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A force function formula for solutions of nonlinear weakly singular Volterra integral equations (WSVIE)

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Abstract. In this paper, we examine the nonlinear Weakly Singular Volterra Integral Equation (WSVIE), $u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} [u(t)]^{\beta} dt$. Al-Jawary and Shehan used the Daftardar-Jafari Method (DJM) and solved the above integral equation for the investigation parameter $\mu > 1$ using specific force functions with μ and β values and obtained unique solutions. We have discovered a force function $f(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu}$ that allows the introduction of noise terms phenomena discovered by Wazwaz that cancel out the terms of the power series in the successive solution terms u_m, $m = 0, 1, 2, ..., n$: we thus obtain a maximum finite power series terms for each solution term called truncation point and denoted by $x^{g(n)}$. Such that the integral solution can be written as $u(x) = u_0 + \sum_{m=1}^{n} u_m$, where n is finite. Simplifying the solution terms, we get the unique solution $u(x) = x^{k_1}$, irrespective of the $n-$value in the truncation point. We have discovered a formula relation between the last solution term u_n and the truncation point as $u_n = a_n x^{g(n)}$. Our results confirm the results of the two solution examples of AL-Jawary and Shehan for the investigation parameter $\mu > 1$. We extend the parameter range to include $\mu > 1$ and $0 < \mu \leq 1$ for our solution. In addition, for any chosen rational parameter k_1, the solution $u(x) = x^{k_1}$ is extrapolated to be valid for all integer parameter values $\beta \geq 2$ and positive rational parameter values $\mu > 0$ and for any finite value of $n \geq 2$.

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1. Introduction

Mathematical problems can be formulated using integral equations. Mathematical formulations of physical problems as differential equations are converted to integral equations[12] or physical problems are formulated as integral transforms using appropriate kernels[29]. Some of the kernels of integral equations found in the literature are the logarithmic kernel, $K(x,t) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\ln(\frac{t}{s})}}(\frac{s}{t})^{\mu}\frac{1}{s}$, Abel's kernel, $K(x,t) = \frac{1}{(x-t)^{\frac{1}{2}}}$, difference kernel, $K(x,t) = (x-t)$, and the reproducing kernel, $K(x,t) = \frac{t^{\mu-1}}{x^{\mu}}$. While a singular integral equation has infinite limits as well as the kernel being undefined at one or two points in the range of integration, a weaker singularity occurs when only the kernel is undefined at some points[17].

The weakly singular Volterra integral equation,

$$u(x) = f(x) + \int_0^x K(x,t)[u(t)]^{\beta}dt \tag{1}$$

is linear if $\beta = 1$ and nonlinear otherwise, has various applications of scientific problems including stereology[23], heat conduction with mixed boundary conditions[13], crystal formation, electrochemistry, superfluidity, and the radiation of heat from a semi-infinite solid state [11].

Various numerical and analytic methods have been used to solve both linear and nonlinear Volterra integral equations. These include the interpolation approach[8, 9], the optimal homotopy asymptotic method[19], the Riemann-Liouville fractional operator[27], the extrapolation technique[22], the Adomian Decomposition Method(ADM)[2] and the Variational Iteration Method(VIM)[24].

In [14], the authors stated the existence, uniqueness, and singularity properties of linear WSVIE for the case when

1. $0 < \mu \leq 1$, if $0 < \mu < 1$, the kernel is singular at $x = 0$ and $t = 0$ for all values of $t > 0$. Equation(1) has an infinite set of solutions, but if $\mu = 1$, then the kernel has a singularity only at $x = 0$, and in this case, when $f \epsilon C^1[0, X]$ with $f(0) = 1$, equation(1) has an infinite set of solutions in C[0,X], which contains only one particular solution belonging to $C^1[0, X]$.

2. When $\mu > 1$, the kernel has a singularity only at $x = 0$, and equation (1) is said to have a unique solution in $C^m[0, X]$, $f \epsilon C^m[0, X]$[17]

In solving the WSVIE, authors mostly use a specific force function or force function formula in their solution. In[2, 15, 18, 30], the authors applied the Adomian Decomposition Method (ADM) for solving linear and nonlinear weakly singular Volterra integral equations with the reproducing kernel. The Homotopy method was used in[7, 16, 20, 21, 26], to solve linear and nolinear Volterra integral equations using specific force functions. In [5, 28, 29, 31], the authors used the Variational Iteration Method (VIM) to solve linear and nonlinear Volterra integral equations using specific force functions. Applying the Series Solution Method (SSM) given in [18, 29, 31], the authors solved the linear WSVIE and obtained an exact solution. In[6], the Modified Adomian Decomposition Method (MADM) introduced by Wazwaz[29] to accelerate the convergence of the ADM was implemented. Daftardar-Jafari discovered the iterative method[12], popularly known as DJM, for solving general functional equations, including nonlinear Volterra integral equations, algebraic equations and systems of ordinary differential equations, nonlinear algebraic equations, and fractional differential equations. In [1, 3, 4, 25], the authors solved various problems using DJM.

In [18], the authors used a force function formula instead of a specific force function to obtain unique solutions for linear WSVIE. In [29],the authors discovered noise term phenomena such that terms in series solutions cancelled out to give an exact solution in a finite number of solution terms. The noise term phenomenon was reinforced in [32].

To the best of our knowledge, only AL-Jawary and Shehan [4] have implemented the DJM to solve both linear and nonlinear WSVIE while using the reproducing kernel $K(x,t) = \frac{t^{\mu-1}}{x^{\mu}}$ in

$$u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}}[u(t)]^{\beta} dt. \tag{2}$$

In [4], the authors provided a limited solution to the WSVIE of equation (1) using two specific force functions, $f(x) = x^{\frac{1}{2}} - \frac{5}{11}x$ and $f(x) = x - \frac{2}{9}x^3$ with specific parameter values of β =2 and 3.

In this paper, our solution is also based on the method of DJM[12], wherein we introduce a force function formula in line with Hasan and Mohammed[18]. We use the force function formula to expand the specific integral values of β =2 and 3 in AL-Jawary and Shehan[4] to $\beta \geq 2$. The force function formula introduces cancellation of terms in the integral series solution to facilitate a unique solution, as discussed in [32] as a noise term phenomenon. We have derived a truncation point formula to augment the force function to minimise length computation. In addition, we have provided solution models to facilitate solution examples. Finally, we have extended the investigation parameter $\mu > 1$ of [4] to $0 < \mu \leq 1$, which the existing literature has not considered.

The paper is organised as follows: In Section 2, the authors provided the Banach space assumptions for the solutions of the nonlinear WSVIE using DJM. In the same section, the authors introduced a force function formula to expand the DJM for solutions of the nonlinear WSVIE. The authors then introduced a truncation point formula that relates the last solution term and derived solution models in sections 2.2.1, 2.2.2, 2.2.3, and 2.2.4. In Sections 3 and 4, the solution models were used to compute solution examples for various parameter values of $\beta \geq 2$, k_1, and μ being rational. In Section 5, results were displayed using tables and summarized. Discussion of the results was done in Section 6 and ended with a conclusion in Section 7.

