. Then $A_R(G; \Sigma)$ is an R-submodule of $C(\Sigma, R)$.

Definition 11. Let G be an ample Hausdorff groupoid, and let (Σ, i, q) be a discrete twist by $T \leq R^{\times}$ over G. Let $P : G \to \Sigma$ be any continuous global section. There is a multiplication called *convolution* on the R-module $A_R(G; \Sigma)$, given by

$$(f *_{\Sigma} g)(\epsilon) := \textstyle\sum_{\gamma \in G^{s(q(\epsilon))}} f(\epsilon P(\gamma)) g(P(\gamma)^{-1}),$$

under which $A_R(G; \Sigma)$ is an R-algebra. We call $A_R(G; \Sigma)$ the *twisted Steinberg algebra* of G associated to the pair (G, Σ).

The following is an example of a twisted Steinberg algebra of a discrete group $\mathbb{Z}$ over a commutative ring R called the twisted discrete group algebra.

Example 4. Let R be a discrete commutative unital ring and consider an ample Hausdorff groupoid $\mathbb{Z}$ with the discrete topology. Let $\sigma : \mathbb{Z} \mapsto R^{\times}$ be a continuous 2-cocycle which is locally constant. Then the set $A_R(\mathbb{Z}, \sigma) = \text{span } \{1_{\{z\}} : \mathbb{Z} \mapsto R \,|\, \{z\} \text{ is compact open bisection of } \mathbb{Z}\}$ with the twisted convolution

$$(f *_{\sigma} g)(z) := \textstyle\sum_{(x,y) \in \mathbb{Z}^{(2)}_{xy=z}} \sigma(x, y) f(x) g(y)$$

is the twisted Steinberg algebra of $\mathbb{Z}$ over R associated to the pair $(\mathbb{Z}, \sigma)$ denoted as $A_R(\mathbb{Z}, \sigma)$.

3. The groupoid $\hat{A} \rtimes \mathcal{R}$ and discrete twist (D, i, q)

In this section, we will define what is $\hat{A} \rtimes \mathcal{R}$ from a groupoid $\mathcal{G}$ and a commutative unital ring R, investigate its properties and construct the discrete twist (D, i, q). Throughout, $\mathcal{G}$ is an ample Hausdorff groupoid.

Let $A = \text{Iso}(\mathcal{G}) = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma)\}$ be the isotropy of $\mathcal{G}$. For $u \in \mathcal{G}^{(0)}$, we let $A_u = \{\gamma \in A : s(\gamma) = u\}$. We define $\hat{A}_u = \{\chi : A_u \to R^{\times} | \chi \text{ is a continuous group homomorphism}\}$ with A_u and $R^{\times}$ having the subspace and discrete topology, respectively.

Define $\mathcal{R} = \mathcal{G}/A = \{\gamma A : \gamma \in \mathcal{G}\}$. Let $\dot{\gamma} = \gamma A \in \mathcal{R}$, $\hat{A} = \{(\chi, u) : u \in \mathcal{G}^{(0)}, \chi \in \hat{A}_u\}$ and $\hat{A} \rtimes \mathcal{R} = \{(\chi, u, \dot{\gamma}) : (\chi, u) \in \hat{A}, r(\gamma) = u\}$.

Theorem 1. *Let $\mathcal{G}$ be an ample Hausdorff groupoid and R be a commutative unital ring. Then $\mathcal{R}$ is an ample Hausdorff groupoid.*

Proof. Let $m : \mathcal{R}^{(2)} \to \mathcal{R}$ be the composition map defined by $m((\dot{\alpha}, \dot{\beta})) = \dot{\alpha\beta} = \alpha\beta A$ where $\alpha\beta \in \mathcal{G}$ and $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $i : \mathcal{R} \to \mathcal{R}$ be defined by $i(\dot{\gamma}) = \dot{\gamma}^{-1} = \gamma^{-1}A$.

Let $(\dot{\alpha}, \dot{\beta}), (\dot{\gamma}, \dot{\mu}) \in \mathcal{R}^{(2)}$ such that $(\dot{\alpha}, \dot{\beta}) = (\dot{\gamma}, \dot{\mu})$. Then $\alpha A = \gamma A$ and $\beta A = \mu A$. So that $m((\dot{\alpha}, \dot{\beta})) = \alpha\beta A = \alpha A \beta A = \gamma A \mu A = \gamma\mu A = m((\dot{\gamma}, \dot{\mu}))$. Thus, m is well-defined.

For $\dot{\alpha}, \dot{\gamma} \in \mathcal{R}$ with $\dot{\alpha} = \dot{\gamma}$, $i(\dot{\gamma}) = \gamma^{-1}A = \alpha^{-1}A = i(\dot{\alpha})$. Also, for $\dot{\gamma}, \dot{\beta} \in \mathcal{R}$ with $\dot{\gamma} = \dot{\beta}$,

$$s(\dot{\gamma}) = \dot{\gamma}^{-1}\dot{\gamma} = (\gamma A)^{-1}\gamma A = \gamma^{-1}A\gamma A = \beta^{-1}A\beta A = \dot{\beta}^{-1}\dot{\beta} = s(\dot{\beta});$$

$$r(\dot{\gamma}) = \dot{\gamma}\dot{\gamma}^{-1} = \gamma A \gamma^{-1} A = \beta A \beta^{-1} A = \dot{\beta}\dot{\beta}^{-1} = r(\dot{\beta}).$$

Thus, the inverse, source and range maps are also well-defined.

Let $(\dot{\alpha}, \dot{\beta}), (\dot{\beta}, \dot{\gamma}) \in \mathcal{R}^{(2)}$. Then $s(\dot{\alpha}) = \dot{\beta}\dot{\beta}^{-1}$ and $s(\dot{\beta}) = \dot{\gamma}\dot{\gamma}^{-1}$ since α and β are composable. Hence,

$$\begin{aligned}
s(\dot{\alpha\beta}) &= (\dot{\alpha\beta})^{-1}(\dot{\alpha\beta}) \\
&= (\alpha\beta A)^{-1}(\alpha\beta A) \\
&= \beta^{-1}\alpha^{-1}A\alpha\beta A \\
&= \alpha^{-1}\alpha\beta^{-1}\beta A \\
&= s(\alpha\beta)A \\
&= s(\beta)A, \\
&= \beta^{-1}\beta A.
\end{aligned}$$

Also, $r(\dot{\gamma}) = \dot{\gamma}\dot{\gamma}^{-1} = \dot{\beta}^{-1}\dot{\beta} = \beta^{-1}\beta\ A$. Hence, $s(\dot{\alpha\beta}) = r(\dot{\gamma})$. Thus, $(\dot{\alpha}\dot{\beta}, \dot{\gamma}) \in \mathcal{R}^{(2)}$. We also have $\dot{\beta}\dot{\gamma} = \dot{\beta\gamma}$. Hence,

$$\begin{aligned}
r(\dot{\beta\gamma}) &= (\dot{\beta\gamma})(\dot{\beta\gamma})^{-1} \\
&= (\beta\gamma A)(\beta\gamma A \\
&= \beta\gamma A\gamma^{-1}\beta^{-1}A \\
&= \beta\beta^{-1}\gamma\gamma^{-1}A \\
&= r(\beta\gamma)A \\
&= r(\beta)A \\
&= \beta\beta^{-1}A.
\end{aligned}$$

Since $s(\dot{\alpha}) = \dot{\alpha}^{-1}\dot{\alpha} = \dot{\beta}\dot{\beta}^{-1} = \beta A\beta^{-1}A = \beta\beta^{-1}A$, then $r(\dot{\beta}\dot{\gamma}) = s(\dot{\alpha})$ and $(\dot{\alpha}, \dot{\beta}\dot{\gamma}) \in \mathcal{R}^{(2)}$. Now, $(\dot{\alpha}\dot{\beta})\dot{\gamma} = (\alpha\beta A)\gamma A = (\alpha A(\beta\gamma A)) = \dot{\alpha}(\dot{\beta\gamma}) = \dot{\alpha}(\dot{\beta}\dot{\gamma})$. Let $\dot{\gamma} \in \mathcal{R}$. Now,

$$(\dot{\gamma}^{-1})^{-1} = (\gamma^{-1}A)^{-1} = (\gamma^{1})^{-1}A = \gamma A = \dot{\gamma}.$$

For $\dot{\gamma} \in \mathcal{R}$, $r(\dot{\gamma}^{-1}) = \dot{\gamma}^{-1}(\dot{\gamma}^{-1})^{-1} = \dot{\gamma}^{-1}\dot{\gamma}$. Hence, $(\dot{\gamma}, \dot{\gamma}^{-1}) \in \mathcal{R}^{(2)}$. Let $(\dot{\beta}, \dot{\gamma}) \in \mathcal{R}^{(2)}$. Then , $(\dot{\beta}\dot{\gamma})\dot{\gamma}^{-1} = \dot{\beta}(\dot{\gamma}\dot{\gamma}^{-1}) = \dot{\beta}r(\dot{\gamma}) = \dot{\beta}s(\dot{\beta}) = \dot{\beta}$ and $\dot{\gamma}^{-1}(\dot{\gamma}\dot{\beta}) = (\dot{\gamma}^{-1}\dot{\gamma})\dot{\beta} = s(\dot{\gamma})\dot{\beta} = r(\dot{\beta})\dot{\beta} = \dot{\beta}$. Thus, $\mathcal{R}$ is a groupoid.

Let $\mathcal{R}$ be a topological space with the quotient topology $\tau_{\mathcal{R}}$. Define the quotient map $\pi_{\mathcal{R}} : \mathcal{G} \to \mathcal{R}$ by $\pi_{\mathcal{R}}(\alpha) = \alpha A = \dot{\alpha}$. Define the topology for $\mathcal{R} \times \mathcal{R}$ and $\mathcal{R}^{(2)}$ as follow: $\tau_{\mathcal{R}\times\mathcal{R}} = \{U \times V : U, V \in \tau_{\mathcal{R}}\}$ and $\tau_{\mathcal{R}^{(2)}} = \{(U \times V) \cap \mathcal{R}^{(2)} : U \times V \in \tau_{\mathcal{R}\times\mathcal{R}}\}$. Let U

be an open set in $\mathcal{R}$. Then $V = \pi_{\mathcal{R}}^{-1}(U)$ is open in $\mathcal{G}$. Let $(\dot{\gamma}, \dot{\beta}) \in m^{-1}(U) \subseteq \mathcal{R}^{(2)}$. Then $\pi(\gamma)\pi(\beta) \in U$, i.e., $(\dot{\gamma}, \dot{\beta}) \in U \times U$ and $(\dot{\gamma}, \dot{\beta}) \in (U \times U) \cap \mathcal{R}^{(2)} \in \tau_{\mathcal{R}^{(2)}}$ since $U \times U \in \tau_{\mathcal{R}\times\mathcal{R}}$. Since $(\dot{\alpha}, \dot{\beta})$ is chosen arbitrarily, every element in $m^{-1}(U)$ is contained in some open set in $\tau_{\mathcal{R}^{(2)}}$ and m is continuous. Also, let M be an open set in $\mathcal{R}$. Then there exists $V = \pi_{\mathcal{R}}^{-1}(M) = \{\gamma \in \mathcal{G} : \pi_{\mathcal{R}}(\gamma) \in M\} \subset \mathcal{G}$. Note that $i^{-1}(M) = \pi_{\mathcal{R}}(V) = \{\gamma \in \mathcal{G} : \pi_{\mathcal{R}}(\gamma^{-1}) \in M\} = \{\gamma \in \mathcal{G} : \gamma^{-1} \in M\}$. For any open set $U' \in \mathcal{R}$, $\pi_{\mathcal{R}}^{-1}(U')$ is open in $\mathcal{G}$. In particular, $V = \pi_{\mathcal{R}}^{-1}(M)$ is open in $\mathcal{G}$. It follows that $\pi_{\mathcal{R}}(V)$ is open in $\mathcal{R}$. Hence, $i^{-1}(M)$ is open in $\mathcal{R}$ and i is continuous. Thus, $\mathcal{R}$ is a topological groupoid.

Suppose that $\dot{\gamma}$ and $\dot{\beta}$ are distinct points in $\mathcal{R}$. Then for $\gamma, \beta \in \mathcal{G}$, $\pi_{\mathcal{R}}(\gamma) = \gamma A$ and $\pi_{\mathcal{R}}(\beta) = \beta A$ where $\gamma A \neq \beta A$. Let $\dot{U}$ be an open set in $\mathcal{G}$ defined as $\dot{U} = \{\alpha \in \mathcal{G} : \pi_{\mathcal{R}}(\alpha) \neq \beta A\}$. Since $\gamma A \neq \beta A$, then $\gamma \in \dot{U}$. Hence, $\dot{U}$ is an open neighborhood in $\mathcal{G}$ containing γ. Now, let $\dot{V}$ be an open set in $\mathcal{G}$ defined as $\dot{V} = \{\alpha \in \mathcal{G} : \pi_{\mathcal{R}}(\alpha) \neq \gamma A\}$. Since $\beta A \neq \gamma A$, then $\beta \in \dot{V}$. Hence, $\dot{V}$ is an open neighborhood containing β. Let $\alpha \in \dot{U} \cap \dot{V}$. Then, $\pi_{\mathcal{R}}(\alpha) \neq \gamma A$ and $\pi_{\mathcal{R}}(\alpha) \neq \beta A$ which is a contradiction since γA and βA are distinct. Thus, we have found open sets $\dot{U}$ and $\dot{V}$ in $\mathcal{G}$ such that $\dot{U} \cap \dot{V}$ is empty. It follows that $\pi(\dot{V})$ and $\pi(\dot{V})$ are open in $\mathcal{R}$ with $\pi(\dot{V}) \cap \pi(\dot{V}) = \varnothing$ and each contains distinct equivalence classes. Thus, $\mathcal{R}$ is Hausdorff.

