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Fuzzy SSPO-separation Axioms and Fuzzy α-SSPO Compactness

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Abstract. In this paper, we introduce the concept of new separation axioms named fuzzy SSPO-separation axioms by using the fuzzy strong semi preo-pen sets and we also introduce and investigate properties of α-SSPO compactness. We define and investigate the relation between fuzzy separation axioms, fuzzy pre-separation axioms, and different forms of fuzzy continuous mappings. We also investigate the existence of a countable base of fuzzy strong semi pre-open sets, we define the concept of SSPO separability, the concept of $\alpha - SSPO$ Lindelof sets and examine their properties. With the concepts of fuzzy strong semi pre-continuity, SSPO-irresolute continuous mappings, and other forms of fuzzy continuity, we investigate the new concept of fuzzy compactness and its properties in regard to the mentioned mappings.

2020 Mathematics Subject Classifications: 54A40, 03E72

Key Words and Phrases: Fuzzy separation axioms, Fuzzy compactness, Fuzzy topological space, Fuzzy strongly semi pre-open set, Fuzzy continuity, Fuzzy SSPO-irresolute continuous mapping, Fuzzy SSPO-irresolute open (closed) mapping, Fuzzy SSPO homeomorphism

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1. Introduction

Separation axioms were introduced to fuzzy topological spaces in [9], [10], [11],[24],[32], and in some more recent works [25], [28]. They were extensions of separation axioms introduced in general Topology. The separation axioms are more restrictive in the fuzzy topologies than in the general topologies. In this sense, the separation axioms are modified, and in many cases weaker conditions have been adapted for fuzzy topological spaces. With the introduction of fuzzy strongly semi pre-open (short SSPO) sets, we introduce the new separation axioms and investigate their relation with other forms of fuzzy separation axioms. By giving several examples we will be able to show that the newly introduced

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axioms are different from the other fuzzy separation axioms introduced by other authors in [32], [27], and [6].

Compactness is another concept that was also introduced to fuzzy topological spaces. The first ideas of compactness in fuzzy topological spaces were introduced by Chang in [4]. Some other results regarding compact fuzzy spaces were introduced by Goguen [7]. Other authors that have treated compactness in fuzzy topological spaces are Lowen in [15] and [16], Wong [31], T.E. Gantner et al [5], and more recently Saleh S. et al in [26]. The concept of α-compactness was introduced by T.E. Gantner et al in [5] and it is among the most acceptable concepts of compactness in fuzzy topological spaces. With the definition of other forms of generalized fuzzy open sets, different authors, [13], [29], [8], have defined generalization of the concept of fuzzy compactness. In our work, we use the same approach as Gantner et al in [5]. Similarly, we will define the concept of α-SSPO shading and α^*-SSPO shading that are collections of fuzzy sets that constitute only from fuzzy strong semi pre-open sets. Following the introduction of α-SSPO shading (α^*-SSPO shading) we introduce the concepts of α-SSPO compact (α^*-SSPO compact) fuzzy sets and fuzzy spaces. The new concept is stronger then the concept of α-compactness (α^*-compactness). We also investigate the existence of a countable base of fuzzy strong semi pre-open sets, SSPO separability and define the concept of α-SSPO Lindelof sets and fuzzy topological spaces as well as examine their properties. With the definition of fuzzy strong semi pre-continuity and SSPO-irresolute mappings, we investigate the new concept of fuzzy compactness and its properties in relation to the mentioned mappings.

Since separation axioms and compactness are among the fundamental principles in the field of Fuzzy Topology, the aim of this research paper is to propose some novel approaches to some theoretical problems in light of new generalized fuzzy opened sets.

2. Preliminaries

The concept of fuzzy set was initially formulated by Zadeh in [33]. Chang in [4] introduced the concept of fuzzy topological spaces (short *fts*).

Definition 1. *[33] Let X be a space of points (objects). A fuzzy set (class) A in X is characterized by a membership (characteristic) function $A(x)$ which associates with each point in X a real number in the interval $[0,1]$, with the value of $A(x)$ representing the "grade of membership" of x in A. In other words, the nearer the value of $A(x)$ to 1, the higher the grade of membership of x in A.*

Definition 2. *[20] Given a fuzzy set A of a fuzzy topological space (X, τ) , the support of the set A is defined as the set $suppA = \{x \in X : A(x) > 0\}$.*

Lemma 1. *([1], [22], [12], [13]) Let $f : X \to Y$ be a mapping. The following statements hold:*

(i) $ff^{-1}(B) \le B$, for every fuzzy set B in Y;

(ii) $f^{-1}f(A) \ge A$, for every fuzzy set A in X;

(iii) $f(A^c) \leq (f(A))^c$, *for every fuzzy set* A *in* X *;*

(iv) $f^{-1}(B^c) = (f^{-1}(B))^c$, *for every fuzzy set* B *in* Y*;*

(v) If A_1, A_2 *are fuzzy sets in* X *such that* $A_1 \leq A_2$, *then* $f(A)_1 \leq f(A)_2$*;*

(vi) If B_1, B_2 *are fuzzy sets in* Y *such that* $B_1 \leq B_2$, *then* $f^{-1}(B_1) \leq f^{-1}(B_2)$*;*

(vii) If f *is an injective mapping, then* $f^{-1}f(A) = A$ *for every fuzzy set* A *in*X*;*

(viii) If f *is a surjective mapping, then* $ff^{-1}(B) = B$ *for every fuzzy set* B *in* Y*;*

(ix) If f *is a bijective mapping, then* $f(A^c) = (f(A))^c$, *for every fuzzy set* A *in* X*;*

(x) $f(\bigwedge_{i\in I} A_i) \leq \bigwedge_{i\in I} f(A_i)$, *for every family* $\{A_i, i \in I\}$ *of fuzzy sets from* X *and* I *representing a set of indexes;*

(xi) $f(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} f(A_i)$, *for every family* $\{A_i, i \in I\}$ *of fuzzy sets from* X*;*

(xii) $f^{-1}(\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} f^{-1}(B_i)$, *for every family* $\{B_i, i \in I\}$ *of fuzzy sets from* Y *and* I *representing a set of indexes;*

(xiii) $f^{-1}(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} f^{-1}(B_i)$, *for every family* $\{B_i, i \in I\}$ *of fuzzy sets from* Y*;*

Definition 3. *([22], [23]) A fuzzy point* x_α *of a fuzzy topological space* X *is a fuzzy set defined as:*

$$x_\alpha(z) = \begin{cases} \alpha & \text{if } z = x \\ 0 & \text{if otherwise} \end{cases}$$

The support of the fuzzy point x_α *is only the element* x *with the value of membership* α. *If* $\alpha = 1$ *then* x_α *is called a singleton.*

Definition 4. *A fuzzy set* A *of the fuzzy topological space* X *is called:*

(i) Fuzzy preopen if and only if $A \leq int(clA)$ *([3], [27]);*

(ii) Fuzzy preclosed if and only if A^c *is a fuzzy preopen set of a fts* X *([3], [14], [27]).*

Given any fuzzy topological space (X, τ) the family of all fuzzy preopen (preclosed) sets is denoted $FPO(\tau)$ $(FPC(\tau))$.

Definition 5. *Let* A *be a fuzzy set of a fts* (X, τ). *Then:*

(i) $pintA = \wedge\{B \leq A; B \in FPO(\tau)\}$, *is called the fuzzy preinterior of the set* A *([27]);*

(ii) $pclA = \vee\{B \geq A; B \in FPC(\tau)\}$, *is called the fuzzy preclosure of the set* A *([27]).*

Definition 6. *A fuzzy set A of a fts X is called:*

- *Fuzzy strongly preopen (strongly preclosed) if and only if*

$$A \leq int(pclA)\ (A \geq cl(pintA))\ ([12]);$$

- *Fuzzy strongly semi pre-open (strongly semi pre-closed) if and only if*

$$A \leq int(pclA) \vee pcl(intA)\ (A \geq cl(pintA) \wedge pint(clA))\ ([17]).$$

The family of all fuzzy strongly preopen (strongly preclosed) sets in (X, τ) is denoted by $FSPO(\tau)$ ($FSPC(\tau)$); The family of all fuzzy strongly semi pre-open (strongly semi pre-closed) sets is denoted $FSSPO(\tau)$ ($FSSPC(\tau)$).

Definition 7. *[17] If A is a fuzzy set of a fts X, then:*

(i) The set: $sspintA = \vee\{B \leq A; B \in FSSPO(\tau)\}$, is called the fuzzy strong semi preinterior of set A.

(ii) The set: $sspclA = \wedge\{B \geq A; B \in FSSPC(\tau)\}$, is called the fuzzy strong semi preclosure of set A.

Lemma 2. *[17] If A is a fuzzy set of a fuzzy topological space (X, τ), then:*

- $sspclA^c = (sspintA)^c$;
- $sspintA^c = (sspclA)^c$.