2. Daftardar-Jafari Method(DJM) for nonlinear WSVIE with reproducing kernel

Following the Daftardar-Jafari Method given in [12], let f, u be in Banach Space B, then the nonlinear WSVIE of equation (1), represented in operator form, is expressed as:

$$u = f + N(u), \tag{3}$$

is in Banach Space B, such that $B \longmapsto B$ with the operator N being,

$$u = N(u) = \int_0^x K(x,t)u[(t)]^\beta dt. \tag{4}$$

The solution of equation (3) can be represented in series form:

$$u = \sum_{n=0}^{\infty} u_n. \tag{5}$$

The decomposition of the nonlinear operator N yields

$$N\left(\sum_{n=0}^{\infty} u_n\right) = N(u_0) + \sum_{n=1}^{\infty}\left\{N\left(\sum_{j=0}^{n} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right\}. \tag{6}$$

From eqns. (5) and (6), eqn. (3) is equivalent to

$$\sum_{n=0}^{\infty} u_n = f + N(u_0) + \sum_{n=1}^{\infty}\left\{N\left(\sum_{j=0}^{n} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right\}. \tag{7}$$

The recurrence relation is defined as:

$$\begin{aligned} &u_0 = f, \\ &u_1 = N(u_0), \\ &u_{n+1} = N\{(u_0 + \ldots + u_n)\} - (u_1 + \ldots + u_{n-1}), n = 1, 2, \ldots && (8) \\ &u = f + \sum_{n=1}^{\infty} u_n. && (9) \end{aligned}$$

2.1. Implementation of the DJM and the force function formula

In this section, we present a solution approach using our new force function formula for the nonlinear WSVIE, leading to a unique solution.
Let us consider the general form of the weakly singular Volterra integral equation in[10]

$$u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^\mu}[u(t)]^\beta dt \tag{10}$$

where

$$u_0(x) = f(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1}, \tag{11}$$

is the force function formula.

$$u_1(x) = N[u_0(t)]$$
$$= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1}\right]^\beta dt. \tag{12}$$
$$u_2(x) = N[u_0(x) + u_1(x)] - N[u_0(x)]$$
$$= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + u_1(t)\right]^\beta dt - u_1. \tag{13}$$
$$u_3(x) = N[u_0(x) + u_1(x) + u_2(x)] - N[u_0(x) + u_1(x)]$$
$$= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + (u_1 + u_2)(t)\right]^\beta dt - \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + u_1(t)\right]^\beta dt. \tag{14}$$

In continuing from equation(14), we generate successive solution terms as:

$$\begin{aligned}
u_0(x) &= f(x),\\
u_1(x) &= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1}\right]^\beta dt,\\
u_2(x) &= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + u_1(t)\right]^\beta dt - \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1}\right]^\beta dt\\
u_3(x) &= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + (u_1 + u_2)t\right]^\beta dt - \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + u_1(t)\right]^\beta dt\\
&\vdots\\
u_m(x) &= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + ... + u_{m-1}(t)\right]^\beta dt - \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + ... + u_{m-2}(t)\right]^\beta dt,\\
&\vdots \qquad (15)\\
u_n(x) &= \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + ... + u_{n-1}(t)\right]^\beta dt - \int_0^x \frac{t^{\mu-1}}{x^\mu}\left[t^{k_1} - \frac{t^{\gamma k_1}}{\mu + \gamma k_1} + ... + u_{n-2}(t)\right]^\beta dt,\\
u(x) &= x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \sum_{m=1}^{n} u_m. \qquad (16)
\end{aligned}$$

which reduces to a unique solution, $u(x) = x^{k_1}$, for every integer value $\gamma = \beta$, $(\beta \geq 2)$, positive rational values of k_1 and $\mu > 0$, and u_n is a finite solution term and is related to the truncation point by the relation $u_n = a_n x^{g(n)}$.

Thus, the force function formula, $f(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu+\gamma k_1}$, generates noise term cancellation to obtain a unique solution when the truncation point is introduced. The relation between the truncation point $x^{g(n)}$ and u_n is given by:

$$u_n(x) = a_n x^{[n(\gamma-1)+1]k_1}, \quad n \geq 2, \tag{17}$$

where,

$$a_n = \frac{\beta^{n-1}}{(\gamma k_1 + \mu)\prod_{m=2}^{n}\Big[[m(\gamma-1)+1]k_1 + \mu\Big]}. \tag{18}$$

For example, when $\beta = \gamma = 2$, $k_1 = 3$, and $\mu = \frac{3}{2}$, if $n = 2$, then the final series solution term is $u_2(x) = \frac{8}{315}x^9$. If $n = 3$, then the final series solution term is $u_3(x) = \frac{16}{8505}x^{12}$, and so on. Equation (17) was discovered through the solution process.

2.2. Solution Models for the nonlinear WSVIE

Based on the solution series of equation (15) in section (2.1) and the subsequent truncation point formula in equations (17) and (18), we provide explicit algebraic solutions for the nonlinear WSVIE for values of $\beta = 2$, 3, 4, and 5 and show that for each case we obtain the unique solution for our force function.

2.2.1. Solution model for 2nd -order nonlinear WSVIE ($\gamma = \beta = 2$)

Consider the 2nd order WSVIE given by:

$$u(x) = a_1(x)^{k_1} - a_2(x)^{\gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^\mu}u(t)^\beta dt, \quad \gamma = \beta = 2, \mu > 0 \quad \text{and} \quad k_1 \in Q^+$$

Truncation point is given by

$u_n(x) = a_n x^{[n(\gamma-1)+1]k_1}, \quad n \geq 2$

$$u_0(x) = f(x) = a_1(x)^{k_1} - a_2(x)^{\gamma k_1} = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1}$$

where

$$a_1 = \mu - (\mu - 1) = 1, \quad a_2 = \frac{1}{\mu + \gamma k_1}$$

$$u_1 = \int_0^x \frac{t^{\mu-1}}{x^\mu}(u_0)(t)^\beta dt$$

$$= \frac{x^{\beta k_1}}{\beta k_1 + \mu} - \beta a_2 \frac{x^{k_1+\gamma k_1}}{k_1 + \gamma k_1 + \mu} + a_2^\beta \frac{x^{\beta\gamma k_1}}{\beta\gamma k_1 + \mu}$$

$$u_2 = \int_0^x \frac{t^{\mu-1}}{x^\mu}(u_0 + u_1)(t)^\beta dt - u_1$$

$$= \beta a_2 \frac{x^{k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)} - a_2^{\beta} \frac{x^{\beta\gamma k_1}}{\beta\gamma k_1+\mu} - 2\beta a_2 \frac{x^{2k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)}$$