Since $\mathcal{R}$ is Hausdorff then $\mathcal{R}^{(0)}$ is Hausdorff. Let $\mathcal{B}$ be a basis for a topology in $\mathcal{G}$. Then $\pi_{\mathcal{R}}(\mathcal{B}) = \{\pi_{\mathcal{R}}(B) : B \in \mathcal{B}\}$ is a basis for the quotient topology in $\mathcal{R}$. Let V be an open subset of R and consider $s|_{\pi_{\mathcal{R}}(\mathcal{B})} : \pi_{\mathcal{R}}(\mathcal{B}) \to V$. We denote $s_1 = s|_{\pi_{\mathcal{R}}(\mathcal{B})}$. Let U be an open subset of V. Then $\pi_{\mathcal{R}}^{-1}(U)$ is open in $\mathcal{G}$. Since $\pi_{\mathcal{R}}(\mathcal{B})$ is a basis for the topology on $\mathcal{R}$, then there exists basic element $\pi_{\mathcal{R}}(B)$ in $\pi_{\mathcal{R}}(\mathcal{B})$ containing $s_1^{-1}(U)$. Now, let $s_1^{-1} = (s|_{\pi\mathcal{R}(\mathcal{B})})^{-1} : V \to \pi_{\mathcal{R}}(\mathcal{B})$. Let $B \subseteq \pi_{\mathcal{R}}(\mathcal{B})$ which is open in $\mathcal{R}$. Then there exists $W \subseteq \mathcal{G}$ such that W is the inverse image of of B under $\pi_{\mathcal{R}}$. Then $W = \pi_{\mathcal{R}}^{-1}(B) = \{\gamma \in \mathcal{G} : \pi_{\mathcal{R}}(\gamma) \in B\}$. Since $\pi_{\mathcal{R}}$ is surjective, $s_1^{-1}(B) = \pi_{\mathcal{R}}(W)$. Since $\pi_{\mathcal{R}}$ is a quotient map then it is an open map. Thus, for any open set U' in $\mathcal{R}$, $\pi_{\mathcal{R}}(U)$ is open in $\mathcal{G}$. In particular, $W = \pi_{\mathcal{R}}^{-1}(B)$ is open in $\mathcal{G}$. It follows that $\pi_{\mathcal{R}}(W)$ is open in $\mathcal{R}$. Hence, $s_1^{-1}(W)$ is open and the source map is a homeomorphism onto an open subset of $\mathcal{R}$. Similarly, the range map is also homeomorphic onto an open subset of $\mathcal{R}$. Therefore, $\mathcal{R}$ is an ample Hausdorff groupoid with respect to the quotient topology. □

From now on, we denote the elements of $\hat{A} \rtimes \mathcal{R}$ by $(\chi, \dot{\gamma})$ with $\chi \in \hat{A}_{r(\gamma)}$. A subset C of $\hat{A}$ is closed if and only if for all sequences where x_n converges to x such that $x_n \in C$, then $x \in C$. We will denote our topology for $\hat{A}$ as $\tau_{\hat{A}} = \{D : D = C^c$, C is closed in $\hat{A}\}$ and C^c stands for the compliment of C. Also, $\tau_{\hat{A}\times\mathcal{R}} = \{U \times V : U \in \tau_{\hat{A}}$ and $V \in \tau_{\mathcal{R}}\}$, and $\tau_{(\hat{A}\rtimes\mathcal{R})\times(\hat{A}\rtimes\mathcal{R})} = \{A \times B : A, B \in \tau_{\hat{A}\rtimes\mathcal{R}}\}$ which gives us the relative topology for $(\hat{A} \rtimes \mathcal{R})^{(2)}$ as $\tau_{(\hat{A}\rtimes\mathcal{R})^{(2)}} = \{(A \times B) \cap (\hat{A} \rtimes \mathcal{R})^{(2)} : A \times B \in \tau_{(\hat{A}\rtimes\mathcal{R})\times(\hat{A}\rtimes\mathcal{R})}\}$. Let $r((\chi, \dot{\gamma})) = (\chi, r(\gamma))$ and $s((\chi, \dot{\gamma})) = (\chi \cdot \gamma, s(\gamma))$, respectively. The set of composable pairs is $(\hat{A} \rtimes \mathcal{R})^{(2)} = \{((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')) | \chi' = \chi \cdot \gamma\}$. Note that $\chi \cdot \gamma(a) = \chi(\gamma a \gamma^{-1})$. Also, for every $u \in \mathcal{G}$, $\chi \cdot u = \chi$ since

$$\chi \cdot u(a) = \chi(uau^{-1}) = \chi(auu^{-1}) = \chi(ar(u)) = \chi(au) = \chi(as(a)) = \chi(a).$$

Also, $i((\chi, \dot{\gamma})) = (\chi \cdot \gamma, \dot{\gamma}^{-1})$ and $m((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')) = (\chi, \dot{\gamma}\dot{\gamma}')$ are the inversion and composition maps, respectively.

Theorem 2. *Let $\mathcal{G}$ be an ample Hausdorff groupoid and R be a commutative unital ring. Then $\hat{A} \rtimes \mathcal{R} = \{(\chi, u, \dot{\gamma}) : (\chi, u) \in \hat{A}, r(\gamma) = u\}$ is a groupoid.*

Proof.
Suppose that $((\chi_1, \dot{\gamma}_1), (\chi_1', \dot{\gamma}_1'))$, $((\chi_2, \dot{\gamma}_2), (\chi_2', \dot{\gamma}_2')) \in (\hat{A} \rtimes \mathcal{R})^{(2)}$ with

$$((\chi_1, \dot{\gamma}_1), (\chi_1', \dot{\gamma}_1')) = ((\chi_2, \dot{\gamma}_2), (\chi_2', \dot{\gamma}_2')).$$

Then, $(\chi_1, \dot{\gamma}_1) = (\chi_2, \dot{\gamma}_2)$ and $(\chi_1', \dot{\gamma}_1') = (\chi_2', \dot{\gamma}_2')$. Now,

$$m(((\chi_1, \dot{\gamma}_1), (\chi_1', \dot{\gamma}_1'))) = (\chi_1, \dot{\gamma}_1\dot{\gamma}_1') = (\chi_2, \dot{\gamma}_2\dot{\gamma}_2') = m(((\chi_2, \dot{\gamma}_2), (\chi_2', \dot{\gamma}_2'))).$$

Let $(\chi, \dot{\gamma}), (\chi', \dot{\gamma}') \in \hat{A} \rtimes \mathcal{R}$ such that $(\chi, \dot{\gamma}) = (\chi', \dot{\gamma}')$. Then

$$i((\chi, \dot{\gamma})) = (\chi \cdot \dot{\gamma}, \dot{\gamma}^{-1}) = i((\chi', \dot{\gamma}'));$$

$$r((\chi, \dot{\gamma})) = (\chi, r(\gamma)) = (\chi', r(\gamma')) = r((\chi', \dot{\gamma}'));$$
$$s((\chi, \dot{\gamma})) = (\chi \cdot \gamma, s(\gamma)) = (\chi' \cdot \gamma', s(\gamma')) = s((\chi', \dot{\gamma}')).$$

Thus, the composition, inverse, range and source maps are well-defined.
Let $((\chi, \dot{\gamma}), (\chi', \dot{\gamma}'))$, $((\chi', \dot{\gamma}'), (\chi'', \dot{\gamma}'')) \in (\hat{A} \rtimes \mathcal{R})^{(2)}$. Then

$$s((\chi, \dot{\gamma})) = (\chi \cdot \gamma, s(\gamma)) = r((\chi', \dot{\gamma}')) = (\chi', r(\gamma'));$$

$$s((\chi', \dot{\gamma}')) = (\chi' \cdot \gamma', s(\gamma')) = r((\chi'', \dot{\gamma}'')) = (\chi'', r(\gamma'')).$$

Now,

$$\begin{aligned} s((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')) = s((\chi, \dot{\gamma}\dot{\gamma}')) &= (\chi \cdot \gamma\gamma', s(\gamma\gamma')) = (\chi \cdot \gamma\gamma', s(\gamma')) \\ &= (\chi \cdot \gamma\gamma', r(\gamma'')) = (\chi' \cdot \gamma', r(\gamma'')) \\ &= (\chi'', r(\gamma'')) = r(\chi'', \dot{\gamma}''); \end{aligned}$$

$$\begin{aligned} r[(\chi', \dot{\gamma}')(\chi'', \dot{\gamma}'')] = r((\chi', \dot{\gamma}'\dot{\gamma}'')) &= (\chi', r(\gamma'\gamma'')) = (\chi', r(\gamma')) \\ &= (\chi \cdot \gamma, s(\gamma)) = s((\chi, \dot{\gamma})). \end{aligned}$$

Hence, $((\chi, \dot{\gamma})(\chi', \dot{\gamma}'), (\chi'', \dot{\gamma}'')) \in (\hat{A} \rtimes \mathcal{R})^{(2)}$ and $((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')(\chi'', \dot{\gamma}'')) \in (\hat{A} \rtimes \mathcal{R})^{(2)}$.
Now, $[(\chi, \dot{\gamma})(\chi', \dot{\gamma}')](\chi'', \dot{\gamma}'') = (\chi, \dot{\gamma}\dot{\gamma}')(\chi'', \dot{\gamma}'') = (\chi, \dot{\gamma}\dot{\gamma}'\dot{\gamma}'') = (\chi, \dot{\gamma})(\chi', \dot{\gamma}'\dot{\gamma}'') = (\chi, \dot{\gamma})[(\chi', \dot{\gamma}')(\chi'', \dot{\gamma}'')]$.

Let $(\chi, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$. Then

$$\begin{aligned} ((\chi,\dot{\gamma})^{-1})^{-1} = (\chi\cdot\gamma, \dot{\gamma}^{-1}) &= (\chi\cdot\gamma\gamma^{-1}, (\dot{\gamma}^{-1})^{-1}) \\ &= (\chi\cdot r(\gamma), \dot{\gamma}) \\ &= (\chi\cdot u, \dot{\gamma}) \\ &= (\chi, \dot{\gamma}); \end{aligned}$$

$$\begin{aligned} r((\chi,\dot{\gamma})^{-1}) = r((\chi\cdot\gamma, \dot{\gamma}^{-1})) &= (\chi\cdot\gamma, r(\gamma^{-1}) \\ &= (\chi\cdot\gamma, s(\gamma)) \\ &= s((\chi,\dot{\gamma})). \end{aligned}$$

Hence, $((\chi,\dot{\gamma}),(\chi,\dot{\gamma})^{-1}) \in (\hat{A} \rtimes \mathcal{R})^{(2)}$. Suppose that $((\chi',\dot{\gamma}'),(\chi,\dot{\gamma})) \in (\hat{A} \rtimes \mathcal{R})^{(2)}$. Then,

$$\begin{aligned} [(\chi',\dot{\gamma}')(\chi,\dot{\gamma})](\chi,\dot{\gamma})^{-1} &= (\chi,\dot{\gamma}\dot{\gamma}')(\chi',\dot{\gamma}')^{-1} \\ &= (\chi,\dot{\gamma}\dot{\gamma}')(\chi'\cdot\gamma', (\dot{\gamma}')^{-1}) \\ &= (\chi,\dot{\gamma}\dot{\gamma}'(\dot{\gamma}')^{-1}) \\ &= (\chi,\dot{\gamma}r((\dot{\gamma}')) \\ &= (\chi,\dot{\gamma}s(\dot{\gamma})) \\ &= (\chi,\dot{\gamma}) \end{aligned}$$

Also,

$$\begin{aligned} (\chi,\dot{\gamma})^{-1}[(\chi,\dot{\gamma})(\chi',\dot{\gamma}')] &= (\chi,\dot{\gamma})^{-1}(\chi,\dot{\gamma}\dot{\gamma}') \\ &= (\chi\cdot\gamma,\dot{\gamma}^{-1})(\chi,\dot{\gamma}\dot{\gamma}') \\ &= (\chi\cdot\gamma,\dot{\gamma}^{-1}\dot{\gamma}\dot{\gamma}') \\ &= (\chi\cdot\gamma, s(\dot{\gamma})\dot{\gamma}') \\ &= (\chi\cdot\gamma, r(\dot{\gamma}')\dot{\gamma}') \\ &= (\chi',\dot{\gamma}') \end{aligned}$$