Definition 8. *Let $f : (X, \tau) \to (Y, \delta)$ be a mapping from a fts (X, τ) to a fts (Y, δ). The mapping f is called:*

(i) Fuzzy continuous if $f^{-1}(B)$ is a fuzzy open set of X, for each $B \in \delta$ ([2], [4], [20], [21]);

(ii) Fuzzy open (closed) if $f(A)$ is a fuzzy open (closed) set of Y, for each $A \in \tau$ ([2], [4], [20], [21]);

(iii) Fuzzy strong semi pre-continuous if $f^{-1}(B) \in FSSPO(\tau)$ for every $B \in \delta$ ([17]);

(iv) Fuzzy SSPO-irresolute continuous if $f^{-1}(B) \in FSSPO(\tau)$ for each $B \in FSSPO(\delta)$ ([17], [18]);

(v) Fuzzy SSPO homeomorphism if it is a bijective mapping and if the mapping f and its inverse are both fuzzy $SSPO-irresolute$ continuous. ([18]).

The concepts of fuzzy separation axioms FT_0 ($FT_1, FT_s, FT_2, FT_{2\frac{1}{2}}, FR, FT_3, FN, FT_4$) will be based on definitions given in [6].
The concepts of $\alpha - shading$ ($\alpha^* - shading$), $\alpha - subshading$ ($\alpha^* - subshading$) and $\alpha - compact$ ($\alpha^* - compact$) will be based on the definitions given in ([5], [13], [19]).

Definition 9. *[5] Let (X, τ) be a fuzzy topological space and let $\alpha \in [0,1]$. A collection $\mathcal{F}$ of fuzzy sets of X is called $\alpha -$ centered ($\alpha^* -$ centered) if for every finite subcollection $\mathcal{U}$ of $\mathcal{F}$, there exist $x \in X$ such that $U(x) \geq 1-\alpha$ (respectively $U(x) > 1-\alpha$), for every $U \in \mathcal{U}$.*

Definition 10. *[30] A fuzzy topological space X is called fuzzy separable if there exists a countable sequence of fuzzy points $\{x_i\}_{i\in\mathbb{N}}$ such that for each fuzzy open set $A \neq 0_X$, there exists $x_i \in A$.*

3. Axioms of fuzzy strong semi pre-separation

Initially, we will define the axioms of fuzzy strong semi pre-separation.

Definition 11. *A fuzzy topological space X is a fuzzy strong semi pre-T_0 (or short $FSSPT_0$) if and only if for every pair of fuzzy points p_1 and p_2 with different supports, there exists a fuzzy strongly semi pre-open set O such that $p_1 \leq O \leq p_2^c$ or $p_2 \leq O \leq p_1^c$.*

It follows directly from the Definition 11 and [6] that every FT_0 space is also an $FSSPT_0$ while the converse is not true in general. We will give the following example to illustrate this fact.

Example 1. *Given a set $X = \{p_1, p_2\}$, fuzzy set $U = \{(p_1, 0), (p_2, 0.6)\}$ and fts $\tau = \{0, U, 1\}$. It is obvious that the fts is not FT_0 but it is an $FSSPT_0$ since for $O = \{(p_1, 0), (p_2, 1)\}$, $O \in FSSPO(\tau)$ it follows that $p_2 \leq O \leq p_1^c$.*

Theorem 1. *If the fuzzy topological space X is an $FSSPT_0$, and given any pair of fuzzy singletons p_1 and p_2 with different supports, then $sspclp_1 \neq sspclp_2$.*

Proof. Since the fuzzy topological space (X, τ) is an $FSSPT_0$, then given two fuzzy singletons p_1 and p_2 with different support, it is obvious that there exist a set $O \in FSSPO(\tau)$, such that $p_1 \leq O \leq p_2^c$. If we use the fact that $sspclp_2 \leq O^c$ and since $p_1 \nleq O^c$, it follows that $sspclp_1 \neq sspclp_2$.

If we refer to Example 1, it is obvious that $sspclp_1 \leq \{(p_1, 1), (p_2, 0.4)\}$ while $sspclp_2 = 1_X$, that is $sspclp_1 \neq sspclp_2$.

Definition 12. *A fuzzy topological space X is a fuzzy strong semi pre-T_1 (or short $FSSPT_1$) if and only if for any pair of fuzzy points p_1 and p_2 which have different supports, there exist fuzzy strongly semi pre-open sets O_1, O_2 such that $p_1 \leq O_1 \leq p_2^c$ and $p_2 \leq O_2 \leq p_1^c$.*

We can formulate and prove the following theorem which gives some characteristic properties for $FSSPT_1$ spaces.

Theorem 2. *The fuzzy topological space X is an $FSSPT_1$ space if and only if each fuzzy singleton is a fuzzy strongly semi pre-closed set.*

Proof. Let us suppose that the given fuzzy topological space X is an $FSSPT_1$ space. If we consider fuzzy singletons p and x with different support, it is obvious that there exist fuzzy strongly semi pre-open sets O_p and O_x such that $p \leq O_p \leq x^c$ and $x \leq O_x \leq p^c$. Now, if we consider the fuzzy set p^c as a fuzzy set that contains all of its fuzzy points, we can write that as $p^c = \bigvee_{x \leq p^c} x \leq \bigvee_{x \leq p^c} O_x$. Since X is an $FSSPT_1$ space, we also have that:

$$O_x \leq p^c \implies \bigvee_{x \leq p^c} O_x \leq p^c$$

From the two last inequalities, we have $p^c = \bigvee_{x \leq p^c} O_x$. In other words, the fuzzy set p^c is a fuzzy strongly semi pre-open set as a union of such sets and subsequently the singleton p is a fuzzy strongly semi pre-closed set.
Conversely, if each fuzzy singleton of a fuzzy topological space X is a fuzzy strongly semi pre-closed set and if we consider any pair of fuzzy singletons p and x with different support, it is obvious that p^c and x^c are fuzzy strongly semi pre-open sets such that $p \leq x^c$ and $x \leq p^c$. If we write $x^c = O_p$ and $p^c = O_x$ we get the following $p \leq O_p \leq x^c$ and $x \leq O_x \leq p^c$, which means that fuzzy topological space X is an $FSSPT_1$.

Corollary 1. *A fuzzy topological space is an $FSSPT_1$ space if and only if for each pair of fuzzy singletons p_1 and p_2 which have different supports, there exist fuzzy strongly semi pre-open sets O_1, O_2 such that $O_1(p_1) = 1$, $O_1(p_2) = 0$ and $O_2(p_1) = 0$, $O_2(p_2) = 1$.*

Proof. If the fuzzy topological space is an $FSSPT_1$ space, then according to Theorem 2, the conditions are met if we put $p_2^c = O_1$ and $p_1^c = O_2$.
Conversely, if for any pair of fuzzy singletons p_1 and p_2, with different supports, there exist fuzzy strong semi pre-open sets O_1, O_2 such that $O_1(p_1) = 1$, $O_1(p_2) = 0$ and $O_2(p_1) = 0$, $O_2(p_2) = 1$, it is obvious that $p_1 \leq O_1 \leq p_2^c$ and $p_2 \leq O_2 \leq p_1^c$, which means that the fuzzy topological space is an $FSSPT_1$ space.

We can easily conclude that any $FSSPT_1$ space is also an $FSSPT_0$ while the converse is not always true. If we consider example 1, it is obvious that the fuzzy topological space (X, τ) is not an $FSSPT_1$ space.

Definition 13. *A fuzzy topological space X is a fuzzy strong semi pre-T_s (or short $FSSPT_s$) if and only if every fuzzy point is a fuzzy strongly semi pre-closed set.*

By Theorem 2 it is obvious that any $FSSPT_s$ space is also an $FSSPT_1$. With the following example, we will show that the converse is not always true.

Example 2. *Given a set $X = \{p, q\}$, and fuzzy sets $U = \{(p, 1), (q, 0)\}$, $V = \{(p, 0), (q, 1)\}$, the fuzzy topological space $\tau = \{0, U, V, 1\}$ is $FSSPT_1$ but it is not $FSSPT_s$. It is obvious that every singleton is a fuzzy strongly semi pre-closed set, and the conclusion follows from Theorem 2.*

Example 3. *Given a set $X = \{p_1, p_2\}$, and fuzzy sets $U = \{(p_1, 0.6), (p_2, 0)\}$, $V = \{(p_1, 0.7), (p_2, 0)\}$ and $W = \{(p_1, 0.8), (p_2, 0.7)\}$. If $\tau = \{0, U, V, W, 1\}$. It can be shown that the fuzzy topological space (X, τ) is $FSSPT_0$ but it is not $FSSPT_1$ and $FSSPT_s$.*

Definition 14. *A fuzzy topological space X is a fuzzy strong semi pre-Hausdorff (or short $FSSPT_2$) if and only if for any pair of fuzzy points p_1 and p_2, which have different supports, there exist fuzzy strongly semi preo-pen sets O_1, O_2 such that $p_1 \leq O_1 \leq p_2^c$, $p_2 \leq O_2 \leq p_1^c$ and $O_1 \leq O_2^c$.*

Theorem 3. *The fuzzy topological space (X, τ) is an $FSSPT_2$ if and only if there exists a fuzzy strongly semi pre-open set O such that $p_1 \leq O \leq sspclO \leq p_2^c$, where p_1 and p_2 are any pair of fuzzy points from X that have different supports.*

Proof. If the fuzzy topological space (X, τ) is an $FSSPT_2$ then it is obvious that for any pair of fuzzy points p_1, p_2 there must exist a set $O \in FSSPO(\tau)$, $p_1 \leq O \leq p_2^c$, and a set $W \in FSSPO(\tau)$ such that $p_2 \leq W \leq p_1^c$ and $O \leq W^c$. It follows that $p_1 \leq O \leq sspclO \leq sspclW^c = W^c \leq p_2^c$.
Conversely, if we denote by $(sspclO)^c = W$, it is obvious that $p_2 \leq W$ and $W \in FSSPO(\tau)$. Now we have a case where $p_1 \leq O \leq p_2^c$, $p_2 \leq W \leq p_1^c$ and also $O \leq W^c$, meaning that (X, τ) is an $FSSPT_2$.