$$+2a_2^{\beta} \frac{x^{k_1+\beta\gamma k_1}}{(\beta\gamma k_1+\mu)(k_1+\beta\gamma k_1+\mu)} + \beta^2 a_2^2 \frac{x^{2k_1+2\gamma k_1}}{(k_1+\gamma k_1+\mu)^2(2k_1+2\gamma k_1+\mu)}$$

$$-2\beta a_2^{\beta+1} \frac{x^{k_1+3\gamma k_1}}{((k_1+\gamma k_1+\mu)(\beta\gamma k_1+\mu)(k_1+3\gamma k_1+\mu)} + \ldots$$

$$u_3(x) = \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0+u_1+u_2)(t)^{\beta} dt - \sum_{i=0}^{2} u_i$$

$$= 2\beta a_2 \frac{x^{2k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)} - 2a_2^{\beta} \frac{x^{k_1+\beta\gamma k_1}}{(\beta\gamma k_1+\mu)(k_1+\beta\gamma k_1+\mu)}$$

$$-4\beta a_2 \frac{x^{3k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)(3k_1+\gamma k_1+\mu)}$$

$$-\beta^2 a_2^2 \frac{x^{2k_1+2\gamma k_1}}{(k_1+\gamma k_1+\mu)^2(2k_1+2\gamma k_1+\mu)}$$

$$+4a_2^{\beta} \frac{x^{2k_1+\beta\gamma k_1}}{(\beta\gamma k_1+\mu)(k_1+\beta\gamma k_1+\mu)(2k_1+\beta\gamma k_1+\mu}$$

$$+2\beta a_2^{\beta+1} \frac{x^{2k_1+\gamma k_1+\beta\gamma k_1}}{(k_1+\gamma k_1+\mu)(\beta\gamma k_1+\mu)(k_1+\gamma k_1+\beta\gamma+\mu)}$$

$$+2\beta^2 a_2^2 \frac{x^{3k_1+2\gamma k_1}}{(k_1+\gamma k_1+\mu)^2(2k_1+\beta\gamma k_1+\mu)(3k_1+2\gamma k_1+\mu)}$$

$$u_4(x) = \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0+u_1+u_2+u_3)(t)^{\beta} dt - \sum_{i=1}^{3} u_i$$

$$= 4\beta a_2 \frac{x^{3k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)(3k_1+\gamma k_1+\mu)}$$

$$-4a_2^{\beta} \frac{x^{2k_1+\beta\gamma k_1}}{(\beta\gamma k_1+\mu)(k_1+\beta\gamma k_1+\mu)(2k_1+\beta\gamma k_1+\mu}$$

$$-8\beta a_2 \frac{x^{4k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)(3\gamma k_1+\gamma k_1+\mu)(4k_1+\gamma k_1+\mu)}$$

$$-2\beta^2 a_2^2 \frac{x^{3k_1+2\gamma k_1}}{(k_1+\gamma k_1+\mu)^2(2k_1+\beta\gamma k_1+\mu)(3k_1+2\gamma k_1+\mu)}$$

$$+8a_2^{\beta} \frac{x^{3k_1+\beta\gamma k_1}}{(\beta\gamma k_1+\mu)(k_1+\beta\gamma k_1+\mu)(2k_1+\beta\gamma k_1+\mu)+(3k_1+\beta\gamma k_1+\mu)}$$

$$u_5(x) = \int_0^x \frac{t^{\mu-1}}{x^{\mu}}(u_0+u_1+u_2+u_3+u_4)(t)^{\beta} dt - \sum_{i=1}^{4} u_i$$

$$= 8\beta a_2 \frac{x^{4k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)(3\gamma k_1+\gamma k_1+\mu)(4k_1+\gamma k_1+\mu)}$$

$$-8a_2^{\beta}\frac{x^{3k_1+\beta\gamma k_1}}{(\beta\gamma k_1+\mu)(k_1+\beta\gamma k_1+\mu)(2k_1+\beta\gamma k_1+\mu)(3k_1+\beta\gamma k_1+\mu)}$$
$$-16\beta a_2\left[\left(\frac{x^{5k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)(3k_1+\gamma k_1+\mu)}\times\right.\right.$$
$$\left.\left.\frac{1}{(4k_1+\gamma k_1+\mu)(5k_1+\gamma k_1+\mu)}\right)\right]$$

The series solutions reduces to

$$u_6(x)=\int_0^x\frac{t^{\mu-1}}{x^{\mu}}(u_0+u_1+u_2+u_3+u_4+u_5(x))(t)^{\beta}dt-\sum_{i=1}^{5}u_i$$
$$=16\beta a_2\left[\left(\frac{x^{5k_1+\gamma k_1}}{(k_1+\gamma k_1+\mu)(2k_1+\gamma k_1+\mu)(3k_1+\gamma k_1+\mu)}\times\right.\right.$$
$$\left.\left.\frac{1}{(4k_1+\gamma k_1+\mu)(5k_1+\gamma k_1+\mu)}\right)\right]+...$$
$$u(x)=\sum_{i=0}^{6}u_i$$
$$\therefore u(x)=x^{k_1}$$

2.2.2. Solution model for 3rd -order nonlinear WSVIE($\gamma=\beta=3$)

Consider the 3rd order WSVIE of the form:

$$u(x)=a_1(x)^{k_1}-a_2(x)^{\gamma k_1}+\int_0^x-\frac{t^{\mu-1}}{x^{\mu}}u(t)^{\beta}dt,\quad \gamma=\beta=3,\mu>0\quad\text{and}\quad k_1\in Q^+\tag{19}$$

Truncation point is given by

$$u_n(x)=a_nx^{[n(\gamma-1)+1]k_1},\quad n\geq 2$$
$$u_0(x)=f(x)=a_1(x)^{k_1}-a_2(x)^{\gamma k_1}=x^{k_1}-\frac{x^{\gamma k_1}}{\mu+\gamma k_1}$$

where,

$$a_1=\mu-(\mu-1),\quad a_2=\frac{1}{\mu+\gamma k_1}$$
$$u_1(x)=\int_0^x\frac{t^{\mu-1}}{x^{\mu}}(u_0)(t)^{\beta}dt$$
$$=\frac{x^{3k_1}}{3k_1+\mu}-3a_2\frac{x^{2k_1+\gamma k_1}}{2k_1+\gamma k_1+\mu}+3a_2^2\frac{x^{k_1+2\gamma k_1}}{k_1+2\gamma k_1+\mu}-a_2^3\frac{x^{3\gamma k_1}}{3\gamma k_1+\mu}$$
$$u_2(x)=\int_0^x\frac{t^{\mu-1}}{x^{\mu}}(u_0+u_1)(t)^{\beta}dt-u_1$$
$$=3a_2\frac{x^{2k_1+\gamma k_1}}{2k_1+\gamma k_1+\mu}+3a_2^2\frac{x^{k_1+2\gamma k_1}}{k_1+2\gamma k_1+\mu}+9a_2\frac{x^{4k_1+\gamma k_1}}{(2k_1+\gamma k_1+\mu)(4k_1+\gamma k_1+\mu)}$$