Therefore, $\hat{A} \rtimes \mathcal{R}$ is a groupoid. □

Lemma 5. *$\hat{A} \rtimes \mathcal{R}$ is an ample Hausdorff groupoid.*

Proof. Let U be open in $\hat{A} \rtimes \mathcal{R}$. Then $U = (A \times B) \cap (\hat{A} \rtimes \mathcal{R})$ where $A \times B$ is open in $\hat{A} \times \mathcal{R}$. Let $((\chi,\dot{\gamma}),(\chi',\dot{\gamma}')) \in m^{-1}(U)$. Then $(\chi,\dot{\gamma}\dot{\gamma}') \in (A \times B)$ and $(\chi,\dot{\gamma}\dot{\gamma}') \in \hat{A} \rtimes \mathcal{R}$. Since $A \times B$ is open in $\hat{A} \times \mathcal{R}$, then there exists open set $(A \times B)'$ containing $(\chi,\dot{\gamma}\dot{\gamma}')$ such that $(A \times B)' \subseteq A \times B$. Also there exists open set W in $\hat{A} \rtimes \mathcal{R}$ containing $(\chi,\dot{\gamma}\dot{\gamma}')$. Since $\hat{A} \rtimes \mathcal{R} \subseteq \hat{A} \times \mathcal{R}$, then there exists open set W' in $\hat{A} \times \mathcal{R}$ containing $(\chi,\dot{\gamma}\dot{\gamma}')$ where $W' = W \cap \hat{A} \rtimes \mathcal{R}$. Consider the set $(A \times B)' \times W' \subseteq (\hat{A} \times \mathcal{R}) \times (\hat{A} \times \mathcal{R})$ where $(A \times B)' \in \tau_{\hat{A}\times\mathcal{R}}$ and $W' \in \tau_{\hat{A}\times\mathcal{R}}$. It follows that $(A \times B)' \times W'$ is open in $(\hat{A} \times \mathcal{R}) \times (\hat{A} \times \mathcal{R})$. Define $M = ((A \times B)' \times W') \cap (\hat{A} \rtimes \mathcal{R})^{(2)} = \{((\chi,\dot{\gamma}),(\chi',\dot{\gamma}')) \in$

$(\hat{A} \rtimes \mathcal{R})^{(2)} : m(((\chi, \dot{\gamma}), (\chi', \dot{\gamma}'))) \in (A \times B)' \cap W'\}$. We claim that $M \subset m^{-1}(U)$. Let $((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')) \in M$. Then $m(((\chi, \dot{\gamma}), (\chi', \dot{\gamma}'))) \in (A \times B)' \cap W' \subseteq (A \times B) \cap \hat{A} \rtimes \mathcal{R}$ and $m(((\chi, \dot{\gamma}), (\chi', \dot{\gamma}'))) \in (A \times B) \cap \hat{A} \rtimes \mathcal{R}$, i.e., $((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')) \in m^{-1}(U)$ and $M \subset m^{-1}(U)$. Since $(A \times B)' \times W'$ is open in $(\hat{A} \times \mathcal{R}) \times (\hat{A} \times \mathcal{R})$, then M is open in $(\hat{A} \rtimes \mathcal{R})^{(2)}$. It follows that $m^{-1}(U)$ is open and m is continuous.

Let U be an open set in $\hat{A} \rtimes \mathcal{R}$. Then $U = V \cap \hat{A} \rtimes \mathcal{R}$ where V is open in $\hat{A} \times \mathcal{R}$. Let $(\chi, \dot{\gamma}) \in i^{-1}(U)$. Then, $i((\chi, \dot{\gamma})) \in V \cap \hat{A} \rtimes \mathcal{R}$. Since V is open in $\hat{A} \times \mathcal{R}$, then there exists open set U_v containing $(\chi, \dot{\gamma})$ where $U_v \subseteq V$. Also, there exists open set W containing $(\chi, \dot{\gamma})^{-1}$ where $W \subseteq \hat{A} \rtimes \mathcal{R}$. Define $U_v' = \{(\chi', \dot{\gamma}') \in \hat{A} \rtimes \mathcal{R} : i((\chi', \dot{\gamma}')) \in U_v \cap W\}$. Let $(\chi', \dot{\gamma}',) \in U_v'$. By definition, $i((\chi', \dot{\gamma}')) \in U_v \cap W \subset V \cap \hat{A} \rtimes \mathcal{R}$ which means that $(\chi', \dot{\gamma}') \in i^{-1}(U)$. Hence, $U_v' \subseteq i^{-1}(U)$. Since $U_v \subseteq V$, then $U_v' \subseteq V \cap W$. Notice that $V \cap W$ is open in $\hat{A} \rtimes \mathcal{R}$. Hence, U_v' is open in $\hat{A} \rtimes \mathcal{R}$. Thus, $i^{-1}(U)$ is open and our inverse map is continuous.

Let $(\chi, \dot{\gamma})$ and $(\chi', \dot{\gamma}')$ be distinct points in $\hat{A} \rtimes \mathcal{R}$. We can choose open sets U and V in $\hat{A} \times \mathcal{R}$ such that $U \cap V = \varnothing$ with $(\chi, \dot{\gamma}) \in U$ and $(\chi', \dot{\gamma}') \in V$. Let $U' = U \cap \hat{A} \rtimes \mathcal{R}$ and $V' = V \cap \hat{A} \rtimes \mathcal{R}$. Since U and V are in $\tau_{\hat{A} \rtimes \mathcal{R}}$, then U' and V' are open sets in $\hat{A} \rtimes \mathcal{R}$ containing $(\chi, \dot{\gamma})$ and $(\chi', \dot{\gamma}')$, respectively. Note that U and V are disjoint, hence U' and V' are also disjoint and we have proved that $\hat{A} \rtimes \mathcal{R}$ is a Hausdorff groupoid.

Since $\hat{A} \rtimes \mathcal{R}$ is Hausdorff, $(\hat{A} \rtimes \mathcal{R})^{(0)}$ is Hausdorff. Let $\mathcal{B}$ be a basis for the product topology on $\hat{A} \times \mathcal{R}$. Then, $\mathcal{B}' = \{B \cap \hat{A} \rtimes \mathcal{R} : B \in \mathcal{B}\}$ is a basis for the relative topology on $\hat{A} \rtimes \mathcal{R}$. Now, let $r_1 = r|_{\mathcal{B}'} : \mathcal{B}' \to U$ where U is an open subset of $\hat{A} \rtimes \mathcal{R}$ and let V be an open subset of U. Then $V = (A \times B) \cap \hat{A} \rtimes \mathcal{R}$ where $A \times B$ is open in $\hat{A} \times \mathcal{R}$. Since $\mathcal{B}'$ is a base for the topology on $\hat{A} \rtimes \mathcal{R}$, then there exists a basic element B containing $(\chi, \dot{\gamma})$ such that $r_1(\chi, \dot{\gamma}) \in V$. Then $r_1^{-1}(U)$ is open in $\hat{A} \rtimes \mathcal{R}$. Now, denote $r_1^{-1} = (r|_{\mathcal{B}'})^{-1} : U \to \mathcal{B}'$. Let B be open subset of $\mathcal{B}'$. Then $B = A \cap (\hat{A} \rtimes \mathcal{R})$ where $A \in \mathcal{B}$. Let $(\chi, \dot{\gamma}) \in r_1(B)$. Then $r_1^{-1}(\chi, \dot{\gamma}) \in A \cap (\hat{A} \rtimes \mathcal{R})$. Since A is open in $\hat{A} \times \mathcal{R}$, then there exists open set A' containing $(\chi, \dot{\gamma})$ where $A' \subseteq A$. Also, there exists open set $W \subseteq \hat{A} \rtimes \mathcal{R}$ containing $(\chi, \dot{\gamma})$. Define $A'' = \{(\chi', \dot{\gamma}') \in U : r_1^{-1}(\chi', \dot{\gamma}') \in A' \cap W\}$. Let $(\chi', \dot{\gamma}') \in A''$. By definition, $r_1^{-1}(\chi', \dot{\gamma}') \in A'' \cap W \subset A \cap (\hat{A} \rtimes \mathcal{R})$. Hence, $r_1^{-1}(\chi', \dot{\gamma}') \in A \cap (\hat{A} \rtimes \mathcal{R})$ which means that $(\chi', \dot{\gamma}') \in r_1(B)$. Hence, $A'' \subset r_1(B)$. Since $A'' \subseteq A' \cup W$ and $A' \subset A$, $A'' \subseteq A \cap W$. Notice that $A \cap W$ is open in $\hat{A} \rtimes \mathcal{R}$. Hence, A'' is open in $\hat{A} \rtimes \mathcal{R}$. Thus, $r_1(B)$ is open and r is a homeomorphism onto an open subset of $\hat{A} \rtimes \mathcal{R}$. Similarly, s is homeomorphic onto an open subset of $\hat{A} \rtimes \mathcal{R}$. □

Proposition 1. *$\hat{A} \rtimes \mathcal{R}$ is a principal groupoid with unit space $\hat{A}$.*

Proof. Let $(\chi, u) \in \hat{A}$ where $u \in \mathcal{G}^{(0)}$. Then, $s(\gamma) = u$ for $\gamma \in A_u$ and $s(\gamma) = r(\gamma)$ for $\gamma \in A$. Hence, $(\chi, u) = (\chi, s(\gamma)) = (\chi, r(\gamma)) = (\chi \cdot \gamma, s(\gamma))$. Thus, $(\chi, u) \in (\hat{A} \rtimes \mathcal{R})^{(0)}$ and $\hat{A} \subseteq (\hat{A} \rtimes \mathcal{R})^{(0)}$. Let $(\chi \cdot \gamma, s(\gamma)) \in (\hat{A} \rtimes \mathcal{R})^{(0)}$ where $(\chi, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$. Then $(\chi \cdot \gamma, s(\gamma)) = (\chi, r(\gamma)) = (\chi \cdot \gamma, u)$ since $r(\gamma) = u$. Thus, $(\hat{A} \rtimes \mathcal{R})^{(0)} \subseteq \hat{A}$ and $\hat{A}$ is the unit space of $\hat{A} \rtimes \mathcal{R}$.

Let $\theta : \hat{A} \rtimes \mathcal{R} \to \hat{A} \times \hat{A}$ be defined by $\theta((\chi, \dot{\gamma})) = (r((\chi, \dot{\gamma})), s((\chi, \dot{\gamma})))$ and let

$(\chi_1, \dot{\gamma}_1), (\chi_2, \dot{\gamma}_2) \in \hat{A} \rtimes \mathcal{R}$ such that $\theta((\chi_1, \dot{\gamma}_1)) = \theta((\chi_2, \dot{\gamma}_2))$. Then

$$(r((\chi_1, \dot{\gamma}_1)), s((\chi_1, \dot{\gamma}_1))) = (r((\chi_2, \dot{\gamma}_2)), s((\chi_2, \dot{\gamma}_2))).$$

Also, $r((\chi_1, \dot{\gamma}_1)) = (\chi_1, r(\gamma_1)) = r((\chi_2, \dot{\gamma}_2)) = (\chi_2, r(\gamma_2))$ and $s((\chi_1, \dot{\gamma}_1)) = (\chi_1 \cdot \gamma_1, s(\gamma_1)) = s((\chi_2, \dot{\gamma}_2)) = (\chi_2 \cdot \gamma_2, s(\gamma_2))$. Hence, $\chi_1 = \chi_2$ and $\gamma_1 = \gamma_2$. Thus, $(\chi_1, \dot{\gamma}_1) = (\chi_2, \dot{\gamma}_2)$ and θ is injective. Therefore, $\hat{A} \rtimes \mathcal{R}$ is a principal groupoid. □

We now introduce a sequence of groupoids and investigate whether it is our desired discrete twist over $\hat{A} \rtimes \mathcal{R}$. Define $\hat{A} * \mathcal{G} \times T = \{(\chi, z, \gamma) : \chi \in \hat{A}_{r(\gamma)}, z \in T, \text{ and } \gamma \in \mathcal{G}\}$. Let $r((\chi, z, \gamma)) = (\chi, r(\gamma))$ and $s((\chi, z, \gamma)) = (\chi \cdot \gamma, s(\gamma))$ be the range and source maps, respectively. The composition map and inverse map is $(\chi, z, \gamma)(\chi', z', \gamma') = (\chi, zz', \gamma\gamma')$ and $(\chi, z, \gamma)^{-1} = (\chi \cdot \gamma, z^{-1}, \gamma^{-1})$, respectively. We note that (χ, z, γ) and (χ', z', γ') are composable pairs if we have $\chi' = \chi \cdot \gamma$ and $\chi \cdot \gamma$ is defined by $\chi \cdot \gamma(a) = \chi(\gamma a \gamma^{-1})$ where $\chi \cdot u = \chi$.