Example 4. *Given a set $X = \{p_1, p_2\}$, and fuzzy sets $U = \{(p_1, 0.6), (p_2, 0)\}$, $V = \{(p_1, 0), (p_2, 0.6)\}$ and $W = \{(p_1, 0.6), (p_2, 0.8)\}$. If $\tau = \{0, U, V, U \vee V, W, 1\}$, it can be shown that the fuzzy topological space (X, τ) is an $FSSPT_2$ and it is not and $FSSPT_s$. It is also obvious that this is an example of a fuzzy space that is not an FT_2 and not an $FSPT_2$ ([13]).*

Example 5. *Let X be an infinite set and let the family of fuzzy sets be defined as:*

$$\tau = \{G | suppG^c \text{ is a finite set}\}.$$

It is clear that (X, τ) is a fuzzy topological space and that each fuzzy point in τ is a fuzzy strongly semi pre-closed set. From the other perspective, it is impossible to find any pair of fuzzy points p_1, p_2 and fuzzy strongly semi pre-open sets O_1, O_2 such that $p_1 \leq O_1 \leq p_2^c$, $p_2 \leq O_2 \leq p_1^c$ and $O_1 \leq O_2^c$, because the last relation would imply that an infinite fuzzy set is the subset of a finite fuzzy set.
The first argument shows that the fuzzy topological space (X, τ) is an $FSSPT_s$ while the second arguments tells us that it is not an $FSSPT_2$ space.

In other words, we have illustrated with two last examples that the classes of $FSSPT_2$ spaces and $FSSPT_s$ spaces are independent.

Definition 15. *A fuzzy topological space X is a fuzzy strong semi pre-Urysohn (or short $FSSPT_{2\frac{1}{2}}$) if and only if for any pair of fuzzy points p_1 and p_2, which have different supports, there exist fuzzy strongly semi pre-open set O_1, O_2 such that $p_1 \leq O_1 \leq p_2^c$, $p_2 \leq O_2 \leq p_1^c$ and $sspclO_1 \leq (sspclO_2)^c$.*

According to the above definition, it is clear that any fuzzy strong semi pre-Urysohn space is also a fuzzy strong semi pre-Hausdorff space. With the following example, we will illustrate that the converse statement does not hold in general case.

Example 6. *Let X be an infinite set and let there p_0 be a fuzzy point with $x_0 \in X$ as its support. Let us define the family of fuzzy sets as follows:*

$$A = \{O | O(x_0) \leq p_0(x_0)\}$$
$$B = \{O | suppO^c \text{ is a finite set}\}$$

It is obvious that $\tau = A \vee B$ is a fuzzy topological space that is a case of an $FSSPT_2$ space which is not an $FSSPT_{2\frac{1}{2}}$.

Definition 16. *A fuzzy topological space X is a fuzzy strong semi pre-regular (or short $FSSPR$) if and only if for every fuzzy points p and every fuzzy strongly semi pre-closed set C in X such that $p \leq C^c$, there exist fuzzy strongly semi pre-open sets O_1, O_2 such that $p \leq O_1$, $C \leq O_2$ and $O_1 \leq O_2^c$.*

The definition of $FSSPR$ spaces can also be given in the equivalent form as it follows. The Fuzzy topological space (X, τ) is an $FSSPR$ space if and only if for every fuzzy point p and every fuzzy strongly semi pre-open set O such that $p \leq O$, there exists a fuzzy strongly semi pre-open set U such that $p \leq U \leq sspclU \leq O$.

Definition 17. *A fuzzy topological space X which is an $FSSPR$ and $FSSPT_s$ is called $FSSPT_3$.*

We can give also a different and weaker condition of fuzzy strong semi pre-regularity with the following definition.

Definition 18. *A fuzzy topological space X is a fuzzy strong semi pre-weakly regular (or short $FSSPWR$) if and only if for every fuzzy points p and every fuzzy closed set C in X such that $p \leq C^c$, there exist fuzzy strongly semi pre-open sets O_1, O_2 such that $p \leq O_1$, $C \leq O_2$ and $O_1 \leq O_2^c$.*

Theorem 4. *Let X be an $FSSPR$ space, then for every fuzzy strongly semi pre-closed set F in X and any fuzzy point $p \leq F^c$, there exist fuzzy strongly semi pre-open sets U, W such that $p \leq U$, $F \leq W$ and $sspclU \leq (sspclW)^c$.*

Proof. According to the statement of the theorem, for every fuzzy point $p \leq F^c$ and the fact that X is an $FSSPR$ space, there exist fuzzy strongly semi pre-open sets O, W such that $p \leq O$, $F \leq W$ and $O \leq W^c$. Also from the equivalent definition of $FSSPR$ spaces, for every fuzzy point p and a fuzzy strongly semi pre-open set O such that $p \leq O$, there exists a fuzzy strongly semi pre-open set U such that $p \leq U \leq sspclU \leq O$. Now the conclusion of the theorem is obvious.

Corollary 2. *Every $FSSPT_3$ space is an $FSSPT_{2\frac{1}{2}}$ space.*

Proof. It follows directly from the previous theorem and from the fact that an $FSSPT_3$ space is an $FSSPR$ and $FSSPT_s$ space.

Theorem 5. *Any fuzzy topological space* (X, τ) *which is an* $FSSPR$ *and* $FSSPT_0$ *space is also an* $FSSPT_{2\frac{1}{2}}$ *space.*

Proof. We are going to take into consideration two fuzzy points p_1 and p_2 with different support. Based on the assumption that (X, τ) is an $FSSPT_0$ space, it follows that there exists a set $U \in FSSPO(\tau)$ such that $p_1 \leq U \leq p_2^c$. If we denote by $F = U^c$, $F \in FSSPC(\tau)$, it is obvious that $p_1 \leq F^c$ and since (X, τ) is an $FSSPR$ space, it implies the existence of fuzzy sets $V, W \in FSSPO(\tau)$ such that $p_1 \leq V$, $F \leq W$, and $V \leq W^c$. Now, from the Theorem 4 we also have that $sspclV \leq (sspclW)^c$ and combining it with the fact that $p_2 \leq U^c = F \leq W$ and as well as $p_1 \leq V$, we reach the desired result. The latest conditions imply that for any given pair of fuzzy points p_1 and p_2 with different support, there exist fuzzy sets $V, W \in FSSPO(\tau)$ such that $p_1 \leq V \leq p_2^c$, $p_2 \leq W \leq p_1^c$ and $sspclV \leq (sspclW)^c$, hence the fuzzy topological space (X, τ) is $FSSPT_{2\frac{1}{2}}$.

Definition 19. *A fuzzy topological space X is a fuzzy strong semi pre-normal (or short* $FSSPN$*) if and only if for every pair of fuzzy strongly semi pre-closed sets* C_1, C_2*, such that* $C_1 \leq C_2^c$*, there exist fuzzy strongly semi pre-open sets* O_1, O_2 *such that* $C_1 \leq O_1$*,* $C_2 \leq O_2$ *and* $O_1 \leq O_2^c$*.*

A fuzzy topological space with $FSSPN$ and $FSSPT_s$ properties is called an $FSSPT_4$ space.
Clearly, any $FSSPT_4$ space is also an $FSSPT_3$ space.
We can formulate the following theorem which gives a necessary and sufficient condition for the existence of $FSSPN$ spaces.

Theorem 6. *The fuzzy topological space* (X, τ) *is an* $FSSPN$ *if and only if for any* $F \in FSSPC(\tau)$ *and a fuzzy set* $O \in FSSPO(\tau)$ *such that* $F \leq O$*, there exists a fuzzy set* $W \in FSSPO(\tau)$ *such that* $F \leq W \leq sspclW \leq O$*.*

Proof. We can use similar argumentation as in Theorem 4.

Definition 20. *A fuzzy topological space X is a fuzzy strong semi pre-weakly normal (or short* $FSSPWN$*) if and only if for every pair of fuzzy strongly semi pre-closed sets* C_1, C_2*, such that* $C_1 \wedge C_2 = \emptyset$*, there exist fuzzy strongly semi pre-open sets* O_1, O_2 *such that* $C_1 \leq O_1$*,* $C_2 \leq O_2$ *and* $O_1 \leq O_2^c$*.*

We can formulate the following theorem in regards to the $FSSPWN$ spaces.