$$-9a_2^2\frac{x^{3k_1+2\gamma k_1}}{(k_1+2\gamma k_1+\mu)(3k_1+2\gamma k_1+\mu)}+a_2^3\frac{x^{3\gamma k_1}}{3\gamma k_1+\mu}$$

$$u_3(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2)(t)^\beta dt-\sum_{i=0}^{2}u_i$$

$$=9a_2\frac{x^{4k_1+\gamma k_1}}{(2k_1+\gamma k_1+\mu)(4k_1+\gamma k_1+\mu)}-9a_2^2\frac{x^{3k_1+2\gamma k_1}}{(k_1+2\gamma k_1+\mu)(3k_1+2\gamma k_1+\mu)}$$

$$-27a_2\frac{x^{6k_1+\gamma k_1}}{(2k_1+\gamma k_1+\mu)(4k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)}$$

$$u_4(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2+u_3)(t)^\beta dt-\sum_{i=1}^{3}u_i$$

$$=27a_2\frac{x^{6k_1+\gamma k_1}}{(2k_1+\gamma k_1+\mu)(4k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)}$$

$$u(x)=\sum_{i=0}^{4}u_i$$

$$\therefore u(x)=x^{k_1}$$

2.2.3. Solution model for 4th -order nonlinear WSVIE$(\gamma=\beta=4)$

Consider the 4th order WSVIE of the form:

$$u(x)=a_1(x)^{k_1}+a_2(x)^{\gamma k_1}+\int_0^x-\frac{t^{\mu-1}}{x^\mu}u(t)^\beta dt,\gamma=\beta=3,\mu>0\quad\text{and}\quad k_1\in Q^{+}\tag{20}$$

Truncation point is given by,

$$u_n(x)=a_nx^{[n(\gamma-1)+1]k_1},n\geq 2$$

$$u_0(x)=f(x)=a_1(x)^{k_1}-a_2(x)^{\gamma k_1}=x^{k_1}-\frac{x^{\gamma k_1}}{\mu+\gamma k_1}$$

where,

$$a_1=\mu-(\mu-1),\quad a_2=\frac{1}{\mu+\gamma k_1}$$

$$u_1=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0)(t)^\beta dt$$

$$=\frac{x^{4k_1}}{4k_1+\mu}-4a_2\frac{x^{3k_1+\gamma k_1}}{3k_1+\gamma k_1+\mu}+6a_2^2\frac{x^{2k_1+2\gamma k_1(10)}}{2k_1+2\gamma k_1+\mu}-4a_2^3\frac{x^{k_1+3\gamma k_1}}{k_1+3\gamma k_1+\mu}$$

$$+a_2^4\frac{x^{4\gamma k_1}}{4\gamma k_1+\mu}$$

$$u_2=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1)(t)^\beta dt-u_1$$

$$=4a_2\frac{x^{3k_1+\gamma k_1}}{3k_1+\gamma k_1+\mu}-6a_2^2\frac{x^{2k_1+2\gamma k_1}}{2k_1+2\gamma k_1+\mu}$$

$$
\begin{aligned}
&-16a_2\frac{x^{6k_1+\gamma k_1}}{(3k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)}+4a_2^3\frac{x^{k_1+3\gamma k_1}}{(k_1+3\gamma k_1+\mu)}\\
&+24a_2^3\frac{x^{5k_1+2\gamma k_1}}{(2k_1+2\gamma k_1+\mu)(5k_1+2\gamma k_1+\mu)}-a_2^4\frac{x^{4\gamma k_1}}{4\gamma k_1+\mu}\\
&-16a_2^3\frac{x^{4k_1+3\gamma k_1}}{(k_1+3\gamma k_1+\mu)(4k_1+3\gamma k_1+\mu)}+96a_2^2\frac{x^{8k_1+2\gamma k_1}}{(3k_1+\gamma k_1+\mu)^2(8k_1+2\gamma k_1+\mu)} \qquad (21)\\
&u_3(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2)(t)^\beta dt-\sum_{i=0}^{2}u_i\\
&=16a_2\frac{x^{6k_1+\gamma k_1}}{(3k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)}-24a_2^3\frac{x^{5k_1+2\gamma k_1}}{(2k_1+2\gamma k_1+\mu)(5k_1+2\gamma k_1+\mu)}\\
&\quad-64a_2\frac{x^{9k_1+\gamma k_1}}{(3k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)(9k_1+\gamma k_1+\mu)}\\
&\quad+16a_2^3\frac{x^{4k_1+3\gamma k_1}}{(k_1+3\gamma k_1+\mu)(4k_1+3\gamma k_1+\mu)}\\
&\quad+96a_2^2\frac{x^{8k_1+2\gamma k_1}}{(2k_1+2\gamma k_1+\mu)(5k_1+2\gamma k_1+\mu)(8k_1+2\gamma k_1+\mu)}\\
&-96a_2^2\frac{x^{8k_1+2\gamma k_1}}{(3k_1+\gamma k_1+\mu)^2(8k_1+2\gamma k_1+\mu)}\\
&u_4(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2+u_3)(t)^\beta dt-\sum_{i=1}^{3}u_i\\
&=64a_2\frac{x^{9k_1+\gamma k_1}}{(3k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)(9k_1+\gamma k_1+\mu)}\\
&\quad-96a_2^2\frac{x^{8k_1+2\gamma k_1}}{(2k_1+2\gamma k_1+\mu)(5k_1+2\gamma k_1+\mu)(8k_1+2\gamma k_1+\mu)}\\
&-256a_2\frac{x^{12k_1+\gamma k_1}}{(3k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)(9k_1+\gamma k_1+\mu)(12k_1+\gamma k_1+\mu)}\\
&u_5(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2+u_3+u_4)(t)^\beta dt-\sum_{i=1}^{4}u_i\\
&-256a_2\frac{x^{12k_1+\gamma k_1}}{(3k_1+\gamma k_1+\mu)(6k_1+\gamma k_1+\mu)(9k_1+\gamma k_1+\mu)(12k_1+\gamma k_1+\mu)}\\
&u(x)=\sum_{i=0}^{5}u_i\\
&\therefore u(x)=x^{k_1}
\end{aligned}
$$

$$(22)$$

2.2.4. Solution model for 5th -order nonlinear WSVIE$(\gamma = \beta = 5)$

Consider the general forcing function for a unique solution of WSVIE given by:

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^\mu} u^\beta(t)dt\gamma = \beta = 3, \mu > 0 \quad \text{and} \quad k_1 \in Q^+ \tag{23}$$