Lemma 6. *$\hat{A} * \mathcal{G} \times T$ is a Hausdorff groupoid.*

Proof. Let (χ, z, γ), $(\chi', z', \gamma') \in \hat{A} * \mathcal{G} \times T$ with $(\chi, z, \gamma) = (\chi', z', \gamma')$. Now,

$$(\chi, z, \gamma)^{-1} = (\chi \cdot \gamma, z^{-1}, \gamma^{-1}) = (\chi' \cdot \gamma', (z^{-1})', \gamma'^{-1}) = (\chi', z', \gamma')^{-1}.$$

Also,

$$r((\chi, z, \gamma)) = (\chi, r(\gamma)) = (\chi, s(\gamma')) = r((\chi', z', \gamma'));$$

$$s((\chi, z, \gamma)) = (\chi \cdot \gamma, s(\gamma)) = (\chi' \cdot \gamma', s(\gamma')) = s((\chi', z', \gamma')).$$

Hence, the inverse range and source maps are well-defined.

Composition is well-defined since for $((\chi_1, z_1, \gamma_1), (\chi_1', z_1', \gamma_1')), ((\chi_2, z_2, \gamma_2), (\chi_2', z_2', \gamma_2')) \in \hat{A} * \mathcal{G} \times T^{(2)}$ with $((\chi_1, z_1, \gamma_1), (\chi_1', z_1', \gamma_1')) = ((\chi_2, z_2, \gamma_2), (\chi_2', z_2', \gamma_2')), m(((\chi_1, z_1, \gamma_1)(\chi_1', z_1', \gamma_1'))) = (\chi_1, z_1 z_1', \gamma_1 \gamma_1') = (\chi_2, z_2 z_2', \gamma_2 \gamma_2') = m(((\chi_2, z_2, \gamma_2), (\chi_2', z_2', \gamma_2')))$.

Now, let $((\chi_1, z_1, \gamma_1), (\chi_2, z_2, \gamma_2))$, $((\chi_2, z_2, \gamma_2), (\chi_3, z_3, \gamma_3)) \in \hat{A} * \mathcal{G} \times T^{(2)}$. Then

$$\begin{aligned} s((\chi_1, z_1, \gamma_1)(\chi_2, z_2, \gamma_2)) &= s((\chi_1, z_1 z_2, \gamma_1 \gamma_2)) \\ &= (\chi_1 \cdot \gamma_1 \gamma_2, s(\gamma_1 \gamma_2)) \\ &= (\chi_2 \cdot \gamma_2, s(\gamma_2)) \\ &= (\chi_3, r(\gamma_3)) \\ &= ((\chi_3, z_3, \gamma_3)). \end{aligned}$$

Also,

$$\begin{aligned} r((\chi_2, z_2, \gamma_2)(\chi_3, z_3, \gamma_3)) &= r((\chi_2, z_2 z_3, \gamma_2 \gamma_3)) \\ &= (\chi_2, r(\gamma_2 \gamma_3)) \\ &= (\chi_2, r(\gamma_2)) \\ &= (\chi_1 \cdot \gamma_1, s(\gamma_1)) \end{aligned}$$

$$= \quad s((\chi_1, z_1, \gamma_1)).$$

Thus, $((\chi_1, z_1, \gamma_1)(\chi_2, z_2, \gamma_2), (\chi_3, z_3, \gamma_3))$,$((\chi_1, z_1, \gamma_1), (\chi_2, z_2, \gamma_2)(\chi_3, z_3, \gamma_3)) \in (\hat{A} * \mathcal{G} \times T)^{(2)}$. Composition in $\hat{A} * \mathcal{G} \times T$ is associative since

$$\begin{aligned} ((\chi_1, z_1, \gamma_1)(\chi_2, z_2, \gamma_2))(\chi_3, z_3, \gamma_3) &= (\chi_1, z_1z_2, \gamma_1\gamma_2)(\chi_3, z_3, \gamma_3) \\ &= (\chi_1, z_1z_2z_3, \gamma_1\gamma_2\gamma_3) \\ &= (\chi_1, z_1, \gamma_1)(\chi_2, z_2z_3, \gamma_2\gamma_3) \\ &= (\chi_1, z_1, \gamma_1)((\chi_2, z_2, \gamma_2)(\chi_3, z_3, \gamma_3)). \end{aligned}$$

For $(\chi, z, \gamma) \in \hat{A} * \mathcal{G} \times T$,

$$\begin{aligned} ((\chi, z, \gamma)^{-1})^{-1} &= (\chi \cdot \gamma, z^{-1}, \gamma^{-1})^{-1} \\ &= (\chi \cdot \gamma \cdot \gamma^{-1}, (z^{-1})^{-1}, (\gamma^{-1})^{-1}) \\ &= (\chi \cdot r(\gamma), z, \gamma) \\ &= (\chi \cdot u, z, \gamma) \\ &= (\chi, z, \gamma). \end{aligned}$$

Also, $r((\chi, z, \gamma)^{-1}) = r((\chi \cdot \gamma, z^{-1}, \gamma^{-1})) = (\chi \cdot \gamma, r(\gamma^{-1})) = (\chi \cdot \gamma, s(\gamma)) = s((\chi, z, \gamma))$. Hence, $((\chi, z, \gamma), (\chi, z, \gamma)^{-1}) \in \hat{A} * \mathcal{G} \times T^{(2)}$. Notice that

$$\begin{aligned} ((\chi_1, z_1, \gamma_1)(\chi_2, z_2, \gamma_2))(\chi_2, z_2, \gamma_2)^{-1} &= (\chi_1, z_1z_2, \gamma_1\gamma_2)(\chi_2, z_2, \gamma_2)^{-1} \\ &= (\chi_1, z_1z_2, \gamma_1\gamma_2)(\chi_2 \cdot \gamma_2, z_2^{-1}, \gamma_2^{-1}) \\ &= (\chi_1, z_1z_2z_3, \gamma_1\gamma_2\gamma_3^{-1}) \\ &= (\chi_1, z_1, \gamma_1 r(\gamma_2)) \\ &= (\chi_1, z_1, \gamma_1 s(\gamma_1)) \\ &= (\chi_1, z_1, \gamma_1). \end{aligned}$$

Also,

$$\begin{aligned} (\chi_1, z_1, \gamma_1)^{-1}((\chi_1, z_1, \gamma_1)(\chi_2, z_2, \gamma_2)) &= (\chi_1, z_1, \gamma_1)^{-1}(\chi_1, z_1z_2, \gamma_1\gamma_2) \\ &= (\chi_1 \cdot \gamma_1, z_1^{-1}, \gamma_1^{-1})(\chi_1, z_1z_2, \gamma_1\gamma_2) \\ &= (\chi_1 \cdot \gamma_1, z_1^{-1}z_1z_2, \gamma_1^{-1}\gamma_1\gamma_2) \\ &= (\chi_2, z_2, s(\gamma_1)\gamma_2) \\ &= (\chi_2, z_2, r(\gamma_2)\gamma_2) \\ &= (\chi_2, z_2, \gamma_2). \end{aligned}$$

Hence, $\hat{A} * \mathcal{G} \times T$ is a groupoid.

Endowed $\hat{A} * \mathcal{G} \times T$ with the product topology define as $\tau_{\hat{A} * \times \mathcal{G}} = \{(A * B \times C) \in \hat{A} * \mathcal{G} \times T : A \in \tau_{\hat{A}}, B \in \tau_{\mathcal{G}}, C \in \tau_T\}$. Since $\mathcal{G}$, T and $\hat{A} \rtimes \mathcal{R}$ are Hausdorff, $\hat{A}$ is also Hausdorff. Let (χ, z, γ) and (χ', z', γ') be distinct points in $\hat{A} * \mathcal{G} \times T$. Since $\hat{A}$ is Hausdorff, then there exists open neighborhoods A_1 and A_2 in $\hat{A}$ containing χ and χ', respectively such that $A_1 \cap A_2 = \varnothing$. Also, there exists open neighborhoods G_1 and G_2 in G containing γ and γ', respectively such that $G_1 \cap G_2 = \varnothing$. For the Hausdorff space T, there exists open neighborhoods T_1 and T_2 in T containing z and z', respectively wherein $T_1 \cap T_2 = \varnothing$. Then by definition of $\tau_{\hat{A} * \mathcal{G} \times T}$, $U = A_1 * G_1 \times T_1$ and $V = A_2 * G_2 \times T_2$ are open neighboorhoods in $\hat{A} * \mathcal{G} \times T$ containing (χ, z, γ) and (χ', z', γ'), respectively such that $U \cap V = \varnothing$. Therefore, $\hat{A} * \mathcal{G} \times T$ is Hausdorff. □

Lemma 7. *Let $\hat{A} \times T = \{(\chi, z, u) : (\chi, u) \in \hat{A}$ and $z \in T\}$. Then $\hat{A} \times T$ is the isotropy group of $\hat{A} * \mathcal{G} \times T$.*

Proof. Note that $\mathrm{Iso}(\hat{A} * \mathcal{G} \times T) = \{(\chi, z, \gamma) \in \hat{A} * \mathcal{G} \times T : s(\chi, z, \gamma) = r(\chi, z, \gamma)\}$. Let $(\chi, z, \gamma) \in \mathrm{Iso}(\hat{A} * \mathcal{G} \times T)$. Then $s(\chi, z, \gamma) = r(\chi, z, \gamma)$, that is, $(\chi \cdot \gamma, s(\gamma)) = (\chi, r(\gamma))$. Note that $\chi \cdot \gamma = \chi$ if and only if $\gamma = u$ where $u \in \mathcal{G}^{(0)}$. Also, $s(\gamma) = r(\gamma)$ means $\gamma = u \in \mathcal{G}^{(0)}$. Thus, $(\chi, z, \gamma) = (\chi, z, u)$ and $\mathrm{Iso}(\hat{A} * \mathcal{G} \times T) = \{(\chi, z, u) \in \hat{A} * \mathcal{G} \times T : (\chi, u) \in \hat{A}, z \in T\} = \hat{A} \times T$. Therefore, $\hat{A} \times T$ is the isotropy group for $\hat{A} * \mathcal{G} \times T$. □

Lemma 8. *Define $\sim$ on $\hat{A} * \mathcal{G} \times T$ by $(\chi, z, \gamma) \sim (\chi', z', \gamma')$ if and only if $\chi = \chi'$ and there exists $a \in A_u$ such that $\chi(a)z = z'$ and $\gamma = a \cdot \gamma'$. Then $\sim$ is an equivalence relation on $\hat{A} * \mathcal{G} \times T$.*

Proof. Let $(\chi, z, \gamma) \in \hat{A} * \mathcal{G} \times T$. Choose $a \in A_u$ such that $\chi(a) = 1 \in R^{\times}$. Then $\chi(a)z = 1(z) = z$ and $s(a) = u$. Since $\gamma \in A_u$, $s(\gamma) = u$, $s(\gamma) = u = s(a) = r(a)$ and a and γ are composable pairs in $\mathcal{G}$. Then, $(aa^{-1})\gamma = \gamma$ where $aa^{-1} \in A_u$. Hence $(\chi, z, \gamma) \sim (\chi, z, \gamma)$.

Let $(\chi, z, \gamma), (\chi', z', \gamma') \in \hat{A} * \mathcal{G} \times T$ such that $(\chi, z, \gamma) \sim (\chi', z', \gamma')$. Then $\chi(a)\chi(a)^{-1}z = \chi(a)^{-1}z'$ and we have $z = \chi(a)^{-1}z'$. Also since $\gamma' \in A_u$ then $s(\gamma') = u = r(a)$, that is, $(\gamma', a) \in \mathcal{G}^{(2)}$ and $aa^{-1}\gamma' = a^{-1}\gamma$ which is $\gamma' = a^{-1}\gamma, a^{-1} \in A_u$. Thus, $(\chi', z', \gamma') \sim (\chi, z, \gamma)$.

Let $(\chi_1, z_1, \gamma_1) \sim (\chi_2, z_2, \gamma_2)$ and $(\chi_2, z_2, \gamma_2) \sim (\chi_3, z_3, \gamma_3)$. Then $\chi_1 = \chi_3$ and $z_3 = \chi_2(a)\chi_1(a)z_1 = \chi_1(a)\chi_1(a)z_1 = \chi_1(a)z_1$. Also, $\gamma_1 = a \cdot \gamma_2 = a \cdot a \cdot \gamma_3 = a \cdot \gamma_3$. Thus $(\chi_1, z_2, \gamma_1) \sim (\chi_3, z_3, \gamma_3)$. Therefore, $\sim$ is an equivalence relation on $\hat{A} * \mathcal{G} \times T$. □

Denote the set of equivalence classes of $\hat{A} * \mathcal{G} \times T$ with respect to the equivalence relation $\sim$ on Lemma 8 by $D = \hat{A} * \mathcal{G} \times T/\sim = \{[\chi, z, \gamma] : (\chi, z, \gamma) \in \hat{A} * \mathcal{G} \times T\}$.