Theorem 7. *Every fuzzy topological space* (X, τ) *which is an* $FSSPN$ *space is also an* $FSSPWN$ *space.*

Proof. In fuzzy topological spaces the following implication is always true:

$$C_1 \wedge C_2 = \emptyset \implies C_1 \leq C_2^c$$

In general, the equivalence is not always valid for fuzzy sets, this means that if the fuzzy topological space is $FSSPN$ then it is also an $FSSPWN$.

4. Axioms of fuzzy strong semi pre-separation and fuzzy strong semi pre-continuous mappings

In this section, we will investigate the relation between fuzzy separation axioms, fuzzy pre-separation axioms and different forms of fuzzy continuity.

Theorem 8. *Let $f : X \to Y$ be a fuzzy strong semi pre-continuous and injective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space Y is an FT_2 (FT_1, FT_0) space then X is an $FSSPT_2$ $(FSSPT_1, FSSPT_0)$ space.*

Proof. Let us suppose that fuzzy points $p, q \leq X$ represent any pair of fuzzy points with different support. According to the assumption of the theorem, the mapping $f : X \to Y$ is an injective mapping, it is obvious that $f(p), f(q)$ are two fuzzy points in Y with different support. Now, since the fuzzy topological space Y is an FT_2, there exist fuzzy open sets U, V such that:

$$f(p) \leq U \leq f(q)^c,\ f(q) \leq V \leq f(p)^c \text{ and } U \leq V^c.$$

Since the mapping f is a fuzzy strong semi pre-continuous mapping then $f^{-1}(U)$, $f^{-1}(V)$ are two fuzzy strongly semi pre-open sets in X such that:

$$p \leq f^{-1}(U) \leq q^c,\ q \leq f^{-1}(V) \leq p^c \text{ and also } f^{-1}(U) \leq f^{-1}(V)^c,$$

that is, the fuzzy topological space X is an $FSSPT_2$ space.
Similarly, we can prove the cases when Y is an FT_1 and FT_0 space.

Theorem 9. *Let $f : X \to Y$ be a fuzzy strong semi pre-open and bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space X is an FT_2 (FT_1, FT_0) space then Y is an $FSSPT_2$ $(FSSPT_1, FSSPT_0)$ space.*

Proof. Let us suppose that $p, q \leq Y$ are two fuzzy points with different support. It is obvious from the conditions of the theorem that $f^{-1}(p), f^{-1}(q) \leq X$ are two fuzzy points with different support. Since the fuzzy topological space X is an FT_2, there are fuzzy open sets U, V such that:

$$f^{-1}(p) \leq U \leq f^{-1}(q)^c,\ f^{-1}(q) \leq V \leq f^{-1}(p)^c \text{ and } U \leq V^c.$$

Based on the assumption of the theorem, the images $f(U), f(V)$ of U and V are fuzzy strongly semi pre-open sets in Y and the following stands:

$$p \leq f(U) \leq q^c,\ q \leq f(V) \leq p^c \text{ and } f(U) \leq f(V)^c,$$

which means that Y is an $FSSPT_2$ space.
In similar manner we can show that the same holds when X is an FT_1 and FT_0 space.

Theorem 10. *Let $f : X \to Y$ be a fuzzy strong semi pre-continuous and injective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space Y is an FT_s space then X is an $FSSPT_s$ space.*

Proof. Let p be any fuzzy point in the fuzzy topological space X then $f(p)$ is a fuzzy point in Y. Since Y is an FT_s space, it means that any fuzzy point is a fuzzy closed set, that is $f(p)$ is a fuzzy closed set in Y. Because f is a fuzzy strong semi pre-continuous mapping and it is an injective mapping then $f^{-1}(f(p)) = p$, and p is a fuzzy strongly semi pre-closed set in X. Since p is any fuzzy point of X, that means that the fuzzy topological space X is an $FSSPT_s$.

Theorem 11. *Let $f : X \to Y$ be a fuzzy strongly semi pre-open and bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space X is an FT_s space then Y is an $FSSPT_s$ space.*

Proof. Similar to Theorem 10.

Theorem 12. *Let $f : X \to Y$ be a fuzzy strong semi pre-continuous and injective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space Y is an $FT_{2\frac{1}{2}}$ space then X is an $FSSPT_{2\frac{1}{2}}$ space.*

Proof. Let us suppose that $p, q \leq X$ are two fuzzy points with different support. Since $f : X \to Y$ is a fuzzy strong semi pre-continuous and an injective mapping, it follows that $f(p), f(q) \leq Y$ are two fuzzy points with different support. Due to the fact that the fuzzy topological space Y is an $FT_{2\frac{1}{2}}$, there are fuzzy open sets U, V such that $f(p) \leq U \leq f(q)^c$, $f(q) \leq V \leq f(p)^c$ and $clU \leq (clV)^c$.
Based on the assumption of the theorem, the images $f^{-1}(U), f^{-1}(V)$, of U and V are fuzzy strongly semi pre-open sets in Y and $p \leq f^{-1}(U) \leq q^c$, $q \leq f^{-1}(V) \leq p^c$. According to the Theorem 4.1. [17][17] we have that:

$$sspclf^{-1}(U) \leq f^{-1}(clU) \leq f^{-1}(clV)^c \leq f^{-1}(intV^c) \leq sspintf^{-1}(V^c) \leq (sspclf^{-1}(V))^c$$

The last expression can also be summarized as $sspclf^{-1}(U) \leq (sspclf^{-1}(V))^c$ which means that X is an $FSSPT_{2\frac{1}{2}}$ space.

Theorem 13. *Let $f : X \to Y$ be a fuzzy strongly semi pre-open and bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space X is an $FT_{2\frac{1}{2}}$ space then Y is an $FSSPT_{2\frac{1}{2}}$ space.*

Proof. Let us suppose that $p, q \leq Y$ are two fuzzy points with different support. It is obvious from the conditions of the theorem that $f^{-1}(p), f^{-1}(q) \leq X$ are two fuzzy points with different support. Since the fuzzy topological space X is an $FT_{2\frac{1}{2}}$, there are fuzzy open sets U, V such that $f^{-1}(p) \leq U \leq f^{-1}(q)^c$, $f^{-1}(q) \leq V \leq f^{-1}(p)^c$ and also $clU \leq (clV)^c$. Based on the assumption of the theorem, the images $f(U), f(V)$ of U and V are fuzzy strongly semi pre-open sets in Y and the following stands:

$$p \leq f(U) \leq q^c, \, q \leq f(V) \leq p^c$$

and

$$sspclf(U) \leq f(clU) \leq f(clV)^c \leq (sspclf(V))^c$$

which means that Y is an $FSSPT_{2\frac{1}{2}}$ space.

Theorem 14. *Let $f : X \to Y$ be a fuzzy closed and fuzzy strong semi pre-continuous and bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space Y is an FR space then X is an $FSSPWR$ space.*

Proof. Proof: Let X be a fuzzy topological space and let p be a fuzzy point, let F be any fuzzy closed set in X such that $p \leq F^c$. Then, $f(p)$ is a fuzzy point in Y and according to the conditions of the theorem $f(p) \leq f(F)^c$. It is obvious that the fuzzy set $f(F)$ is a fuzzy closed set in Y. Since Y is an FR space, then there exist fuzzy open sets U, V such that $f(p) \leq U$, $f(F) \leq V$ and $U \leq V^c$. If we refer again to the conditions of the theorem, then we have:

$$p \leq f^{-1}(U),\ F \leq f^{-1}(V) \text{ and } f^{-1}(U) \leq f^{-1}(V)^c.$$

It is obvious that $f^{-1}(U)$ and $f^{-1}(V)$ are fuzzy strongly semi pre-open sets in X. Hence the fuzzy topological space X is an $FSSPWR$ space.

Theorem 15. *Theorem Let $f : X \to Y$ be a fuzzy continuous and fuzzy strongly semi pre-open and bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space X is an FR space then Y is an $FSSPWR$ space.*

Proof. Similar to Theorem 14.

Theorem 16. *Let $f : X \to Y$ be a fuzzy $SSPO$-irresolute and injective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space Y is an $FSSPT_{2\frac{1}{2}}$ ($FSSPT_2, FSSPT_s, FSSPT_1, FSSPT_0$) space then X is also an $FSSPT_{2\frac{1}{2}}$ ($FSSPT_2, FSSPT_s, FSSPT_1, FSSPT_0$) space.*

Proof. Let us suppose that fuzzy points $p, q \leq X$ represent any pair of fuzzy points with different support. According to the assumption of the theorem, the mapping $f : X \to Y$ is an injective mapping, it is obvious that $f(p), f(q)$ are two fuzzy points in Y with different support. Now, since the fuzzy topological space Y is an $FSSPT_{2\frac{1}{2}}$, there exist fuzzy strongly semi pre-open sets U, V such that:

$$f(p) \leq U \leq f(q)^c,\ f(q) \leq V \leq f(p)^c \text{ and } sspclU \leq (sspclV)^c.$$

Since the mapping f is a fuzzy $SSPO$-irresolute, it follows that $f^{-1}(U), f^{-1}(V)$ are two fuzzy strongly semi pre-open sets in X such that:

$$p \leq f^{-1}(U) \leq q^c,\ q \leq f^{-1}(V) \leq p^c.$$

From the conditions set out by Theorem 1 in [18], we can prove that:

$$sspclf^{-1}(U) \leq f^{-1}(sspclU) \leq f^{-1}((sspclV)^c) = f^{-1}(sspintV^c) \leq$$
$$\leq sspint(f^{-1}(V^c)) = (sspcl(f^{-1}(V))^c.$$

We have shown that for the given fuzzy strongly semi pre-open sets $f^{-1}(U), f^{-1}(V)$ it follows that $sspclf^{-1}(U) \leq (sspcl(f^{-1}(V))^c$, which means that the fuzzy topological space X is an $FSSPT_{2\frac{1}{2}}$ space.