Truncation point is given by,

$u_n(x) = a_n x^{[n(\gamma-1)+1]k_1}, \quad n \geq 2$

$$u_0(x) = f(x) = a_1(x)^{k_1} + a_2(x)^{\gamma k_1} = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1}$$

where,

$$a_1 = \mu - (\mu - 1), \quad a_2 = \frac{1}{\mu + \gamma k_1}$$

$$u_1(x) = \int_0^x \frac{t^{\mu-1}}{x^\mu}(u_0)^\beta(t)dt$$

$$= \frac{x^{5k_1}}{5k_1 + \mu} - 5a_2 \frac{x^{4k_1+\gamma k_1}}{4k_1 + \gamma k_1 + \mu} + 10a_2^2 \frac{x^{3k_1+2\gamma k_1}}{3k_1 + 2\gamma k_1 + \mu}$$

$$-a_2^3 \frac{x^{2k_1+3\gamma k_1}}{2k_1 + 3\gamma k_1 + \mu} + 5a_2^4 \frac{x^{k_1+4\gamma k_1}}{k_1 + 4\gamma k_1 + \mu} + ...$$

$$u_2(x) = \int_0^x \frac{t^{\mu-1}}{x^\mu}(u_0 + u_1)(t)^\beta dt - u_1$$

$$= 5a_2 \frac{x^{4k_1+\gamma k_1}}{4k_1 + \gamma k_1 + \mu} - 10a_2^2 \frac{x^{3k_1+2\gamma k_1}}{3k_1 + 2\gamma k_1 + \mu}$$

$$-25a_2 \frac{x^{8k_1+\gamma k_1}}{(4k_1 + \gamma k_1 + \mu)(8k_1 + \gamma k_1 + \mu)} + 10a_2^3 \frac{x^{2k_1+3\gamma k_1}}{2k_1 + 3\gamma k_1 + \mu}$$

$$+50a_2^2 \frac{x^{7k_1+2\gamma k_1}}{(3k_1 + 2\gamma k_1 + \mu)(7k_1 + 2\gamma k_1 + \mu)} - 5a_2^4 \frac{x^{k_1+4\gamma k_1}}{k_1 + 4\gamma k_1 + \mu}$$

$$-50a_2^3 \frac{x^{6k_1+3\gamma k_1}}{(2k_1 + 3\gamma k_1 + \mu)(6k_1 + 3\gamma k_1 + \mu)}$$

$$+250a_2^2 \frac{x^{11k_1+2\gamma k_1}}{(4k_1 + \gamma k_1 + \mu)^2(11k_1 + 2\gamma k_1 + \mu)}$$

$$u_3(x) = \int_0^x \frac{t^{\mu-1}}{x^\mu}(u_0 + u_1 + u_2)(t)^\beta dt - \sum_{i=1}^{2} u_i$$

$$= 25a_2 \frac{x^{8k_1+\gamma k_1}}{(4k_1 + \gamma k_1 + \mu)(8k_1 + \gamma k_1 + \mu)}$$

$$-50a_2^2 \frac{x^{7k_1+2\gamma k_1}}{(3k_1 + 2\gamma k_1 + \mu)(7k_1 + 2\gamma k_1 + \mu)}$$

$$-125a_2 \frac{x^{12k_1+\gamma k_1}}{(4k_1 + \gamma k_1 + \mu)(8k_1 + \gamma k_1 + \mu)(12k_1 + \gamma k_1 + \mu)}$$

$$
\begin{aligned}
&+50a_2^3\frac{x^{6k_1+3\gamma k_1}}{(2k_1+3\gamma k_1+\mu)(6k_1+3\gamma k_1+\mu)}\\
&-250a_2^2\frac{x^{11k_1+2\gamma k_1}}{(4k_1+\gamma k_1+\mu)^2(11k_1+2\gamma k_1+\mu)}\\
&+250a_2^2\frac{x^{11k_1+2\gamma k_1}}{(3k_1+2\gamma k_1+\mu)(7k_1+2\gamma k_1+\mu)(11k_1+2\gamma k_1+\mu)}\\
&u_4(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2+u_3)(t)^\beta dt-\sum_{i=1}^{3}u_i\\
&=125a_2\frac{x^{12k_1+\gamma k_1}}{(4k_1+\gamma k_1+\mu)(8k_1+\gamma k_1+\mu)(12k_1+\gamma k_1+\mu)}\\
&-250a_2^2\frac{x^{11k_1+2\gamma k_1}}{(3k_1+2\gamma k_1+\mu)(7k_1+2\gamma k_1+\mu)(11k_1+2\gamma k_1+\mu)}\\
&-625a_2\frac{x^{16k_1+\gamma k_1}}{(4k_1+\gamma k_1+\mu)(8k_1+\gamma k_1+\mu)(12k_1+\gamma k_1+\mu)(16k_1+\gamma k_1+\mu)}\\
&u_5(x)=\int_0^x\frac{t^{\mu-1}}{x^\mu}(u_0+u_1+u_2+u_3+u_4)(t)^\beta dt-\sum_{i=1}^{4}u_i\\
&=625a_2\frac{x^{16k_1+\gamma k_1}}{(4k_1+\gamma k_1+\mu)(8k_1+\gamma k_1+\mu)(12k_1+\gamma k_1+\mu)(16k_1+\gamma k_1+\mu)}\\
&u(x)=\sum_{i=0}^{5}u_i\\
&\therefore u(x)=x^{k_1}
\end{aligned}
$$

3. Examples for β solution models using the investigation parameter $\mu > 1$

In this section, we implement the DJM and the force function formula for solutions of nonlinear WSVIE.

Example 1(a). Consider the 2nd order nonlinear WSVIE given by

$$
u(x)=x^{k_1}-\frac{x^{\gamma k_1}}{\gamma k_1+\mu}+\int_0^x\frac{t^{\mu-1}}{x^\mu}u(t)^\beta dt \tag{24}
$$

Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

Solution,

$$
\begin{aligned}
&u_0(x)=f(x)=x^{\frac{1}{2}}-\frac{5}{11}x,\\
&u_1(x)=\frac{5}{11}x-\frac{100}{297}x^{\frac{3}{2}}+\frac{125}{1936}x^2
\end{aligned}
$$

$$u_2(x) = \frac{100}{297}x^{\frac{3}{2}} - \frac{125}{594}x^2 - \frac{125}{1936}x^2 + \frac{625}{17908}x^{\frac{5}{2}} + \frac{50000}{1852389}x^3 + ...$$
$$u_3(x) = \frac{125}{594}x^2 - \frac{625}{17908}x^{\frac{5}{2}} - \frac{1250}{10989}x^{\frac{5}{2}} - \frac{50000}{1852389}x^3 + \frac{3125}{188034}x^3 + ...$$
$$u_4(x) = \frac{1250}{10989}x^{\frac{5}{2}} - \frac{3125}{188034}x^3 - \frac{12500}{230769}x^3 + ...$$
$$u_5(x) = \frac{12500}{230769}x^3 + ...$$
$$u(x) = u_0 + ... + u_5 = x^{\frac{1}{2}}$$