Define the following structure for D and verify whether it is a groupoid. The range and source maps will be $r([\chi, z, \gamma]) = [\chi, r(\gamma)]$ and $s([\chi, z, \gamma]) = [\chi \cdot \gamma, s(\gamma)]$, respectively. The composition and inverse map is $[\chi, z, \gamma][\chi', z', \gamma'] = [\chi, zz', \gamma\gamma']$ and $[\chi, z, \gamma]^{-1} = [\chi \cdot \gamma, z^{-1}, \gamma^{-1}]$, respectively. We note that $[\chi, z, \gamma]$ and $[\chi', z', \gamma']$ are composable pairs if $\chi' = \chi \cdot \gamma$ where $\chi \cdot \gamma$ is defined by $\chi \cdot \gamma(a) = \chi(\gamma a \gamma^{-1})$ and $\chi \cdot u = \chi$.

Theorem 3. *D is a Hausdorff étale groupoid with respect to the quotient topology with $D^{(0)} = i(\hat{A} \times \{1\})$.*

Proof. Let $[\chi, z, \gamma]$, $[\chi', z', \gamma'] \in D$ with $[\chi, z, \gamma] = [\chi', z', \gamma']$. Now,

$$[\chi, z, \gamma]^{-1} = [\chi \cdot \gamma, z^{-1}, \gamma^{-1}] = [\chi' \cdot \gamma', (z^{-1})', \gamma'^{-1}] = [\chi', z', \gamma']^{-1}.$$

Also,

$$r([\chi, z, \gamma]) = [\chi, r(\gamma)] = [\chi, s(\gamma')] = r([\chi', z', \gamma']);$$

$$s([\chi, z, \gamma]) = [\chi \cdot \gamma, s(\gamma)] = [\chi' \cdot \gamma', s(\gamma')] = s([\chi', z', \gamma']).$$

Hence, the inverse, range and source maps are well-defined.

Let $([\chi_1, z_1, \gamma_1], [\chi_1', z_1', \gamma_1'])$, $([\chi_2, z_2, \gamma_2], [\chi_2', z_2', \gamma_2']) \in D^{(2)}$ with $([\chi_1, z_1, \gamma_1], [\chi_1', z_1', \gamma_1']) = ([\chi_2, z_2, \gamma_2], [\chi_2', z_2', \gamma_2'])$. Composition is well-defined since

$$\begin{aligned} m(([\chi_1, z_1, \gamma_1], [\chi_1', z_1', \gamma_1'])) &= [\chi_1, z_1 z_1', \gamma_1 \gamma_1'] \\ &= [\chi_2, z_2 z_2', \gamma_2 \gamma_2'] \\ &= m(([\chi_2, z_2, \gamma_2], [\chi_2', z_2', \gamma_2'])). \end{aligned}$$

Now, let $([\chi_1, z_1, \gamma_1], [\chi_2, z_2, \gamma_2])$, $([\chi_2, z_2, \gamma_2], [\chi_3, z_3, \gamma_3]) \in D^{(2)}$. Then

$$\begin{aligned} s([\chi_1, z_1, \gamma_1][\chi_2, z_2, \gamma_2]) &= s([\chi_1, z_1 z_2, \gamma_1 \gamma_2]) \\ &= [\chi_1 \cdot \gamma_1 \gamma_2, s(\gamma_1 \gamma_2)] \\ &= [\chi_2 \cdot \gamma_2, s(\gamma_2)] \\ &= [\chi_3, r(\gamma_3)] \\ &= r([\chi_3, z_3, \gamma_3]). \end{aligned}$$

Also,

$$\begin{aligned} r([\chi_2, z_2, \gamma_2][\chi_3, z_3, \gamma_3]) &= r([\chi_2, z_2 z_3, \gamma_2 \gamma_3]) \\ &= [\chi_2, r(\gamma_2 \gamma_3)] \\ &= [\chi_2, r(\gamma_2)] \\ &= [\chi_1 \cdot \gamma_1, s(\gamma_1)] \\ &= s([\chi_1, z_1, \gamma_1]). \end{aligned}$$

Thus, $([\chi_1, z_1, \gamma_1][\chi_2, z_2, \gamma_2], [\chi_3, z_3, \gamma_3])$ and $([\chi_1, z_1, \gamma_1], [\chi_2, z_2, \gamma_2][\chi_3, z_3, \gamma_3])$ are composable pairs. To show that composition in D is associative,

$$\begin{aligned} ([\chi_1, z_1, \gamma_1][\chi_2, z_2, \gamma_2])[\chi_3, z_3, \gamma_3] &= [\chi_1, z_1 z_2, \gamma_1 \gamma_2][\chi_3, z_3, \gamma_3] \\ &= [\chi_1, z_1 z_2 z_3, \gamma_1 \gamma_2 \gamma_3] \\ &= [\chi_1, z_1, \gamma_1][\chi_2, z_2 z_3, \gamma_2 \gamma_3] \\ &= [\chi_1, z_1, \gamma_1]([\chi_2, z_2, \gamma_2][\chi_3, z_3, \gamma_3]). \end{aligned}$$

For $[\chi, z, \gamma] \in D$ we have,

$$\begin{aligned}
([\chi, z, \gamma]^{-1})^{-1} &= [\chi \cdot \gamma, z^{-1}, \gamma^{-1}]^{-1} \\
&= [\chi \cdot \gamma \cdot \gamma^{-1}, (z^{-1})^{-1}, (\gamma^{-1})^{-1}] \\
&= [\chi \cdot r(\gamma), z, \gamma] \\
&= [\chi \cdot u, z, \gamma] \\
&= [\chi, z, \gamma].
\end{aligned}$$

Also, $r([\chi, z, \gamma]^{-1}) = r([\chi \cdot \gamma, z^{-1}, \gamma^{-1}]) = [\chi \cdot \gamma, r(\gamma^{-1})] = [\chi \cdot \gamma, s(\gamma)] = s([\chi, z, \gamma])$. Hence, $([\chi, z, \gamma], [\chi, z, \gamma]^{-1}) \in D^{(2)}$. Notice that

$$\begin{aligned}
([\chi_1, z_1, \gamma_1][\chi_2, z_2, \gamma_2])[\chi_2, z_2, \gamma_2]^{-1} &= [\chi_1, z_1 z_2, \gamma_1 \gamma_2][\chi_2, z_2, \gamma_2]^{-1} \\
&= [\chi_1, z_1 z_2, \gamma_1 \gamma_2][\chi_2 \cdot \gamma_2, z_2^{-1}, \gamma_2^{-1}] \\
&= [\chi_1, z_1 z_2 z_3, \gamma_1 \gamma_2 \gamma_3^{-1}] \\
&= [\chi_1, z_1, \gamma_1 r(\gamma_2)] \\
&= [\chi_1, z_1, \gamma_1 s(\gamma_1)] \\
&= [\chi_1, z_1, \gamma_1].
\end{aligned}$$

Also,

$$\begin{aligned}
[\chi_1, z_1, \gamma_1]^{-1}([\chi_1, z_1, \gamma_1][\chi_2, z_2, \gamma_2]) &= [\chi_1, z_1, \gamma_1]^{-1}[\chi_1, z_1 z_2, \gamma_1 \gamma_2] \\
&= [\chi_1 \cdot \gamma_1, z_1^{-1}, \gamma_1^{-1}][\chi_1, z_1 z_2, \gamma_1 \gamma_2] \\
&= [\chi_1 \cdot \gamma_1, z_1^{-1} z_1 z_2, \gamma_1^{-1} \gamma_1 \gamma_2] \\
&= [\chi_2, z_2, s(\gamma_1)\gamma_2] \\
&= [\chi_2, z_2, r(\gamma_2)\gamma_2] \\
&= [\chi_2, z_2, \gamma_2].
\end{aligned}$$

Hence, D is a groupoid.

Let D be a topological space with the quotient topology τ_D. We define the quotient map $\pi_D : \hat{A} * \mathcal{G} \times T \to D$ by $\pi_D((\chi, z, \gamma)) = [\chi, z, \gamma]$ for $(\chi, z, \gamma) \in \hat{A} * \mathcal{G} \times T$ and $[\chi, z, \gamma] \in D$ and $\tau_{\hat{A} * \mathcal{G} \times T} = \{U \times V : U \in \tau_{\hat{A}}, V \in \tau_{\mathcal{G} \times T}\}$ where $\tau_{\mathcal{G} \times T}$ is the product topology with the topology in $\mathcal{G}$ and T having the discrete topology.

Let $[\chi, z, \gamma]$ and $[\chi', z', \gamma']$ be distinct elements of D. Then there exists $(\chi, z, \gamma) \nsim (\chi', z', \gamma')$ in $\hat{A} * \mathcal{G} \times T$, that is, $(\chi, z, \gamma) \neq (\chi', z', \gamma')$ such that $\pi_D((\chi, z, \gamma)) = [\chi, z, \gamma]$ and $\pi_D((\chi', z', \gamma')) = [\chi', z', \gamma']$. Since $\hat{A} * \mathcal{G} \times T$ is Hausdorff, there exists open neighborhoods U and V in $\hat{A} * \mathcal{G} \times T$ containing (χ, z, γ) and (χ', z', γ'), respectively such that $U \cap V = \varnothing$. Then, $\pi_D(U)$ and $\pi_D(V)$ are open neighborhoods in D containing $[\chi, z, \gamma]$ and $[\chi', z', \gamma']$, respectively such that $\pi_D(U) \cap \pi_D(V) = \varnothing$.

Let $[\chi, z, \gamma] \in D$ and U and V be open subsets of D where $[\chi, z, \gamma] \in U$. We need to show that $r : U \to V$ is a homeomorphism. Let V_1 be an open subset of V such that $r^{-1}(V_1) \subseteq U$. Then, $M = \pi_D^{-1}(V_1)$ is open in $\hat{A} * \mathcal{G} \times T$ so that $r^{-1}(V_1) = \pi_D(M)$ is open D.

Hence, r is continuous. Similarly, r^{-1} is continuous. Thus, r is a local homeomorphism. Therefore, D is a Hausdorff étale groupoid.

Let $[\chi, z, \gamma] \in D$ such that $s([\chi, z, \gamma]) = r([\chi, z, \gamma])$. Note that

$$\begin{aligned} s([\chi, z, \gamma]) &= [\chi, z, \gamma]^{-1}[\chi, z, \gamma] \\ &= [\chi \cdot \gamma, z^{-1}, \gamma^{-1}][\chi, z, \gamma] \\ &= [\chi \cdot \gamma, z^{-1}z, \gamma^{-1}\gamma] \\ &= [\chi \cdot \gamma, 1, s(\gamma)] \end{aligned}$$

$$\begin{aligned} r([\chi, z, \gamma]) &= [\chi, z, \gamma][\chi, z, \gamma]^{-1} \\ &= [\chi, z, \gamma][\chi \cdot \gamma, z^{-1}, \gamma^{-1}] \\ &= [\chi, zz^{-1}, \gamma\gamma^{-1}] \\ &= [\chi, 1, r(\gamma)]. \end{aligned}$$

Hence, $[\chi \cdot \gamma, 1, s(\gamma)] = [\chi, 1, r(\gamma)]$, that is, $\chi \cdot \gamma = \chi$ and $r(\gamma) = s(\gamma)$. Then $\gamma = u \in \mathcal{G}^{(0)}$. Hence, the elements in $D^{(0)}$ will look like $[\chi, 1, u]$. Now, $i(\hat{A} \times \{1\}) = i(\chi, 1, u) = [\chi, 1, u]$. Therefore, $D^{(0)} = i(\hat{A} \times \{1\})$. □

Note that $\hat{A} \times T$ is the isotropy group of $\hat{A} * \mathcal{G} \times T$ by Lemma 7. Hence, $\hat{A} \times T$ is a group bundle by Remark 1. Then, define the sequence $\hat{A} \times T \overset{i}{\hookrightarrow} D \overset{q}{\hookrightarrow} \hat{A} \rtimes \mathcal{R}$ where D is a Hausdorff étale groupoid by Proposition 3, and the maps i and q are defined by $i((\chi, z, u)) = [\chi, z, u]$ and $q([\chi, z, \gamma]) = (\chi, \dot{\gamma})$, respectively.

Lemma 9. *The maps i and q are continuous groupoid homomorphism that restricts to homeomorphism of unit spaces.*

Proof. Let $(\chi, z, u), (\chi', z', u') \in \hat{A} \times T$ such that $(\chi, z, u) = (\chi', z', u')$. Then $[\chi, z, u] = [\chi', z', u']$. Thus, $i((\chi, z, u)) = i((\chi', z', u'))$ and i is well-defined. Let $[\chi_1, z_1, \gamma_1]$ and $[\chi_2, z_2, \gamma_2]$ be elements in D such that $q[\chi_1, z_1, \gamma_1] \neq q[\chi_2, z_2, \gamma_2]$. Then $(\chi_1, \dot{\gamma}_1) \neq (\chi_2, \dot{\gamma}_2)$. If $\chi_1 \neq \chi_2$, then $(\chi_1, z_1, \gamma_1) \nsim (\chi_2, z_2, \gamma_,)$. If $\dot{\gamma}_1 \neq \dot{\gamma}_2$, then $\gamma_1 A \neq \gamma_2 A$. Since $A_u \subset A$, then we cannot find $a \in A_u$ such that $\gamma_1 = a \cdot \gamma_2$. Hence, $(\chi_1, z_1, \gamma_1) \nsim (\chi_2, z_2, \gamma_2)$. In both cases $(\chi_1, z_1, \gamma_1) \nsim (\chi_2, z_2, \gamma_2)$ which means that $[\chi_1, z_1, \gamma_1] \neq [\chi_2, z_2, \gamma_2]$. Thus, q is well-defined.