In the similar way we can prove the cases when Y is an $FSSPT_2, FSSPT_s, FSSPT_1, FSSPT_0$ space.

Theorem 17. *Let $f : X \to Y$ be a fuzzy SSPO-irresolute open and bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space X is an $FSSPT_{2\frac{1}{2}}$ ($FSSPT_2, FSSPT_s, FSSPT_1, FSSPT_0$) space then Y is also an $FSSPT_{2\frac{1}{2}}$ ($FSSPT_2, FSSPT_s, FSSPT_1, FSSPT_0$) space.*

Proof. Let us show only the case when the fuzzy topological space X is an $FSSPT_s$ space. Other cases are proved in similar manner (similar to Theorem 16).
Let $q \leq Y$ be any fuzzy point of Y. The preimage of this point satisfies the following $f^{-1}(q) \leq X$. Based on the conditions of the theorem and from the fact that X is an $FSSPT_s$ space, that is, any fuzzy point $f^{-1}(q) \leq X$ is a fuzzy strongly semi pre-closed set in X. It follows that the image $f(f^{-1}(q)) = q \leq Y$ of the fuzzy point $f^{-1}(q)$ is also a fuzzy strongly semi pre-closed set in Y. In other words any fuzzy point of Y is also a fuzzy strongly semi pre-closed set and therefore Y is also an $FSSPT_s$ space.

Theorem 18. *Let $f : X \to Y$ be a fuzzy SSPO-irresolute closed and fuzzy strong semi pre-continuous bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space Y is an $FSSPN$ ($FSSPWN, FSSPT_3, FSSPR$) space then X is also an $FSSPN$ ($FSSPWN, FSSPT_3, FSSPR$) space.*

Proof. Let F_1, F_2 be two fuzzy strongly semi pre-closed sets in X such that $F_1 \leq F_2^c$. Obviously, due to the conditions of the theorem, $f(F_1), f(F_2)$ are two fuzzy strongly semi pre-closed sets in Y such that $f(F_1) \leq (f(F_2))^c$. Since Y is an $FSSPN$ space, there are fuzzy strongly semi pre-open sets W_1, W_2 such that:

$$f(F_1) \leq W_1,\ f(F_2) \leq W_2 \text{ and } W_1 \leq W_2^c.$$

From the assumption that f is a fuzzy strong semi pre-continuous mapping, it follows that $f^{-1}(W_1)$ and $f^{-1}(W_2)$ are two fuzzy strongly semi pre-open sets in X and the following stands:

$$F_1 \leq f^{-1}(W_1),\ F_2 \leq f^{-1}(W_2) \text{ and } f^{-1}(W_1) \leq (f^{-1}(W_2))^c$$

Therefore the fuzzy topological space X is an $FSSPN$ space.
Similarly, we can prove the other cases when Y is an $FSSPWN, FSSPT_3$ and $FSSPR$ space.

Theorem 19. *Let $f : X \to Y$ be a fuzzy SSPO-irresolute open and fuzzy strong semi pre-continuous bijective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy topological space X is an $FSSPN$ ($FSSPWN, FSSPT_3, FSSPR$) space then Y is also an $FSSPN$ ($FSSPWN, FSSPT_3, FSSPR$) space.*

Proof. Let V_1, V_2 be two fuzzy strongly semi pre-closed sets in Y such that $V_1 \leq V_2^c$. From the assumptions of the theorem, since f is a fuzzy strong semi pre-continuous mapping, it follows that $f^{-1}(V_1)$, $f^{-1}(V_2)$ are two fuzzy strongly semi pre-closed sets in X such that $f^{-1}(V_1) \leq f^{-1}(V_1)^c$. Now, since X is an $FSSPN$ space, there are fuzzy strongly semi pre-open sets U_1, U_2 such that:

$$f^{-1}(V_1) \leq U_1,\ f^{-1}(V_2) \leq U_2 \text{ and } U_1 \leq U_2^c.$$

From the assumption that f is a fuzzy $SSPO$-irresolute open mapping, it follows that $f(U_1)$ and $f(U_2)$ are fuzzy strongly semi pre-open sets in Y and the following conditions are fulfilled:

$$V_1 \leq f(U_1),\ V_2 \leq f(U_2) \text{ and } f(U_1) \leq f(U_2)^c.$$

Hence the fuzzy topological space Y is an $FSSPN$ space.

In similar way we can prove the other cases when the fuzzy topological space X is an $FSSPWN, FSSPT_3$, and $FSSPR$ space.

5. A novel form of fuzzy compactness

In this section, we will introduce a novel form of compactness in fuzzy topological spaces. The properties of this new form of fuzzy compactness, similarities, and differences with other forms of fuzzy compactness will also be investigated. We will initially give the following definition.

Definition 21. *Let (X, τ) be a fuzzy topological space and let $\alpha \in [0, 1]$. A collection $\mathcal{S}$ of fuzzy strongly semi pre-open sets of (X, τ) is called an $\alpha - SSPO$ shading (respectively $\alpha^* - SSPO$ shading) of the fuzzy set A if, for every $a \in suppA$, there exist a set $W \in \mathcal{S}$ such that $W(a) > \alpha$ (respectively $W(a) \geq \alpha$). A subcollection $\mathcal{C}$ of sets from $\alpha - SSPO$ shading (respectively $\alpha^* - SSPO$ shading) $\mathcal{S}$ which is also an $\alpha - SSPO$ shading (respectively $\alpha^* - SSPO$ shading) for the given fuzzy set A, is called an $\alpha - SSPO$ subshading (respectively $\alpha^* - SSPO$ subshading) of the collection $\mathcal{S}$.*

With the concept of $\alpha - SSPO$ shading ($\alpha^* - SSPO$ shading), which are analogous to the concept of open covers in the ordinary topology, we can define the concept of $\alpha - SSPO$ compactness.

Definition 22. *The fuzzy set A of the fuzzy topological space (X, τ) is called $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) if every $\alpha - SSPO$ shading (respectively $\alpha^* - SSPO$ shading) of the set A has a finite $\alpha - SSPO$ subshading (respectively $\alpha^* - SSPO$ subshading). If instead of any set A we consider the set X in general, then we can state that space (X, τ) is $\alpha - SSPO$ compact ($\alpha^* - SSPO compact$).*

Definition 23. *The fuzzy set A of the fuzzy topological space (X,τ) is called countable $\alpha - SSPO$ compact (respectively countable $\alpha^* - SSPO$ compact) if every countable $\alpha - SSPO$ shading (respectively $\alpha^* - SSPO$ shading) of the set A has a finite $\alpha - SSPO$ subshading (respectively $\alpha^* - SSPO$ subshading).*
If instead of the fuzzy set A we consider the space X then we can state that the fuzzy topological space X is a countable $\alpha - SSPO$ compact (respectively countable $\alpha^* - SSPO$ compact).

From definition 23 it is obvious that if the fuzzy topological space X is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then it is also countable $\alpha - SSPO$ compact (countable $\alpha^* - SSPO$ compact).
Directly from the definition, we can conclude that any fuzzy point is $\alpha - SSPO$ compact set and $\alpha^* - SSPO$ compact set. It is also obvious that any fuzzy set in X is $1 - SSPO$ compact and $0^* - SSPO$ compact. Also from the definition 22 it follows that any $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) space is also an α- compact (α^*-compact) space. The converse is not always true as it can be presented with the following example.

Example 7. *If X is any infinite set and if $\alpha \in [0,1]$, for any $p \in X$ we will define the following sets:*

$$U_p^\alpha(x) = \begin{cases} 1 & \text{if } x = p \\ \alpha & \text{if } x \neq p \end{cases}$$

Let as denote with $\mathcal{T}_\alpha$ the fuzzy topology on X which is generated by $\{U_p^\alpha(x) : p \in X\}$. In [5] it was shown that $(X,\mathcal{T}_\alpha)$ is β-compact for $\beta = 1$ or $0 \leq \beta < \alpha$ and is β^*-compact for $0 \leq \beta \leq \alpha$. It is obvious that $(X,\mathcal{T}_\alpha)$ is $\beta - SSPO$ compact only for $\beta = 1$ or $0 \leq \beta < \alpha$ and is $\beta^* - SSPO$ compact for $0 \leq \beta \leq \alpha$. Moreover, if $\gamma < \alpha$, then $(X,\mathcal{T}_\gamma)$ is α-compact and α^*-compact, see [5], but it is neither $\alpha - SSPO$ compact nor $\alpha^* - SSPO$ compact.