Example 1(b). Consider the 3rd order nonlinear WSVIE given by

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt \tag{25}$$

For third order nonlinear parameter, $\gamma = \beta = 3$, $k_1 = \frac{1}{2}$, $\mu = 2$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$u_0(x) = x^{\frac{1}{2}} - \frac{2}{7}x^{\frac{3}{2}}$$
$$u_1(x) = \frac{2}{7}x^{\frac{3}{2}} - \frac{4}{21}x^{\frac{5}{2}} + \frac{24}{539}x^{\frac{7}{2}} - \frac{16}{4459}x^{\frac{9}{2}}$$
$$u_2(x) = \frac{4}{21}x^{\frac{5}{2}} - \frac{24}{539}x^{\frac{7}{2}} - \frac{8}{77}x^{\frac{7}{2}} + \frac{144}{7007}x^{\frac{9}{2}} + \frac{16}{4459}x^{\frac{9}{2}} + ...$$
$$u_3(x) = \frac{8}{77}x^{\frac{7}{2}} - \frac{144}{7007}x^{\frac{9}{2}} - \frac{48}{1001}x^{\frac{9}{2}} + ...$$
$$u_4(x) = \frac{48}{1001}x^{\frac{9}{2}} + ...$$
$$u(x) = u_0 + ... + u_6 = x^{\frac{1}{2}}$$

Example 1(c). For $k_1 = \frac{1}{2}$, $\mu = \frac{3}{2}$,$\gamma = \beta = 4$,
we substitute the parameter values in the 4th-order solution model to obtain the solution $u(x) = x^{\frac{1}{2}}$
Example 1(d). For $k_1 = \frac{1}{2}$, $\mu = 3$,$\gamma = \beta = 5$,
we substitute the parameter values in the 5th-order solution model to obtain the unique solution, $u(x) = x^{\frac{1}{2}}$
Example 2(a). Consider the nonlinear WSVIE of the form

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^3 dt \tag{26}$$

For 3rd order nonlinear parameter, $\gamma = \beta = 3$, $\mu = \frac{3}{2}$ and $k_1 = 1$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
&u_0(x) = f(x)x - \frac{2}{9}x^3\\
&u_1(x) = \frac{2}{9}x^3 - \frac{4}{39}x^5 + \frac{8}{459}x^7 - \frac{16}{15309}x^9\\
&u_2(x) = \frac{4}{39}x^5 - \frac{8}{459}x^7 - \frac{8}{221}x^7 + ...\\
&u_3(x) = \frac{8}{221}x^7 - \frac{16}{3216}x^9 - \frac{16}{1547}x^9 + ...\\
&u_4(x) = \frac{16}{1547}x^9 + ...\\
&u = u_0 + ... + u_4 = x.
\end{aligned}$$

Example 2(b). Consider the fourth order nonlinear WSVIE given by

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu} + \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t)^\beta dt \tag{27}$$

For fourth order nonlinear parameter , $\gamma = \beta = 4$, $k_1 = 1$, $\mu = \frac{7}{2}$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
&u_0(x) = f(x) = x - \frac{2}{15}x^4\\
&u_1(x) = \frac{2}{15}x^4 - \frac{16}{315}x^7 + \frac{16}{2025}x^{10} - \frac{64}{111375}x^{13} + ...\\
&u_2(x) = \frac{16}{315}x^7 - \frac{16}{2025}x^{10} - \frac{128}{8505}x^{10} + \frac{64}{111375}x^{13} + \frac{256}{1002375}x^{13} + ...\\
&u_3(x) = \frac{128}{8505}x^{10} - \frac{256}{1002375}x^{13} - \frac{1024}{280665}x^{13} + ...\\
&u_4(x) = \frac{1024}{280665}x^{13} + ...\\
&u = u_0 + ... + u_4 = x.
\end{aligned}$$

Example 2(c).For $k_1 = 1$, $\mu = 3$,$\gamma = \beta = 2$,
we substitute the parameter values in the 2nd-order solution model to obtain the solution $u(x) = x$
Example 2(d). For $k_1 = 1$, $\mu = \frac{4}{3}$,$\gamma = \beta = 5$,
we substitute the parameter values in the 5th-order solution model to obtain the solution

$u(x) = x$

Example 3(a). Consider the 3rd order nonlinear WSVIE given by

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt \tag{28}$$

For the 3rd order nonlinear parameter, $\gamma = \beta = 3$, $k_1 = \frac{1}{4}$, $\mu = \frac{5}{2}$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x^{\frac{1}{4}} - \frac{4}{13}x^{\frac{3}{4}} \\
u_1(x) &= \frac{4}{13}x^{\frac{3}{4}} - \frac{16}{65}x^{\frac{5}{4}} + \frac{192}{2873}x^{\frac{7}{4}} - \frac{256}{41743}x^{\frac{9}{4}} \\
u_2(x) &= \frac{16}{65}x^{\frac{5}{4}} - \frac{192}{2873}x^{\frac{7}{4}} - \frac{192}{1105}x^{\frac{7}{4}} + \frac{2304}{54587}x^{\frac{9}{4}} + \frac{256}{41304}x^{\frac{9}{4}} + ... \\
u_3(x) &= \frac{192}{1105}x^{\frac{7}{4}} - \frac{2304}{54587}x^{\frac{9}{4}} - \frac{2304}{20995}x^{\frac{9}{4}} + ... \\
u_4(x) &= \frac{2304}{20995}x^{\frac{9}{4}} + ... \\
u &= u_0 + ... + u_4 = x^{\frac{1}{4}}.
\end{aligned}$$

Example 3(b). Consider the nonlinear WSVIE of the form,

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt \tag{29}$$

For fifth order nonlinear parameter, $\gamma = \beta = 5$, $k_1 = \frac{1}{4}$ and $\mu = \frac{5}{4}$. Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x^{\frac{1}{4}} - \frac{2}{5}x^{\frac{5}{4}} \\
u_1(x) &= \frac{2}{5}x^{\frac{5}{4}} - \frac{4}{7}x^{\frac{9}{4}} + \frac{16}{45}x^{\frac{13}{4}} - \frac{32}{275}x^{\frac{17}{4}} + \frac{32}{1625}x^{\frac{21}{4}} + ... \\
u_2(x) &= \frac{4}{7}x^{\frac{9}{4}} - \frac{16}{45}x^{\frac{13}{4}} - \frac{40}{63}x^{\frac{13}{4}} + \frac{32}{275}x^{\frac{17}{4}} + \frac{32}{99}x^{\frac{17}{4}} - \frac{32}{1625}x^{\frac{21}{4}} + \frac{64}{715}x^{\frac{21}{4}} + \frac{320}{637}x^{\frac{21}{4}} + ... \\
u_3(x) &= \frac{40}{63}x^{\frac{13}{4}} - \frac{32}{99}x^{\frac{17}{4}} - \frac{400}{693}x^{\frac{17}{4}} + \frac{64}{715}x^{\frac{21}{4}} - \frac{320}{637}x^{\frac{21}{4}} + \frac{320}{1287}x^{\frac{21}{4}} + ... \\
u_4(x) &= \frac{400}{693}x^{\frac{17}{4}} - \frac{320}{1287}x^{\frac{21}{4}} - \frac{4000}{9009}x^{\frac{21}{4}} + ... \\
u_5(x) &= \frac{4000}{9009}x^{\frac{21}{4}} + ...
\end{aligned}$$