We need $\hat{A} \times T \subset \hat{A} * \mathcal{G} \times T$ to show that i is continuous. Let $(\chi, z, u) \in \hat{A} \times T$ where $(\chi, u) \in \hat{A}$, and $u \in \mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)} \subset \mathcal{G}$, then $(\chi, z, u) \in \hat{A} * \mathcal{G} \times T$. Since $\pi_D : \hat{A} * \mathcal{G} \times T \to D$ is continuous, $i = \pi_D|_{\hat{A} \times T} : \hat{A} \times T \to D$ is also continuous.

Let $q \circ \pi_D : \hat{A} * \mathcal{G} \times T \to \hat{A} \rtimes \mathcal{R}$ be defined by $(q \circ \pi_D)(\chi, z, \gamma) = q(\pi_D(\chi, z, \gamma))$ and let U be an open subset of $\hat{A} \rtimes \mathcal{R}$. Then $U = (A \times B) \cap \hat{A} \rtimes \mathcal{R}$ where $A \times B$ is open in $\hat{A} \times \mathcal{R}$. Let $(\chi, z, \gamma) \in (q \circ \pi_D)^{-1}(U)$. Then $q \circ \pi_D(\chi, z, \gamma) \in U$, that is, $q(\pi_D(\chi, z, \gamma)) = q([\chi, z, \gamma]) = (\chi, \dot{\gamma}) \in A \times B \cap \hat{A} \rtimes \mathcal{R}$. Then $(\chi, \dot{\gamma}) \in A \times B$, that is, $\chi \in A$ and $\dot{\gamma} \in B$. Since $\pi_{\mathcal{R}}$ is continuous, $\pi_{\mathcal{R}}^{-1}(B)$ is open in $\mathcal{G}$ containing γ. Let

$$M = A * B_2 \times \{z\} = \{(\chi, z, \gamma) \in \hat{A} * \mathcal{G} \times T : \pi_D(\chi, z, \gamma) \in (q \circ \pi_D)^{-1}(U)\}.$$

Then $M \subseteq (q \circ \pi_D)^{-1}(U)$ and M is open in $\hat{A} * \mathcal{G} \times T$. Thus, $(q \circ \pi_D)^{-1}(U)$ is open and $q \circ \pi_D$ is continuous. Since π_D is continuous , q is continuous.

Consider $i \times i : (\hat{A} \times T)^{(2)} \to D^{(2)}$ defined by

$$(i \times i)(((\chi, z, u), (\chi', z', u'))) = (([\chi, z, u], [\chi', z', u'])) \in D^{(2)}.$$

Also, $q \times q : D^{(2)} \to (\hat{A} \times \mathcal{R})^{(2)}$ defined by

$$(q \times q)(([\chi, z, \gamma], [\chi', z', \gamma'])) = ((\chi, \dot{\gamma}), (\chi', \dot{\gamma}')) \in (\hat{A} \rtimes \mathcal{R})^{(2)}.$$

Now,

$$i((\chi, z, u)(\chi', z', u')) = i((\chi, zz', uu')) = [\chi, zz', uu'];$$
$$i((\chi, z, u))i((\chi', z', u')) = [\chi, z, u][\chi', z', u'] = [\chi, zz', uu'].$$

Also,

$$q([\chi, z, \gamma][\chi', z', \gamma']) = q([\chi, zz', \gamma\gamma']) = (\chi, \dot{\gamma}\dot{\gamma}');$$
$$q([\chi, z, \gamma])q([\chi', z', \gamma']) = (\chi, \dot{\gamma})(\chi', \dot{\gamma}') = (\chi, \dot{\gamma}\dot{\gamma}').$$

Thus, i and q are continuous groupoid homomorphism.

Let $i|_{(\hat{A} \times T)^{(0)}} : (\hat{A} \times T)^{(0)} \to D^{(0)}$. Since i is continuous by Lemma 9, then $i|_{(\hat{A} \times T)^{(0)}}$ is continuous. Let V be open in $(\hat{A} \times T)^{(0)}$. Then $V = A \times B$ where A is open in $\hat{A}$ and B is open in T. Let $M = A * \mathcal{G}^{(0)} \times B = \{(\chi, z, u) \in \hat{A} * \mathcal{G} \times T : \pi_D(\chi, z, u) \in i(V)\}$. Then $M = \pi_D^{-1}(i(V))$ and M is open in $\hat{A} * \mathcal{G} \times T$ since A is open in $\hat{A}$, $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$ and B is open in T. Hence, $i(V)$ is open in $D^{(0)}$ and i^{-1} is continuous. Thus, $i|_{\hat{A} \times T}$ is a homeomorphism of unit spaces.

Now, let $q|_{D^{(0)}} : D^{(0)} \to (\hat{A} \rtimes \mathcal{R})^{(0)}$. Since q is continuous by Lemma 9, then $q|_{D^{(0)}}$ is continuous. Let Y be an open subset of $D^{(0)}$. Then $\pi_D^{-1}(Y)$ is open in $\hat{A} * \mathcal{G} \times T$, that is, $\pi_D^{-1}(Y) = A * B \times C$ where A is open in $\hat{A}$, B is open in $\mathcal{G}$ and C is open in T. Let $M = A \times \pi_{\mathcal{R}}(B) \cap (\hat{A} \rtimes \mathcal{R})^{(0)} = \{(\chi, \dot{\gamma}) \in (\hat{A} \rtimes \mathcal{R})^{(0)} : q^{-1}(\chi, \dot{\gamma}) \in Y\}$. Then $M = q(Y)$ and M is open in $(\hat{A} \rtimes \mathcal{R})^{(0)}$ since A is open in $\hat{A}$ and $\pi_{\mathcal{R}}(B)$ is open in $\mathcal{R}$. Then $q(Y)$ is open in $(\hat{A} \rtimes \mathcal{R})^{(0)}$ and q^{-1} is continuous. Therefore, $q|_{D^{(0)}}$ is a homeomorphism. □

In order to show that (D, i, q) is a discrete twist over $\hat{A} \rtimes \mathcal{R}$. We prove first the following results.

Theorem 4. *The sequence* $\hat{A} \times T \overset{i}{\hookrightarrow} D \overset{q}{\hookrightarrow} \hat{A} \rtimes \mathcal{R}$ *is exact, that is,*

(i) $i(\{(\chi, u) \times T\}) = q^{-1}((\chi, u))$ *for* $(\chi, u) \in (\hat{A} \rtimes \mathcal{R})^{(0)}$,

(ii) i *is injective, and*

(iii) q *is a quotient map.*

Proof.

(i) Let $(\chi, u) \in \hat{A}$. Then $i(\{(\chi, u)\} \times T\}) = [\chi, z, u]$ for some $z \in T$ and $q^{-1}((\chi, u)) = [\chi, z, u]$ for some $z \in T$. Hence, $i(\{(\chi, u) \times T\}) = q^{-1}((\chi, u))$ for $(\chi, u) \in (\hat{A} \rtimes \mathcal{R})^{(0)}$.

(ii) Let $(\chi_1, z_1, u_1), (\chi_2, z_2, u_2) \in \hat{A} \rtimes T$ with $q((\chi_1, z_1, u_1)) = q((\chi_2, z_2, u_2))$. Then, $[\chi_1, z_1, \gamma_1] = [\chi_2, z_2, \gamma_2]$, that is, $(\chi_1, z_1, \gamma_1) \sim (\chi_2, z_2, \gamma_2)$. Then $\chi_1 = \chi_2$ and we can choose $a \in A_U$ such that $\chi(a) = 1 \in R^\times$ such that $z_1 = \chi(a)z_2 = (1)z_2 = z_2$. Also, choose $a' \in A_U$ such that $a; \in \mathcal{G}^{(0)}$. Then $\gamma_1 = a \cdot \gamma_2 = \gamma_2$. Thus, $(\chi_1, z_1, \gamma_1) = (\chi_2, z_2, \gamma_2)$ and i is injective.

(iii) By Lemma 9, q is continuous. Let $(\chi, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$. Then $[\chi, z, \gamma]$ where $r(\gamma) = u$ is the pre-image of $(\chi, \dot{\gamma})$ in D. Thus, q is surjective and a quotient map. □

Theorem 5. *D is a locally trivial G-bundle in the sense that for each $(\chi, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$, there is an open bisection B_α of $\hat{A} \rtimes \mathcal{R}$ containing $(\chi, \dot{\gamma})$, and a continuous map $P_\alpha : B_\alpha \to D$ such that*

(i) $q \circ P_\alpha = id_{B_\alpha}$

(ii) the map $(\beta, z) \to i(r(\beta), z)P_\alpha(\beta)$ *is a homeomorphism from* $B_\alpha \times T$ *to* $q^{-1}(B_\alpha)$.

Proof.

(i) Let $(\chi, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$ and B_α be an open bisection of $\hat{A} \rtimes \mathcal{R}$ containing $(\chi, \dot{\gamma})$ and let $P_\alpha : B_\alpha \to D$ defined by $P_\alpha((\chi, \dot{\gamma})) = [\chi, z, \gamma]$. Let U be an open subset of D. Then $\pi_D^{-1}(U)$ is open in $\hat{A} * \mathcal{G} \times T$, that is, $\pi_D^{-1}(U) = A * B \times C$ where A is open in $\hat{A}$, B is open in $\mathcal{G}$ and C is open in T. Let $(\chi, z, \gamma) \in \pi_D^{-1}(U)$. Then $(\chi, z, \gamma) \in A * B \times C$, that is, $\chi \in A$ and $\gamma \in B$. Since B is open in $\mathcal{G}$, then $\pi_R(B)$ is open in D containing $\dot{\gamma}$. Let $M = A \times \pi_R(B) \cap \hat{A} \rtimes \mathcal{R}$. Then $(\chi, \dot{\gamma}) \in M$ and M is open in $\hat{A} \rtimes \mathcal{R}$. Since $(\chi, \dot{\gamma})$ is chosen arbitrarily, then for every element in $P_\alpha^{-1}(U)$ there exists an open neighborhood M containing $(\chi, \dot{\gamma})$. Thus, $P_\alpha^{-1}(U)$ is open and P_α is continuous. Now, $q \circ P_\alpha : B_\alpha \to \hat{A} \rtimes \mathcal{R}$. Then, $q \circ P_\alpha((\chi, \dot{\gamma})) = q(p_\alpha((\chi, \dot{\gamma}))) = q([\chi, z, \gamma]) = (\chi, \dot{\gamma})$. Hence, the image of B_α in P_α is just itself and we have $q \circ P_\alpha = id_{B_\alpha}$.

(ii) Let $\theta : \beta_\alpha \times T \to q^{-1}(\beta_\alpha)$ where $\beta_\alpha \times T \subseteq \hat{A} \rtimes \mathcal{R} \times T$ and $q^{-1}(\beta_\alpha) \subseteq D$. Let U be an open subset of $q^{-1}(\beta_\alpha)$. Then there exists $U' \in \hat{A} * \mathcal{G} \times T$ such that $U = U'/\sim \in D$ where U' is open in $\hat{A} * \mathcal{G} \times T$. Here, U' is the pre-image of U under π_D. Since U' is open in $\hat{A} * \mathcal{G} \times T$, then $U' = (A \times B) \times C$ where A is open in $\hat{A}$, B is open in $\mathcal{G}$ and C is open in the discrete topology for T. Note that $\theta^{-1}(U) = \{(\chi, z, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R} \times T : \theta(\chi, z, \dot{\gamma}) \in U)\}$. Since $U = U'/\sim$, then there exists an element (χ, z, γ) in U' whose equivalence class in D is in U. Since $(\chi, z, \gamma) \in U'$, then $\chi \in \hat{A}$, $z \in C$ which is open in T, and $\gamma \in B$ which is open in $\mathcal{G}$. Since we have B to be an open set in $\mathcal{G}$ containing γ, then there exists open set M in $\mathcal{R}$ containing $\dot{\gamma}$. Hence, $(A \times M) \times C$ is an open set in $\hat{A} \rtimes \mathcal{R} \times T$. Since $(\chi, z, \dot{\gamma})$ is arbitrary, we have shown that every element in $\theta^{-1}(U)$ is contained in some open set in $\hat{A} \rtimes \mathcal{R} \times T$. Thus, θ is continuous.