Theorem 20. *The fuzzy topological space (X,τ) is $\alpha - SSPO$ compact (respectively $\alpha^* - SSPO$ compact) if and only if for every α-centered (α^*-centered) family $\mathcal{F}$ consisting of fuzzy strongly semi pre-closed sets in (X,τ), there exists $x \in X$ such that $F(x) \geq 1 - \alpha$ ($F(x) > 1 - \alpha$), for every $F \in \mathcal{F}$.*

Proof. Let us suppose that $\mathcal{F}$ is an α-centered family consisting of fuzzy strongly semi pre-closed sets in (X,τ) such that for each $x \in X$, there exists a set $F \in \mathcal{F}$ such that $F(x) < 1-\alpha$. Then the family of sets $W = \{F^c, F \in \mathcal{F}\}$ is an $\alpha - SSPO$ shading of (X,τ) and it is evident that it does not have a finite $\alpha - SSPO$ subshading. If it had a finite $\alpha - SSPO$ subshading $F_1^c, F_2^c, \ldots, F_k^c$, then due to the fact that $\mathcal{F}$ is α-centered there exists $x \in X$, such that $F_j(x) \geq 1 - \alpha$ for all $j = 1, 2, \ldots, k$ and consequently $F_j^c(x) \leq \alpha$ for all $j = 1, 2, \ldots, k$.
Conversely, let us suppose that family $\mathcal{S}$ of fuzzy strongly semi pre-open sets of (X,τ) is an α-SSPO shading of X and that it has no finite α-SSPO subshading. Then the collection of fuzzy strongly semi pre-closed sets $\mathcal{F} = \{S^c, S \in \mathcal{S}\}$ is α-centered because for

$S_1^c, S_2^c, \ldots, S_k^c \in \mathcal{F}$ there must exist $x \in X$ such that $S_j(x) \leq \alpha$ for all $j = 1, 2, \ldots, k$ (or otherwise the family $\mathcal{S}$ has a finite $\alpha - SSPO$ subshading), and therefore $S_j^c(x) \geq 1 - \alpha$ for all $j = 1, 2, \ldots, k$. On the other side, given any $x \in X$ there exists $S \in \mathcal{S}$ such that $S(x) > \alpha$ and consequently $S^c \in \mathcal{F}$ and as well $S^c(x) < 1 - \alpha$.
In the same manner we can prove the case when the fuzzy topological space (X, τ) is $\alpha^* - SSPO$ compact.

Corollary 3. *The fuzzy topological space (X, τ) is $\alpha - SSPO$ compact (respectively $\alpha^* - SSPO$ compact) if and only if for every α-centered (α^*-centered) family $\mathcal{F}$ consisting of fuzzy sets in (X, τ), there exists $x \in X$ such that $sspclF(x) \geq 1 - \alpha$ ($sspclF(x) > 1 - \alpha$), for every $F \in \mathcal{F}$.*

Proof. Follows directly from Theorem 20.

Theorem 21. *The fuzzy topological space (X, τ) is countable $\alpha - SSPO$ compact (respectively countable $\alpha^* - SSPO$ compact) if and only if for every countable α-centered (countable α^*-centered) family $\mathcal{F}$ consisting of fuzzy strongly semi pre-closed sets in (X, τ), there exists $x \in X$ such that $F(x) \geq 1 - \alpha$ ($F(x) > 1 - \alpha$), for every $F \in \mathcal{F}$.*

Proof. Similar to Theorem 20

Theorem 22. *Let A be an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in (X, τ) and let $B \in FSSPC(\tau)$, then the fuzzy set $A \wedge B$ is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in the fuzzy topological space (X, τ).*

Proof. Let us suppose that $\mathcal{U} = \{U_i, i \in I\}$ is an $\alpha - SSPO$ shading of the fuzzy set $A \wedge B$. It follows that the collection of sets $\{U_i, i \in I\} \vee B^c$ is an $\alpha - SSPO$ shading of the fuzzy set A. The last is true because if $a \in suppA$ then $a \in supp(A \wedge B)$ or $B(a) = 0$. If $a \in supp(A \wedge B)$ then there exists $U_j \in \mathcal{U}$ such that $U_j(a) > \alpha$, otherwise, if $B(a) = 0$ then $B^c(a) = 1 > \alpha$. In other words the collection $\{U_i, i \in I\} \vee B^c$ is an $\alpha - SSPO$-shading of the fuzzy set A and since A is $\alpha - SSPO$ compact, there exists a finite $\alpha - SSPO$ subshading $\{U_i, i = 1, 2, \ldots, k\} \vee B^c$. It is evident that $\{U_i, i = 1, 2, \ldots, k\}$ is a finite $\alpha - SSPO$ subshading of $A \wedge B$, that is $A \wedge B$ is $\alpha - SSPO$ compact.

In similar way we can show the case when the fuzzy set A is $\alpha^* - SSPO$ compact.

Corollary 4. *Let X be an $\alpha - SSPO$-compact ($\alpha^* - SSPO$ compact) fuzzy topological space, then any fuzzy set $B \in FSSPC(\tau)$ is an $\alpha - SSPO$-compact ($\alpha^* - SSPO$-compact) fuzzy set in the fuzzy topological space (X, τ).*

Proof. It is obvious, from Theorem 22, if we substitute the fuzzy $\alpha - SSPO$-compact set A with X.

Theorem 23. *Let A, B be $\alpha - SSPO$-compact ($\alpha^* - SSPO$ compact) fuzzy sets in (X, τ), then the fuzzy set $A \vee B$ is also an $\alpha - SSPO$-compact ($\alpha^* - SSPO$ compact) fuzzy set in the fuzzy topological space (X, τ).*

Proof. Let us suppose that $\{W_i, i \in I\}$ is an $\alpha - SSPO$ shading of the fuzzy set $A \vee B$. It follows that $\{W_i, i \in I\}$ is also an $\alpha - SSPO$ shading of the fuzzy sets A and B. According to the assumption of the theorem, A and B are two $\alpha - SSPO$ compact fuzzy sets in (X, τ), therefore there exists a finite $\alpha - SSPO$ subshading $W_{i_1}, W_{i_2}, ..., W_{i_k}$ of A as well as a finite $\alpha - SSPO$ subshading $W_{j_1}, W_{j_2}, ..., W_{j_m}$ of B. Now if we consider the finite collection of sets $W_{i_1}, W_{i_2}, ..., W_{i_k}, W_{j_1}, W_{j_2}, ..., W_{j_m}$ it is obvious that it consists a finite $\alpha - SSPO$ subshading of $\{W_i, i \in I\}$ and obviously $A \vee B$ is an $\alpha - SSPO$ compact fuzzy set in (X, τ). The latter is true since for any $x \in supp(A \vee B) = x \in (suppA \cup suppB)$, there exists $W_x \in \{W_{i_1}, W_{i_2}, ..., W_{i_k}, W_{j_1}, W_{j_2}, ..., W_{j_m}\}$ such that $W_x(x) > \alpha$.

In similar way we can show the case when the fuzzy sets A, B are $\alpha^* - SSPO$ compact.

Corollary 5. *Let A be a fuzzy set in fuzzy topological space (X, τ). If the fuzzy set A has a finite support then A is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in (X, τ).*

Corollary 6. *Let X be a finite fuzzy topological space, then X is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in (X, τ).*

Theorem 24. *If $f : X \to Y$ is a fuzzy SSPO-irresolute mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy set A is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in X then $f(A)$ is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in Y.*

Proof. Let us suppose that $\mathcal{U} = \{U_i, i \in I\}$ is an $\alpha - SSPO$ shading of the fuzzy set $f(A)$ in Y. Then the family $W = \{f^{-1}(U_i), U_i \in \mathcal{U}\}$ is a collection of fuzzy strongly semi pre-open sets of X. Since for every $x \in suppA$ we have that $f(x) \in f(suppA) = suppf(A)$ and since $\{U_i, i \in I\}$ is an $\alpha - SSPO$ shading of $f(A)$, there exists $U_j \in \mathcal{U}$ such that $U_j(f(x)) > \alpha$ and subsequently $f^{-1}(U_j)(x) = U_j(f(x)) > \alpha$, which means that the family W is an $\alpha - SSPO$ shading of A. Since A is $\alpha - SSPO$ compact it follows that W contains a finite $\alpha - SSPO$ subshading, $\{f^{-1}(U_i), i \in J\}$, where J is a finite set of indexes. Therefore we can conclude that the finite collection $\{U_i, i \in J\}$ is an $\alpha - SSPO$ subshading of the $\alpha - SSPO$ shading $\mathcal{U}$. This is true due to the fact that for every $y \in suppf(A)$ there exists $x \in suppA$ such that $f(x) = y$. Now, since $\{f^{-1}(U_i), i \in J\}$ is a finite $\alpha - SSPO$ subshading of A, there exists $m \in J$ such that $f^{-1}(U_m)(x) > \alpha$ and therefore $f^{-1}(U_m)(x) = U_m(f(x)) = U_m(y) > \alpha$. We showed that $f(A)$ is an $\alpha - SSPO$ compact set in Y.
The other case is proven in a similar way.