$u = u_0 + ... + u_5 = x^{\frac{1}{4}}.$

Example 3(c). For $k_1 = \frac{1}{4}$, $\mu = 4$,$\gamma = \beta = 4$,
we substitute the parameter values in the 4th-order solution model to obtain the solution,
$u(x) = x^{\frac{1}{4}}$
Example 3(d). For $k_1 = \frac{1}{4}$, $\mu = \frac{5}{2}$,$\gamma = \beta = 2$,
we substitute the parameter values in the 2nd-order solution model to obtain the solution,
$u(x) = x^{\frac{1}{2}}$

Remark 1. *The solution for the above six examples is obtained as $u(x) = x^{k_1}$, irrespective of the values of the assigned parameters defined for k_1, β and μ. See verification of the series solution for various β solution models in Section 2.2.*

4. Examples for various β solution models using the investigation parameter $0 < \mu \leq 1$

In this section, we examine the solutions to nonlinear WSVIE problems using the investigation parameter μ being $0 < \mu \leq 1$ and the nonlinear integer parameter $\beta \geq 2$
Example 4(a). Consider the nonlinear WSVIE of the form

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt \tag{30}$$

For the 2nd-order nonlinear parameter, $\beta = \gamma = 2$, $k_1 = \frac{3}{2}$ and $\mu = \frac{1}{3}$. Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x^{\frac{3}{2}} - \frac{3}{10}x^3 \\
u_1(x) &= \frac{3}{10}x^3 - \frac{18}{145}x^{\frac{9}{2}} + \frac{27}{1900}x^6 \\
u_2(x) &= \frac{18}{145}x^{\frac{9}{2}} - \frac{27}{1900}x^6 - \frac{108}{2755}x^6 + \frac{81}{22325}x^{\frac{15}{2}} - \frac{243}{147175}x^9 - \frac{1458}{4476875}x^{\frac{21}{2}} + ... \\
u_3(x) &= \frac{2916}{148555}x^6 - \frac{81}{22325}x^{\frac{15}{2}} - \frac{1296}{129485}x^{\frac{15}{2}} + \frac{243}{147175}x^9 + \frac{243}{312550}x^9 + \frac{1458}{4476875}x^{12} + \frac{2916}{9566375}x^{\frac{21}{2}} + ... \\
u_4(x) &= \frac{1296}{129485}x^{\frac{15}{2}} - \frac{243}{312550}x^9 - \frac{972}{713545}x^9 - \frac{2916}{9566375}x^{\frac{21}{2}} + \frac{1458}{10157875}x^{\frac{21}{2}} + ... \\
u_5(x) &= \frac{972}{713545}x^9 - \frac{1458}{10157875}x^{\frac{21}{2}} - \frac{23328}{58915675}x^{\frac{21}{2}} + ... \\
u_6(x) &= \frac{23328}{58915675}x^{\frac{21}{2}} + ... \\
u(x) &= \sum_{n=0}^{6} u_n = x^{\frac{3}{2}}
\end{aligned}$$

Example 4(b). Consider the nonlinear WSVIE of the form

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt \tag{31}$$

For 3rd order nonlinear parameter, $\beta = \gamma = 3$, $k_1 = \frac{3}{2}$, $\mu = 1$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x^{\frac{3}{2}} - \frac{2}{11}x^{\frac{9}{2}} \\
u_1(x) &= \frac{2}{11}x^{\frac{9}{2}} - \frac{12}{187}x^{\frac{15}{2}} + \frac{24}{2783}x^{\frac{21}{2}} - \frac{16}{38599}x^{\frac{27}{2}} \\
u_2(x) &= \frac{12}{187}x^{\frac{15}{2}} - \frac{24}{2783}x^{\frac{21}{2}} - \frac{72}{4301}x^{\frac{21}{2}} + \frac{144}{80707}x^{\frac{27}{2}} + \frac{16}{38599}x^{\frac{27}{2}} + ... \\
u_3(x) &= \frac{72}{4301}x^{\frac{21}{2}} - \frac{144}{80707}x^{\frac{27}{2}} - \frac{432}{124729}x^{\frac{27}{2}} + ... \\
u_4(x) &= \frac{432}{124729}x^{\frac{27}{2}} + ... \\
u(x) &= \sum_{n=0}^{4} u_n = x^{\frac{3}{2}}.
\end{aligned}$$

Example 5(a). Consider the nonlinear WSVIE of the form,

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt$$

For the 3rd order nonlinear parameter, $\beta = \gamma = 3$, $k_1 = \frac{1}{4}$, $\mu = \frac{1}{2}$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x^{\frac{1}{4}} - \frac{4}{5}x^{\frac{3}{4}} \\
u_1(x) &= \frac{4}{5}x^{\frac{3}{4}} - \frac{48}{65}x^{\frac{5}{4}} + \frac{64}{75}x^{\frac{7}{4}} - \frac{256}{1375}x^{\frac{9}{4}} \\
u_2(x) &= \frac{48}{65}x^{\frac{5}{4}} - \frac{64}{75}x^{\frac{7}{4}} - \frac{64}{35}x^{\frac{7}{4}} + \frac{256}{1375}x^{\frac{9}{4}} + \frac{256}{275}x^{\frac{9}{4}} + ... \\
u_3(x) &= \frac{64}{35}x^{\frac{7}{4}} - \frac{256}{275}x^{\frac{9}{4}} - \frac{6912}{3465}x^{\frac{9}{4}} + ... \\
u_4(x) &= \frac{6912}{3465}x^{\frac{9}{4}} + ... \\
u(x) &= \sum_{n=0}^{4} u_n = x^{\frac{1}{4}}.
\end{aligned}$$

Example 5(b). Consider the nonlinear WSVIE of the form,

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t)^\beta dt \tag{32}$$

5th-order nonlinear parameter with $\beta = \gamma = 5$, $k_1 = 1$ and $\mu = \frac{3}{4}$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x - \frac{4}{23}x^5 \\
u_1(x) &= \frac{4}{23}x^5 - \frac{80}{897}x^9 + \frac{128}{5819}x^{13} - \frac{2560}{863857}x^{17} + ... \\
u_2(x) &= \frac{80}{897}x^9 - \frac{128}{5819}x^{13} - \frac{320}{9867}x^{13} + \frac{2560}{863857}x^{17} - \frac{12800}{2065745}x^{17} + ... \\
u_3(x) &= \frac{320}{9867}x^{13} + \frac{12800}{2065745}x^{17} - \frac{6400}{700557}x^{17} + ... \\
u_4(x) &= \frac{6400}{700557}x^{17} + ... \\
u(x) &= \sum_{i=0}^{4} u_n = x.
\end{aligned}$$

Example 6(a). Consider the nonlinear WSVIE of the form,

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^\mu} u(t)^\beta dt \tag{33}$$

4th order nonlinear parameter with $\beta = \gamma = 4$, $k_1 = \frac{1}{4}$ and $\mu = \frac{4}{5}$.
Following the algorithm in equation (15), the relation between the final series solution term and the truncation point is determined using equation (17).