Note that $\theta^{-1} : q^{-1}(\beta_\alpha) \to \beta_\alpha \times T$. Let U be an open subset of $\beta_\alpha \times T$. Then, $U = V \times T'$ where V is open in $\hat{A} \rtimes \mathcal{R}$ and T' is open in T. Let $[\chi, z, \gamma] \in \theta(U)$. Then

$\theta^{-1}([\chi, z, \gamma]) \in V \times T'$, that is, $(\chi, z, \gamma) \in V \times T'$. Hence, $z \in T'$ and $(\chi, \dot{\gamma}) \in V$. Since V is open in $\hat{A} \rtimes \mathcal{R}$, then $V = A \times B \cap \hat{A} \rtimes \mathcal{R}$ where $\chi \in A$, A open in $\hat{A}$ and $\dot{\gamma} \in B$, B open in $\mathcal{R}$. Thus, $\pi_{\mathcal{R}}^{-1}(B)$ is open in $\mathcal{G}$. Let $\pi_D^{-1}(\theta(U)) = M = A * \pi_{\mathcal{R}}^{-1}(B) \times T'$ which is open in $\hat{A} * \mathcal{G} \times T$. Hence, $\theta(U)$ is open $q^{-1}(B_\alpha)$ and θ^{-1} is continuous. Therefore, θ is a homeomorphism. □

Theorem 6. *The image $i(\hat{A} \times T)$ is central in D in the sense that $i(r([\chi, z, \gamma]), z)[\chi, z, \gamma] = [\chi, z, \gamma]i(s([\chi, z, \gamma]), z)$ for all $[\chi, z, \gamma] \in D$ and $z \in T$.*

Proof. Let $[\chi, z, \gamma] \in D$ and $z' \in T$. Now,

$$\begin{aligned} i(r([\chi, z, \gamma]), z')[\chi, z, \gamma] &= i((\chi, r(\gamma)), z')[\chi, z, \gamma] \\ &= [\chi, z', r(\gamma)][\chi, z, \gamma] \\ &= [\chi, z'z, r(\gamma)\gamma] \\ &= [\chi, z'z, \gamma]. \end{aligned}$$

Also,

$$\begin{aligned} [\chi, z, \gamma]i(s([\chi, z, \gamma]), z') &= [\chi, z, \gamma]i((\chi \cdot \gamma, s(\gamma), z')) \\ &= [\chi, z, \gamma][\chi \cdot \gamma, z', s(\gamma))] \\ &= [\chi, zz', \gamma s(\gamma)] \\ &= [\chi, zz', \gamma]. \end{aligned}$$

Hence, $i(r([\chi, z, \gamma]), z')[\chi, z, \gamma] = [\chi, z, \gamma]i(s([\chi, z, \gamma]), z')$ and so the image of i is central in D. □

The following corollary follows from Theorems 4, 5 and Lemma 6.

Corollary 1. *(D, i, q) is a discrete twist over $\hat{A} \rtimes \mathcal{R}$.*

4. Non-isomorphic property of $A_{\mathbb{Z}}(\mathbb{Z})$ and $A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$

In this section we present a case in which the non-twisted Steinberg algebra ($A_R(\mathcal{G})$) and twisted Steinberg algebra ($A_R(D; \hat{A} \rtimes \mathcal{R})$) is not isomorphic when $G = \mathbb{Z}$ and $R = \mathbb{Z}$.

Let $\mathcal{G} = \mathbb{Z}$ and $R = \mathbb{Z}$. The set of multiplicative units of $\mathbb{Z}$ is $\mathbb{Z}^\times = \{-1, 1\} = T$ and the unit space of $\mathbb{Z}$ is $\mathbb{Z}^{(0)} = \{x \in \mathbb{Z} : x = s(y) = r(y), y \in \mathbb{Z}\} = \{0\}$. The source and range maps are $s(x) = (-x) + x = \{0\}$ and $r(x) = x + (-x) = \{0\}$, respectively. The isotropy group for $\mathbb{Z}$ is $A = \mathbb{Z}$. Also, $\mathcal{R} = \mathbb{Z}\mathbb{Z} = \{x + \mathbb{Z} : x \in \mathbb{Z}\} = \{\dot{0}\}$ where $\{\dot{0}\} = 0 + \mathbb{Z}$. For $u \in \mathbb{Z}^{(0)}$, we have $A_0 = \mathbb{Z}$. Also, $\hat{A}_0 = \{\chi_1, \chi_2 | \chi_i : \mathbb{Z} \to \{1, -1\}$ is a continuous group homomorphism$\}$, $i = 1, 2$ where $\chi_1 : \mathbb{Z} \to \mathbb{Z}^\times$ defined by $\chi_1(a) = 1$ and $\chi_2 : \mathbb{Z} \to \mathbb{Z}^\times$ defined by

$$\chi_2(a) = \begin{cases} 1 & \text{if}\, a \in 2\mathbb{Z} \\ -1 & \text{if}\, a \in 2\mathbb{Z} + 1. \end{cases}$$

Note that

$$\begin{aligned}\hat{A} * \mathbb{Z} \times T &= \{(\chi, z, x) : \chi \in \hat{A}_0, z \in T, x \in \mathbb{Z}\} \\ &= \{(\chi_1, 1, x), (\chi_1, -1, x), (\chi_2, 1, x), (\chi_2, -1, x)\}.\end{aligned}$$

So, $D = (\hat{A} * \mathbb{Z} \times T/\sim) = \{[\chi_1, 1, x], [\chi_1, -1, x], [\chi_2, 1, x], [\chi_2, -1, x] | x \in \mathbb{Z}\}$.
Claim 1: $D = \{[\chi_1, 1, 0], [\chi_1, -1, 0], [\chi_2, 1, 0], [\chi_2, -1, 0]\}$.
For $i = 1, 2$,

$$\begin{aligned}[\chi_i, 1, x] &= \{(\chi', z', x') | \chi_i = \chi', \exists\, a \in \mathbb{Z} \,\text{where}\, \chi_i(a)(1) = z' \,\text{and}\, x = a \cdot x'\} \\ &= \{(\chi_i, z', x') | \exists\, a \in \mathbb{Z} \,\text{where}\, 1 = \chi_i(a)(1) = z' \,\text{and}\, x = a \cdot x'\} \\ &= \{(\chi_i, 1, x') | \exists\, a \in \mathbb{Z},\, x = a \cdot x'\} \\ &= \{(\chi_i, 1, x/a) | a \in \mathbb{Z}, x \in \mathbb{Z}\} \\ &= \{(\chi_i, 1, x) | a = 1, x \in \mathbb{Z}\} \\ &= \{\cdots (\chi_i, 1, -1), (\chi_i, 1, 0), (\chi_i, 1, 1) \cdots\}\end{aligned}$$

$$\begin{aligned}[\chi_i, -1, x] &= \{(\chi', z', x') | \chi_i = \chi', \exists\, a \in \mathbb{Z} \,\text{where}\, \chi_i(a)(-1) = z' \,\text{and}\, x = a \cdot x'\} \\ &= \{(\chi_i, z', x') | \exists\, a \in \mathbb{Z} \,\text{where}\, -1 = \chi_i(a)(-1) = z' \,\text{and}\, x = a \cdot x'\} \\ &= \{(\chi_i, -1, x') | \exists\, a \in \mathbb{Z},\, x = a \cdot x'\} \\ &= \{(\chi_i, -1, x/a) | a \in \mathbb{Z}, x \in \mathbb{Z}\} \\ &= \{(\chi_i, -1, x) | a = 1, x \in \mathbb{Z}\} \\ &= \{\cdots (\chi_i, -1, -1), (\chi_i, -1, 0), (\chi_i, -1, 1) \cdots\}\end{aligned}$$

Hence, $(\chi_i, 1, 0) \in [\chi_i, 1, x]$ and $(\chi_i, -1, 0) \in [\chi_i, -1, x]$ imply that $[\chi_i, 1, x] = [\chi_i, 1, 0]$ and $[\chi_i, -1, x] = [\chi_i, -1, 0]$. Therefore,

$$D = \{[\chi_1, 1, 0], [\chi_1, -1, 0], [\chi_2, 1, 0], [\chi_2, -1, 0]\}.$$

and claim 1 is proved.
Now, the source of $[\chi_1, -1, 0]$ in D is,

$$\begin{aligned}s([\chi_1, -1, 0]) &= [\chi_1, -1, 0]^{-1}[\chi_1, -1, 0] \\ &= [\chi_1 \cdot 0, (-1)^{-1}, 0][\chi_1, -1, 0] \\ &= [\chi_1, (-1)^{-1}(-1), 0(0)] \\ &= [\chi_1, 1, 0].\end{aligned}$$

Also, the range of $[\chi_1, -1, 0]$ in D is,

$$\begin{aligned}r([\chi_1, -1, 0]) &= [\chi_1, -1, 0][\chi_1, -1, 0]^{-1} \\ &= [\chi_1, -1, 0][\chi_1 \cdot 0, (-1)^{-1}, 0]\end{aligned}$$

$$
\begin{aligned}
&= [\chi_1, (-1)(-1)^{-1}, (0)0] \\
&= [\chi_1, 1, 0]
\end{aligned}
$$

For $[\chi_2, -1, 0]$ in D,

$$
\begin{aligned}
s([\chi_2, -1, 0]) &= [\chi_2, -1, 0]^{-1}[\chi_2, -1, 0] \\
&= [\chi_2 \cdot 0, (-1)^{-1}, 0][\chi_2, -1, 0] \\
&= [\chi_2, (-1)^{-1}(-1), 0(0)] \\
&= [\chi_2, 1, 0].
\end{aligned}
$$

and

$$
\begin{aligned}
r([\chi_2, -1, 0]) &= [\chi_2, -1, 0][\chi_2, -1, 0]^{-1} \\
&= [\chi_2, -1, 0][\chi_2 \cdot 0, (-1)^{-1}, 0] \\
&= [\chi_2, (-1)(-1)^{-1}, (0)0] \\
&= [\chi_2, 1, 0]
\end{aligned}
$$

Our groupoid D is best understood with this illustration:

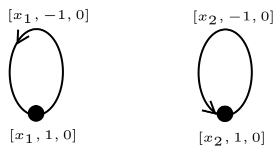

Figure 1: Morphisms in D

Hence, the unit space for D is $D^{(0)} = \{[\chi_1, 1, 0], [\chi_2, 1, 0]\}$.

When $\mathbb{Z}$ is endowed with the discrete topology, its base will be composed of singletons $\{z\}$, for all $z \in \mathbb{Z}$. Let $1_{\{z\}}$ denotes the characteristic function of $\{z\}$ from $\mathbb{Z}$ to $\mathbb{Z}$. The Steinberg algebra associated to $\mathbb{Z}$ is

$$A_{\mathbb{Z}}(\mathbb{Z}) := \text{span}\{1_{\{z\}} : \mathbb{Z} \to \mathbb{Z} | \{z\} \text{ is a compact open bisection of } \mathbb{Z}\}$$

equipped with pointwise addition,

$$a_1 1_{\{z_1\}} + a_2 1_{\{z_2\}} = (a_1 + a_2) 1_{\{z_1+z_2\}}$$

and multiplication as follows;

$$a_1 1_{\{z_1\}} \cdot a_2 1_{\{z_2\}} = a_1 a_2 1_{\{z_1+z_2\}}.$$

Claim 2: $\hat{A} \rtimes \mathcal{R} = (\hat{A} \rtimes \mathcal{R})^{(0)}$

The source and range of $(\chi_1, 0, \dot{0}) \in \hat{A} \rtimes \mathcal{R}$ are:

$$s((\chi_1, 0, \dot{0})) = (\chi_1, 0, \dot{0})^{-1}(\chi_1, 0, \dot{0}) = (\chi_1 \cdot 0, 0, \dot{0})(\chi_1, 0, \dot{0}) = (\chi_1, 0, \dot{0})$$

$$r((\chi_1, 0, \dot{0})) = (\chi_1, 0, \dot{0})(\chi_1, 0, \dot{0})^{-1} = (\chi_1, 0, \dot{0})(\chi_1 \cdot 0, 0, \dot{0}) = (\chi_1, 0, \dot{0}).$$

Also, the source and range for $(\chi_2, 0, \dot{0}) \in \hat{A} \rtimes \mathcal{R}$ are:

$$s((\chi_2, 0, \dot{0})) = (\chi_2, 0, \dot{0})^{-1}(\chi_2, 0, \dot{0}) = (\chi_2 \cdot 0, 0, \dot{0})(\chi_2, 0, \dot{0}) = (\chi_2, 0, \dot{0})$$

$$r((\chi_2, 0, \dot{0})) = (\chi_2, 0, \dot{0})(\chi_2, 0, \dot{0})^{-1} = (\chi_2, 0, \dot{0})(\chi_2 \cdot 0, 0, \dot{0}) = (\chi_2, 0, \dot{0}).$$

Hence, $(\hat{A} \rtimes \mathcal{R})^{(0)} = \{(\chi_1, 0, \dot{0}), (\chi_2, 0, \dot{0})\} = \hat{A} \rtimes \mathcal{R}$.