Corollary 7. *If $f : X \to Y$ is a fuzzy SSPO-irresolute mapping from the fuzzy topological space X to a fuzzy topological space Y. If X is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then $f(X)$ is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set in Y.*

Proof. It follows directly from Theorem 24.

Corollary 8. *Let $f : X \to Y$ is a fuzzy SSPO-irresolute and surjective mapping from the fuzzy topological space X to a fuzzy topological space Y. If X is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then Y is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) fuzzy set.*

Proof. It follows from Corollary 7 and since $f(X) = Y$ when f is a surjective mapping.

Theorem 25. *Let $f : X \to Y$ be a fuzzy strong semi pre-continuous and surjective mapping from the fuzzy topological space X to a fuzzy topological space Y. If X is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then Y is an α-compact (α^*-compact).*

Proof. Let us suppose that $\mathcal{U} = \{U_i, i \in I\}$ is an α-shading of Y. Then the family $W = \{f^{-1}(U_i), U_i \in \mathcal{U}\}$ is a collection of fuzzy strongly semi pre-open sets of X. Since for every $x \in X$ we have that $f(x) \in f(suppX) = suppY$ and since $\{U_i, i \in I\}$ is an α-shading of Y there exists a fuzzy open set $U_j \in \mathcal{U}$ such that $U_j(f(x)) > \alpha$ and subsequently $f^{-1}(U_j)(x) = U_j(f(x)) > \alpha$, which means that the family W is an $\alpha-SSPO$ shading of X. Since X is $\alpha - SSPO$ compact it follows that W contains a finite $\alpha - SSPO$ subshading, $\{f^{-1}(U_i), i \in K\}$, where K is a finite set of indexes. Therefore we can conclude that the finite collection $\{f(f^{-1}(U_i) = U_i, i \in K\}$ is an α-subshading of the α-shading $\mathcal{U}$. The last stands because f is a surjective mapping and due to the fact that for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Now, since $\{f^{-1}(U_i), i \in K\}$ is a finite $\alpha - SSPO$ subshading of X, there exists $m \in K$ such that $f^{-1}(U_m)(x) > \alpha$ and therefore $f^{-1}(U_m)(x) = U_m(f(x)) = U_m(y) > \alpha$. We showed that for any α-shading $\mathcal{U} = \{U_i, i \in I\}$ of Y there exists a finite α-subshading $\{U_i, i \in K\}$ and hence Y is an α-compact set. The proof of the case when X is $\alpha^* - SSPO$ compact is similar and is therefore omitted.

Theorem 26. *Let the mapping $f : X \to Y$ be a fuzzy SSPO homeomorphism from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy set A is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then $f(A)$ is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact).*

Proof. Similar to Theorem 25.

Corollary 9. *Let the mapping $f : X \to Y$ be a fuzzy SSPO homeomorphism from the fuzzy topological space X to a fuzzy topological space Y. If X is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then Y is an $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact).*

Theorem 27. *Let the mapping $f : X \to Y$ be a fuzzy SSPO homeomorphism from the fuzzy topological space X to a fuzzy topological space Y. If X is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then Y is an α-compact (α^*-compact).*

Proof. It follows immediately from Theorem 25.

Corollary 10. *Let the mapping $f : X \to Y$ be a fuzzy SSPO homeomorphism from the fuzzy topological space X to a fuzzy topological space Y. If Y is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) then X is an α-compact (α^*-compact).*

Proof. It follows immediately from Theorem 27. and from the fact that $f^{-1}: Y \to X$ is also a fuzzy $SSPO$ homeomorphism.

Definition 24. *The family B of fuzzy strongly semi pre-open sets of the fuzzy topological space (X, τ) is called a base of fuzzy strongly semi pre-open sets in (X, τ) if every fuzzy strongly semi pre-open set of (X, τ) can be written as union of members of B.*

Theorem 28. *If the fuzzy topological space (X, τ) has a countable base of fuzzy strongly semi pre-open sets then any fuzzy set A in (X, τ) is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) if and only if it is countable $\alpha - SSPO$ compact (countable $\alpha^* - SSPO$ compact).*

Proof. It is certain that every $\alpha - SSPO$ compact set in the fuzzy topological space (X, τ) is also a countable $\alpha - SSPO$ compact fuzzy set.
Conversely, let us suppose that the fuzzy set A in (X, τ) is countable $\alpha - SSPO$ compact. Let us suppose that the family of fuzzy strongly semi pre-open sets $\mathcal{U} = \{U_i, i \in I\}$ is an $\alpha - SSPO$ shading of A. Since (X, τ) has a countable base $B = \{W_i, i \in \mathbb{N}\}$ of fuzzy strongly semi pre-open sets W_i, then any fuzzy strongly semi pre-open set can be represented as union of sets from B. Now let $a \in suppA$, there exists $U_j \in \mathcal{U}$, for some $j \in I$, such that $U_j(a) > \alpha$. There are fuzzy strongly semi pre-open sets $W_{i_k}), k = 1, 2, \ldots, m$ (note that m must not be a finite number) from B such that $U_j = \vee_{k=1}^{m} W_{i_k}$. The fact that $U_j(a) > \alpha$ implies the existence of $W_{i_s}, i_s \in \{1, 2, \ldots, m\}$ such that $W_{i_s}(a) > \alpha$. We can now claim that the family of fuzzy sets $B^0 = \{W_{i_k}, k = 1, 2, \ldots, m\}$ is a countable $\alpha - SSPO$ shading of A in (X, τ). Since, from our assumption, A is countable $\alpha - SSPO$ compact, there exist a finite $\alpha - SSPO$ subshading $B^1 \leq B^0$. If we consider the finite collection of fuzzy strongly semi pre-open sets:

$$U^0 = \{U_i : W_{i_s} \leq U_i, W_{i_s} \in B^0\}$$

It is obvious that U^0 is a finite $\alpha - SSPO$ subshading of U and as consequence A is $\alpha - SSPO$ compact.
In similar way we can prove the case when A is $\alpha^* - SSPO$ compact.

Theorem 29. *Let the mapping $f : X \to Y$ be a fuzzy strong semi pre-continuous and surjective fuzzy SSPO-irresolute open mapping from the fuzzy topological space X to a fuzzy topological space Y. If the space X has a countable base consisting of fuzzy strongly semi pre-open sets then Y also has a countable base consisting of fuzzy strongly semi pre-open sets.*

Proof. Let us suppose that $\mathcal{B} = \{B_i, i \in \mathbb{N}\}$ is a base for fuzzy strongly semi pre-open sets of X. Based on the assumption of the theorem it follows that $f(B_i), \forall i \in \mathbb{N}$, are fuzzy strongly semi pre-open sets in Y. If we consider the collection of fuzzy sets $\mathcal{M} = \{f(B_i), i \in \mathbb{N}\}$, and given any fuzzy strongly semi pre-open set W in Y, then again due to the conditions of the theorem, $f^{-1}(W)$ is a fuzzy strongly semi pre-open set in X and it can be written in the following manner $f^{-1}(W) = \vee_{i \in \mathbb{N}} B_i$.
From the fact that mapping f is surjective, we get $W = f(f^{-1}(W)) = f(\vee_{i \in \mathbb{N}} B_i) = \vee_{i \in \mathbb{N}} f(B_i)$. Therefore $\mathcal{M}$ is a base of fuzzy strongly semi pre-open sets in Y.

Definition 25. *Fuzzy topological space* (X, τ) *is fuzzy* $SSPO$*-separable if and only if there exists a countable sequence of fuzzy points* $\{p_i\}_{i \in \mathbb{N}}$ *such that for each* $U \in FSSPO(\tau), U \neq 0_X$*, there exists a fuzzy point* p_j *such that* $p_j \in U$*, for some* $j \in \mathbb{N}$*.*

It is obvious that the concept of fuzzy $SSPO$-separability is the generalization of fuzzy separability. If a fuzzy topological space is fuzzy $SSPO$-separable then it is also fuzzy separable.

Theorem 30. *If the fuzzy topological space* (X, τ) *has a countable base* B *of fuzzy strongly semi pre-open sets then* (X, τ) *is an* $SSPO$*-separable space.*

Proof. Let us suppose that $\mathcal{B} = \{B_i, i \in \mathbb{N}\}$ is a countable base for fuzzy strongly semi pre-open sets in (X, τ). Let us consider any member of $\mathcal{B}$, let it be denoted as B_j, such that $B_j \neq 0_X$, then there exists a fuzzy point $x_j \in X$ such that $B_j(x_j) > 0$. If we now define a fuzzy point as follows:

$$\begin{cases} p_j(x) = B_j(x_j) & \text{if } x = x_j \\ p_j(x) = 0 & \text{if } x \neq x_j \end{cases}$$

We can conclude that $p_j \leq B_j$. Let us consider the corresponding countable sequence of fuzzy points $\{p_i\}_{i \in \mathbb{N}}$. Given any fuzzy strongly semi pre-open set U in (X, τ), it must contain a certain $B_s \in \mathcal{B}$ and therefore there exists a fuzzy point $p_s \leq B_s$ such that $p_s \leq U$. In other words X is $SSPO$-separable space.