$$\begin{aligned}
u_0(x) &= f(x) = x^{\frac{1}{4}} - \frac{5}{9}x \\
u_1(x) &= \frac{5}{9}x - \frac{400}{459}x^{\frac{7}{4}} + \frac{500}{891}x^{\frac{5}{2}} - \frac{10000}{59049}x^{\frac{13}{4}} + ... \\
u_2(x) &= \frac{400}{459}x^{\frac{7}{4}} - \frac{500}{891}x^{\frac{5}{2}} - \frac{16000}{15147}x^{\frac{5}{2}} + \frac{10000}{59049}x^{\frac{13}{4}} - \frac{200000}{649539}x^{\frac{13}{4}} + ... \\
u_3(x) &= \frac{16000}{15147}x^{\frac{5}{2}} - \frac{200000}{649539}x^{\frac{13}{4}} - \frac{1280000}{1226907}x^{\frac{13}{4}} + ... \\
u_4(x) &= \frac{1280000}{1226907}x^{\frac{13}{4}} + ... \\
u(x) &= \sum_{n=0}^{4} u_n = x^{\frac{1}{4}}.
\end{aligned}$$

Example 6(b). Consider the nonlinear WSVIE of the form

$$u(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu + \gamma k_1} + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)^{\beta} dt \tag{34}$$

5th-order nonlinear parameter $\beta = 5$ with $\beta = \gamma = 5$, $k_1 = 2$, $\mu = \frac{1}{4}$.
Following the algorithm in equation (15), we obtain the solution in series, and the truncation point with the highest power is determined using equation (17).
Solution in series,

$$\begin{aligned}
u_0(x) &= f(x) = x^2 - \frac{4}{41}x^{10} \\
u_1(x) &= \frac{4}{41}x^{10} - \frac{80}{2993}x^{18} + \frac{128}{35301}x^{26} + ... \\
u_2(x) &= \frac{80}{2993}x^{18} - \frac{128}{35301}x^{26} - \frac{320}{62853}x^{26} + ... \\
u_3(x) &= \frac{320}{62853}x^{26} + ... \\
u(x) &= \sum_{n=0}^{\infty} u_n = x^2.
\end{aligned}$$

Remark 2. *The solution for the above six examples is obtained as $u(x) = x^{k_1}$, irrespective of the values of the assigned parameters defined for k_1, β and μ. See verification of the series solution for various β solution models in Section 3.*

5. Summary of results of solved examples

The tables 1 and 2 below show row 1 as example of numbering; the second, third, and fourth rows are parameter values in the nonlinear WSVIE; and the fifth row is the solution.
Tables 1 and 2 show results for $\mu > 1$ and $0 < \mu \leq 1$, respectively.
Table 1: Solutions of worked examples for the case of $\mu > 1$.

Example	1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	2(d)	3(a)	3(b)	3(c)	3(d)
k_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
β	2	3	4	5	3	4	2	5	3	5	4	2
μ	$\frac{6}{5}$	2	$\frac{3}{2}$	3	$\frac{3}{2}$	$\frac{7}{2}$	3	$\frac{4}{3}$	$\frac{5}{2}$	$\frac{5}{4}$	4	$\frac{5}{2}$
u(x)	$x^{\frac{1}{2}}$	$x^{\frac{1}{2}}$	$x^{\frac{1}{2}}$	$x^{\frac{1}{2}}$	x	x	x	x	$x^{\frac{1}{4}}$	$x^{\frac{1}{4}}$	$x^{\frac{1}{4}}$	$x^{\frac{1}{4}}$

Table 2: Solutions of worked examples for the case of $0 < \mu \leq 1$.

Example	4(a)	4(b)	5(a)	5(b)	6(a)	6(b)
k_1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	1	$\frac{1}{4}$	2
β	2	3	3	5	4	5
μ	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{4}$
u(x)	$x^{\frac{3}{2}}$	$x^{\frac{3}{2}}$	$x^{\frac{1}{4}}$	x	$x^{\frac{1}{4}}$	x^2

From Tables 1 and 2 above, the solution is obtained as $u(x) = x^{k_1}$ for the indicated integer values of $\beta \geq 2$, rational values of μ being $\mu > 1$ and $0 < \mu \leq 1$

6. Discussion

In this paper, we used DJM and the force function formula in the solution process. In line with [18], the force function used is $f(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\mu+\gamma k_1}$. As discussed in [32], we obtained the relation between the last solution term and the truncation point to be

$$u_n(x) = \frac{\beta^{n-1} x^{[n(\gamma-1)+1]k_1}}{(\gamma k_1 + \mu) \prod_{m=2}^{n} \Big[[m(\gamma - 1)]k_1 + \mu \Big]}.$$

The relation between the last term in the solution series and the truncation point was discovered through the solution process in the appendix, and the relation holds for all integer values of $\beta \geq 2$ and rational values of k_1 and $\mu > 0$. Due to noise term cancellation, our solution becomes

$$u(x) = u_0(x) + \sum_{m=1}^{n} u_m = x^{k_1}.$$

We extended the range of investigation parameter values from $\mu > 1$ to $0 < \mu \leq 1$.
From Table 1, our solution examples $1(a)$ and $2(a)$ confirm the solutions of Al-Jawary and Shehan[4], who respectively used specific force functions, $f(x) = x^{\frac{1}{2}} - \frac{5}{11}x$ with $\beta = 2$ and $\mu = \frac{6}{5}$,
and $f(x) = x - \frac{2}{9}x^3$ with $\beta = 3$ and $\mu = \frac{3}{2}$ for their nonlinear WSVIE solutions.
Our solutions depend on the force function formula parameter k_1 and produce the unique solution $u = x^{k_1}$ irrespective of the nonlinear integer parameter value of $\beta \geq 2$, positive rational values of k_1 and $\mu > 0$, and for any finite value of $n \geq 2$.

7. Conclusion

In this paper, we have solved the nonlinear WSVIE of equation (2) with the reproducing kernel, $K(x,t) = \frac{t^{\mu-1}}{x^{\mu}}$, by extending the range of the investigation parameter $\mu > 1$ to $0 < \mu \leq 1$. We have discovered a force function formula, $f(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu}$, used in equation (2). We are able to determine a formula relation between the final series solution term and the truncation point. For the purpose of verifying different β solutions, the authors have derived solution models in Section 3 that facilitate the computation of solution examples. The examples of solutions validate the outcomes found in [4]. From the various β solution models and table of solutions, we extrapolate that for the force function, $f(x) = x^{k_1} - \frac{x^{\gamma k_1}}{\gamma k_1 + \mu}$ where k_1 is rational, we shall always get the unique solution $u(x) = x^{k_1}$, irrespective of any chosen parameter values of the integer $\beta \geq 2$, positive rational parameter value $\mu > 0$, and for any finite value of $n \geq 2$. The authors are working on extending the range of μ values from $\mu > 0$ to $\mu \leq 0$

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