Theorem 7. *If $\hat{A} \rtimes \mathcal{R} = (\hat{A} \rtimes \mathcal{R})^{(0)}$, then $A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R}) \cong A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$.*

Proof. Let $F : A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R}) \to A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$ be defined by

$$F(f)([\chi_i, z, \gamma]) = z \cdot f(r((\chi_i, \dot{\gamma})))$$

where $z \in T, (\chi_i, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$. Linearity holds since for all $f, g \in A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R})$ and $n \in \mathbb{Z}$,

$$\begin{aligned} F(f+g)([\chi_i, z, \gamma]) &= z(f+g)(r((\chi_i, \dot{\gamma}))) \\ &= zf(r((\chi_i, \dot{\gamma}))) + zg(r((\chi_i, \dot{\gamma}))) \\ &= (F(f) + F(g))([\chi_i, z, \gamma]) \end{aligned}$$

and

$$F(nf) = znf(r((\chi_i, \dot{\gamma}))) = nzf(r(\chi_i, \dot{\gamma}))) = nF(f).$$

Now, observe that

$$F(fg)([\chi_i, z, \gamma]) = z(fg)(r((\chi_i, \dot{\gamma}))) = zf(r((\chi_i, \dot{\gamma})))g(r((\chi_i, \dot{\gamma}))).$$

Also,

$$(F(f)F(g))([\chi_i, z, \gamma])(\chi_i, \dot{\gamma}) = \sum_{\substack{((\chi_i', \dot{\gamma}'), (\chi_i'', \dot{\gamma}'') \in (\hat{A} \rtimes \mathcal{R})^{(2)}, \\ (\chi_i', \dot{\gamma}'))(\chi_i'', \dot{\gamma}'') = (\chi_i, \dot{\gamma})}} F(f)(\chi_i', \dot{\gamma}')F(g)(\chi_i'', \dot{\gamma}'')^{-1}$$

Since $\hat{A} \rtimes \mathcal{R} = (\hat{A} \rtimes \mathcal{R})^{(0)}$, then for all $(\chi_i, \dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$ the only composable pairs in $\hat{A} \rtimes \mathcal{R}$ is of the form $((\chi_i, \dot{\gamma}), (\chi_i, \dot{\gamma}))$ where $s(\chi_i, \dot{\gamma}) = (\chi_i, \dot{\gamma}) = r(\chi_i, \dot{\gamma})$. But $(\chi_1, \dot{0})(\chi_2, \dot{0}) \neq (\chi_1, \dot{0})$. Hence,

$$\begin{aligned} (F(f)F(g))([\chi_i, z, \gamma])(\chi_1, \dot{0}) &= \sum_{\substack{((\chi_1, \dot{0}), (\chi_1, \dot{0}) \in (\hat{A} \rtimes \mathcal{R})^{(2)}, \\ (\chi_1, \dot{0}))(\chi_1, \dot{0}) = (\chi_1, \dot{0})}} F(f)(\chi_1, \dot{0})F(g)(\chi_1, \dot{0})^{-1} \\ &= F(f)(\chi_1, \dot{0})F(g)(\chi_1, \dot{0})^{-1} \\ &= F(f)(\chi_1, \dot{0})F(g)(\chi_1, \dot{0}) \\ &= zf(r((\chi_1, \dot{0})))g(r((\chi_1, \dot{0}))). \end{aligned}$$

Also, since $(\chi_2,\dot{0})(\chi_1,\dot{0}) \neq (\chi_2,\dot{0})$, we get

$$\begin{aligned}(F(f)F(g))([\chi_i,z,\gamma])(\chi_2,\dot{0}) &= \sum_{\substack{((\chi_2,\dot{0}),(\chi_2,\dot{0})\in(\hat{A}\rtimes\mathcal{R})^{(2)},\\ (\chi_2,\dot{0}))(\chi_2,\dot{0})=(\chi_2,\dot{0})}} F(f)(\chi_2,\dot{0})F(g)(\chi_2,\dot{0})^{-1}\\ &= F(f)(\chi_2,\dot{0})F(g)(\chi_2,\dot{0})^{-1}\\ &= F(f)(\chi_2,\dot{0})F(g)(\chi_2,\dot{0})\\ &= zf(r((\chi_2,\dot{0})))g(r((\chi_2,\dot{0}))).\end{aligned}$$

Thus, for all $(\chi_i,\dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$,

$$F(f)F(g)([\chi_i,z,\gamma])(\chi_i,\dot{\gamma}) = zf(r((\chi_i,\dot{\gamma})))zg(r((\chi_i,\dot{\gamma}))) = F(fg)([\chi_i,z,\gamma])$$

and F is a $\mathbb{Z}$-module homomorphism.

Suppose that $F(f)([\chi_i,z,\gamma]) = 0$ for $[\chi_i,z,\gamma] \in D$. Then, $zf(r((\chi_i,\dot{\gamma}))) = 0$ for all $z \in T$ and $\gamma \in \mathbb{Z}$ which means that $f(r((\chi_i,\dot{\gamma}))) = 0$. Since $\hat{A} \rtimes \mathcal{R} = (\hat{A} \rtimes \mathcal{R})^{(0)}$, then for all $(\chi_i,\dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$, $r((\chi_i,\dot{\gamma})) = (\chi_i,\dot{\gamma})$. Hence for all $(\chi_i,\dot{\gamma}) \in \hat{A} \rtimes \mathcal{R}$, $f((\chi_i,\dot{\gamma})) = f(r((\chi_i,\dot{\gamma}))) = 0$ which implies that $f = 0$. Thus, F is injective. Now let $h \in A_{\mathbb{Z}}(D;\hat{A}\rtimes\mathcal{R})$ and define $f_h : \hat{A} \rtimes \mathcal{R} \to \mathbb{Z}$ by $f_h((\chi_i,\dot{0})) = h([\chi_i,1,0])$. Notice that every element in $A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R})$ is a mapping from $\hat{A} \rtimes \mathcal{R}$ to $\mathbb{Z}$. We are left to show that f_h is continuous and $\text{supp}(f_h)$ is compact. Since $\hat{A} \rtimes \mathcal{R}$ and $\mathbb{Z}$ are both discrete, then f_h is continuous. Since $h \in A_{\mathbb{Z}}(D;\hat{A} \rtimes \mathcal{R})$, $h = a_1 1_{\{[\chi_1,1,0]\}} + a_2 1_{\{[\chi_1,-1,0]\}} + a_3 1_{\{[\chi_2,1,0]\}} + a_4 1_{\{[\chi_2,-1,0]\}}$. Thus,

$$\begin{aligned}f_h(\chi_i,\dot{\gamma}) &= h([\chi_i,z,\gamma])\\ &= (a_1 1_{\{[\chi_1,1,0]\}} + a_2 1_{\{[\chi_1,-1,0]\}} + a_3 1_{\{[\chi_2,1,0]\}} + a_4 1_{\{[\chi_2,-1,0]\}})([\chi_i,z,\gamma])\end{aligned}$$

Hence,

$$\begin{aligned}f_h(\chi_1,\dot{0}) &= h([\chi_1,1,0])\\ &= (a_1 1_{\{[\chi_1,1,0]\}} + a_2 1_{\{[\chi_1,-1,0]\}} + a_3 1_{\{[\chi_2,1,0]\}} + a_4 1_{\{[\chi_2,-1,0]\}})([\chi_1,1,0])\\ &= a_1\end{aligned}$$

$$\begin{aligned}f_h(\chi_2,\dot{0}) &= h([\chi_2,1,0])\\ &= (a_1 1_{\{[\chi_1,1,0]\}} + a_2 1_{\{[\chi_1,-1,0]\}} + a_3 1_{\{[\chi_2,1,0]\}} + a_4 1_{\{[\chi_2,-1,0]\}})([\chi_2,1,0])\\ &= a_3\end{aligned}$$

If $a_1 = a_3 = 0$, then $\text{supp}(f_h) = \varnothing$ which is closed and bounded, that is, compact. If $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$, then $\text{supp}(f_h) = \{(\chi_i,\dot{\gamma}) \in \hat{A} \rtimes \mathcal{R} : f_h((\chi_i,\dot{\gamma})) \neq 0))\} = \hat{A} \rtimes \mathcal{R}$ which is compact. Thus, $f_h \in A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R})$. Now for $[\chi_i,z,\gamma] \in D$ and since $[\chi_i,z,0]$ and $[\chi_i,z,\gamma]$ are the same equivalence classes in D,

$$F(f_h)([\chi_i,z,\gamma]) = zf_h(r((\chi_i,\dot{\gamma})))$$

$$\begin{aligned} &= zf_h(\chi_i, r(\gamma)) \\ &= zf_h(\chi_i, \gamma) \\ &= zh([\chi, 1, 0]) \\ &= h([\chi_i, z, 0]) \\ &= h([\chi_i, z, \gamma]) \\ &= h. \end{aligned}$$

Then F is surjective. Therefore, $A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R}) \cong A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$. □

Since $\hat{A} \rtimes \mathcal{R} = \{(\chi_1, \dot{0}), (\chi_2, \dot{0})\}$ is a topological space with the discrete topology, $\{(\chi_1, \dot{0})\}$ and $\{(\chi_2, \dot{0})\}$ are the basic elements of its base which are compact and an open bisection since they are singletons. Hence, we introduce our characterictic functions that spans $A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R})$ as $1_{\{(\chi, \dot{\gamma})\}} : \hat{A} \rtimes \mathcal{R} \to \mathbb{Z}$ defined by

$$1_{\{(\chi,\dot{\gamma})\}}(g) = \begin{cases} 1 & \text{if}\, g \in \{(\chi, \dot{\gamma})\} \\ 0 & \text{if}\, g \notin \{(\chi, \dot{\gamma})\}. \end{cases}$$

Define the Steinberg algbera of $\hat{A} \rtimes \mathcal{R}$ over $\mathbb{Z}$ as $A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R}) = \text{span}\ \{1_{\{(\chi_1,\dot{0})\}}, 1_{\{(\chi_2,\dot{0})\}}\}$ equipped with the pointwise addition and multiplication as follows:

$$a_1 1_{\{(\chi_1,\dot{0})\}} + a_2 1_{\{(\chi_1,\dot{0})\}} = (a_1 + a_2) 1_{\{(\chi_1,\dot{0})\}+\{(\chi_2,\dot{0})\}}$$
$$a_1 1_{\{(\chi_1,\dot{0})\}} \cdot a_2 1_{\{(\chi_2,\dot{0})\}} = (a_1 \cdot a_2) 1_{\{(\chi_1,\dot{0})\}\{(\chi_2,\dot{0})\}}.$$

By Theorem 7, $A_{\mathbb{Z}}(\hat{A} \rtimes \mathcal{R}) \cong A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$. Then the twisted Steinberg algebra of $\hat{A} \rtimes \mathcal{R}$ over the pair $(D, \mathbb{Z})$ is defined as $A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R}) \cong \text{span}\ \{1_{\{(\chi_1,\dot{0})\}}, 1_{\{(\chi_2,\dot{0})\}}\}$.

Thus, dimension of $A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$ is less than or equal to 2. Note that $A_{\mathbb{Z}}(\mathbb{Z})$ is a $\mathbb{Z}$-module. Since $A_{\mathbb{Z}}(\mathbb{Z})$ is generated by the characteristic functions of the form $1_{\{z\}}$ where $z \in \mathbb{Z}$ which is infinite, then $A_{\mathbb{Z}}(\mathbb{Z})$ is infinite dimensional. Hence, if we map $A_{\mathbb{Z}}(\mathbb{Z})$ to $A_{\mathbb{Z}}(D; \hat{A} \rtimes \mathcal{R})$ it will never be injective. Thus, isomorphism fails to hold.

Note that from Section 3, the isotropy group of $\mathbb{Z}$ is itself, that is, $\text{Iso}(\mathbb{Z}) = \mathbb{Z}$ and the unit space of $\mathbb{Z}$ is $\mathbb{Z}^{(0)} = \{0\}$. Now, the interior of $\text{Iso}(\mathbb{Z}) = \mathbb{Z} \neq \mathbb{Z}^{(0)}$. By Definition 4, $\mathbb{Z}$ is not an effective groupoid.

Conjecture:
Let $\mathcal{G}$ be an effective ample Hausdorff groupoid and R be a unital commutative ring. Then $A_R(\mathcal{G}) \cong A_R(D; \hat{A} \rtimes \mathcal{R})$.

Conclusion: We have defined a groupoid $\hat{A} \rtimes \mathcal{R}$ coming from the isotropy of an ample Hausdorff groupoid $\mathcal{G}$. We have examined the properties of the groupoid $\hat{A} \rtimes \mathcal{R}$. We have successfully constructed a discrete twist on $\hat{A} \rtimes \mathcal{R}$ thereby the presence of a twisted Steinberg algebra over $\hat{A} \rtimes \mathcal{R}$ via the discrete twist (D, i, q). Finally, we have shown that for $\mathcal{G} = \mathbb{Z}$ and $R = \mathbb{Z}$ isomorphism between $A_R(\mathcal{G})$ and $A_R(D; \hat{A} \rtimes \mathcal{R})$ fails to hold.

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