The converse of this theorem does not stand. Let X be an infinite set, and let $fts(X, \tau)$ be such that any fuzzy open set in τ contains a fuzzy singleton $p \in X$ (or a countable set of singletons). In this case (X, τ) does not contain a fuzzy countable base consisting of fuzzy open sets and it does not contain a countable base of fuzzy strongly semi pre-open sets.

Theorem 31. *Let the mapping* $f : X \to Y$ *be a fuzzy* $SSPO$*-irresolute and surjective mapping from the fuzzy topological space* X *to a fuzzy topological space* Y*. If the space* X *is fuzzy* $SSPO$*-separable then* Y *is a fuzzy* $SSPO$*-separable space.*

Proof. Let us consider a countable sequence of fuzzy points $\{p_i\}_{i \in \mathbb{N}}$ from X such that for any fuzzy strongly semi pre-open set W in X, $W \neq 0_X$, there exists a fuzzy point p_i such that $p_i \leq W$. The sequence $\{f(p_i)\}_{i \in \mathbb{N}}$ is a countable sequence of fuzzy points in Y. Let us suppose that V is a fuzzy strongly semi pre-open set in Y such that $V \neq 0_Y$. From the assumption of the theorem $f^{-1}(V)$ is a fuzzy strongly semi pre-open set in X and $f^{-1}(V) \neq 0_X$. Because the space X is fuzzy $SSPO$-separable then there exists a fuzzy point p_s such that $p_s \leq f^{-1}(V)$. Now $f(p_s) \leq f(f^{-1}(V)) = V$, and we have shown that $\{f(p_i)\}_{i \in \mathbb{N}}$ is a countable sequence of fuzzy points in Y such that for any fuzzy strongly semi pre-open set V in Y , $V \neq 0_Y$, there exists a fuzzy point $f(p_s)$ such that $p_s \leq V$, that is Y is a fuzzy $SSPO$-separable space.

Theorem 32. *Let the mapping $f : X \to Y$ be a fuzzy strong semi pre-continuous and surjective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the space X is fuzzy $SSPO$-separable then the space Y is fuzzy separable space.*

Proof. Similar to Theorem 31

Theorem 33. *Let the mapping $f : X \to Y$ be a fuzzy $SSPO$ homeomorphism from the fuzzy topological space X to a fuzzy topological space Y. If the space X is $SSPO$-separable then Y will also be $SSPO$-separable space.*

Proof. Let us consider a countable sequence of fuzzy points $\{p_i\}_{i\in\mathbb{N}}$ from X such that for any fuzzy strongly semi pre-open set W in X, $W \neq 0_X$, there exists a fuzzy point p_i such that $p_i \leq W$. Then the sequence $\{f(p_i)\}_{i\in\mathbb{N}}$ is a countable sequence of fuzzy points in Y. Let us suppose that V is a fuzzy strongly semi pre-open set in Y such that $V \neq 0_Y$. From the assumption of the theorem $f^{-1}(V)$ is a fuzzy strongly semi pre-open set in X and $f^{-1}(V) \neq 0_X$. Due to the fact that the space X is fuzzy $SSPO$-separable then there exists a fuzzy point p_s such that $p_s \leq f^{-1}(V)$. Now $f(p_s) \leq f(f^{-1}(V)) = V$ and we have shown that $\{f(p_i)\}_{i\in\mathbb{N}}$ is a countable sequence satisfying the conditions of definition 25, therefore Y is a fuzzy $SSPO$-separable space.

Definition 26. *The fuzzy set A of the fuzzy topological space (X, τ) is called $\alpha - SSPO$ Lindelof (respectively $\alpha^* - SSPO$ Lindelof) if every $\alpha - SSPO$ shading ($\alpha^* - SSPO$ shading) of the set A has a countable $\alpha - SSPO$ subshading (countable $\alpha^* - SSPO$ subshading).*
If instead of the fuzzy set A we consider space X then we can state that the fuzzy topological space X is $\alpha - SSPO$ Lindelof (respectively $\alpha^* - SSPO$ Lindelof).

It is certain that from the above definition we can conclude that every $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact) space is also an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) space.
Every $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) space is an α- Lindelof (α^*-Lindelof) space.

Theorem 34. *If the fuzzy topological space (X, τ) has a countable base B of fuzzy strongly semi pre-open sets then (X, τ) is an $\alpha - SSPO$ Lindelof (respectively $\alpha^* - SSPO$ Lindelof) space.*

Proof. Similar to Theorem 28.

Theorem 35. *Let the fuzzy set A of the fuzzy topological space (X, τ) be an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) set. The fuzzy set A is countable $\alpha - SSPO$ compact (countable $\alpha^* - SSPO$ compact) if and only if A is $\alpha - SSPO$ compact ($\alpha^* - SSPO$ compact).*

Proof. It is certain that every $\alpha - SSPO$ compact set in the fuzzy topological space (X, τ) is also a countable$\alpha - SSPO$ compact fuzzy set.
Conversely, let us suppose that the fuzzy set A in (X, τ) is a countable $\alpha - SSPO$ compact set. Let us suppose that the family of fuzzy strongly semi pre-open sets $\mathcal{U} = \{U_i, i \in I\}$ is an $\alpha - SSPO$ shading of A. Since the fuzzy set A of (X, τ) is an $\alpha - SSPO$ Lindelof then there exists a countable $\alpha - SSPO$ subshading V_1 of $\alpha - SSPO$ shading $\mathcal{U}$. Based on the assumption that the fuzzy set A is countable $\alpha - SSPO$ compact set, then there exists a finite $\alpha - SSPO$ subshading V_2 of $\alpha - SSPO$ shading $\mathcal{U}$. It is clear that given an $\alpha - SSPO$ shading $\mathcal{U}$ of the fuzzy set A there exists a finite $\alpha - SSPO$ subshading V_2 of $\mathcal{U}$ and subsequently the fuzzy set A is $\alpha - SSPO$ compact.

In similar way we can prove the case when A is an $\alpha^* - SSPO$ Lindelof set.

Theorem 36. *Let A be an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in (X, τ) and let $B \in FSSPC(\tau)$, then $A \wedge B$ is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in the fuzzy topological space (X, τ).*

Proof. Similar to Theorem 22.

Corollary 11. *Let X be an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) space, then any fuzzy set $B \in FSSPC(\tau)$ is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in the fuzzy topological space (X, τ).*

Proof. It follows directly from Theorem 36.

Theorem 37. *Let A, B be $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy sets in (X, τ), then the fuzzy set $A \vee B$ is also an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in (X, τ).*

Proof. In similar way as Theorem 23.

Theorem 38. *If $f : X \to Y$ is a fuzzy SSPO-irresolute mapping from the fuzzy topological space X to a fuzzy topological space Y. If the fuzzy set A is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in X then $f(A)$ is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in Y.*

Proof. In a similar way as Theorem 24.

Corollary 12. *If $f : X \to Y$ is a fuzzy SSPO-irresolute mapping from the fuzzy topological space X to a fuzzy topological space Y. If X is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) then $f(X)$ is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in Y.*

Proof. It follows directly from Theorem 38.

Corollary 13. *Let $f : X \to Y$ is a fuzzy SSPO-irresolute and surjective mapping from the fuzzy topological space X to a fuzzy topological space Y. If X is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) then Y is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof).*

Proof. It follows from Theorem 38.

We can also show that the following assertion are true.

Theorem 39. *Let the mapping $f : X \to Y$ be a fuzzy strong semi pre-continuous and surjective mapping from the fuzzy topological space X to a fuzzy topological space Y. If the space X is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) then Y will be an α- Lindelof (α^*- Lindelof) space.*

Theorem 40. *Let the mapping $f : X \to Y$ be a fuzzy SSPO homeomorphism from the fuzzy topological space X to a fuzzy topological space Y and let A be a fuzzy set in X. If the set A is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) set in X then $f(A)$ will also be an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) fuzzy set in Y.*

Corollary 14. *Let the mapping $f : X \to Y$ be a fuzzy SSPO homeomorphism from the fuzzy topological space X to a fuzzy topological space$Y$. If X is an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof) then Y will also be an $\alpha - SSPO$ Lindelof ($\alpha^* - SSPO$ Lindelof).*

6. Conclusion

In this paper we have investigated properties of a new form of fuzzy pre-separation axioms as well as the new form of fuzzy compactness induced by the new class of fuzzy generalized opened sets. We also investigated their properties in regards to the fuzzy strong semi pre-continuous functions as well as the fuzzy $SSPO$-irresolute mappings. We have shown that the concept of fuzzy strong pre-separation axioms is stronger than the ordinary fuzzy separation axioms. From the properties that we investigated, the concept of $\alpha - SSPO$ Lindelof space, fuzzy $SSPO$-separability and the existence of a base consisting of fuzzy strongly semi pre-open sets, the strongest concept appears to be the concept of the existence of a base consisting of fuzzy strongly semi pre-open sets.

Our future work will be focused on introducing new form of fuzzy connectedness which will be stronger than the concepts of fuzzy connectedness introduced by other authors. Based on this work, we also intent to introduce the new concept of generalized open sets in the Intuitionistic Fuzzy Topological spaces.

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