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Adomian Modification Methods via Orthogonal Polynomials: A Comparative Study

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Abstract. The present manuscript proposes different modification procedures for the standard Adomian Decomposition Method (ADM). These procedures are based on the application of orthogonal polynomials that play vital parts in approximation theories. Moreover, the study also scrutinizes four nonlinear inhomogeneous initial-value problems, and distinctively examines their respective absolute error differences. Remarkably, different computational benefits of the proposed modification are noted with regard high-level of accuracy and fewer computational steps.

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1. Introduction

The celebrated Adomian Decomposition Method (ADM) [2] has in the past and present decades been greatly utilized to solve a variety of functional equations. The method that was proposed by George Adomian (in the 1980s) has further undergone different stages of reformations, modifications, and improvements. Indeed, there exist a huge number of related literature with regards to the development of ADM associated with its applicability in solving various forms of IVPs of both the ordinary and partial differential equation types [1, 3, 4, 13]. On the other hand, orthogonal functions are regarded with high admiration in the fields of numerical methods, and approximation theories among others. However, in line with their applications, Hosseini [7] demonstrated the relevance of Chebyshev's polynomials in improving the known accuracy of the standard ADM. In fact, different nonlinear and linear models were examined via the method to have good approximate solutions. We mention also the excellent work of Liu [8] where Legendre's polynomials were coupled in the ADM instead of the ordinary Adomian procedure. Additionally, ADM

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was equally enhanced using the Gegenbauer's and Jacobi's orthogonal polynomials [6] to solve some important models of mathematical physics; one may in the same fashion read about the relevance of Laguerre's and Hermite's orthogonal polynomials in optimizing the standard ADM procedure in [11, 12].

However, the present manuscript proposes different modification procedures for the standard ADM. These procedures are based on the application of orthogonal polynomials that play vital parts in approximation theories as rightly mentioned. More specifically, the following orthogonal polynomials: Legendre's, Chebyshev's, Laguerre's, Hermite's, Gegenbauer's, and lastly the Jacobi's polynomials will be considered to devise modification methods for the standard ADM. Moreover, the present study will scrutinize four test problems and distinctively examines their respective absolute error differences. Additionally, we organize the paper in the following manner: Section 2 gives the standard ADM procedure; while its modifications based on orthogonal polynomials are presented in Section 3. Section 4 makes consideration to certain illustrative test examples; while Section 5 gives certain concluding comments.

2. Standard ADM procedure

The present section gives a generalized derivation procedure for tackling nonlinear Initial-Value Problems (IVPs) based on the ADM. To do so, let us consider the following differential equation

$$G(u(x)) = g(x), \tag{1}$$

with G representing a generalized ordinary (or partial) differential operator, and $g(x)$ as a source term. This operator being general, it can equally be expressed to involve both linear and nonlinear operators. Thus, we decompose the operator further, and rewrite the above equation as follows

$$Lu + Ru + Nu = g, \tag{2}$$

where L is the highest linear operator that is invertible, with $R < L$; while N is specifically the nonlinear operator. More so, we rewrite the latter equation as follows

$$Lu = g - Ru - Nu, \tag{3}$$

such that applying the inverse linear operator L^{-1} to both sides of the above equation yields

$$u = \phi(x) + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{4}$$

where $\phi(x)$ is the function emanating from the prescribed initial data.
Further, the iterative procedure by the name ADM decomposes the solution $u(x)$ using an infinite series of the following form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{5}$$

while the nonlinear component Nu is equally decomposed using the following infinite series

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ...), \tag{6}$$

where A_n's are polynomials devised by Adomian, and recursively determined using the following scheme

$$A_n(u_0, u_1, ...) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{j=0}^{n} \lambda^j u_j\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, ... \tag{7}$$

Therefore, upon substituting Eqs. (5) and (6) into Eq. (4), one gets

$$\sum_{n=0}^{\infty} u_n(x) = \phi(x) + L^{-1}g(x) - L^{-1}R\sum_{n=0}^{\infty} u_n(x) - L^{-1}\sum_{n=0}^{\infty} A_n(u_0, u_1, ...), \tag{8}$$

Furthermore, the ADM procedure swiftly reveals the generalized recursive solution for the problem from the above equation as follows

$$\begin{cases} u_0 = \phi(x) + L^{-1}g(x), \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n(u_0, u_1, ...), \quad n \geq 0, \end{cases} \tag{9}$$

where A_n's are the Adomian polynomials computed from Eq. (7). Expressing few of these terms, we get

$$\begin{aligned} A_0(u_0) &= N(u_0), \\ A_1(u_0, u_1) &= \frac{dN(u_0)}{du_0}u_1, \\ A_2(u_0, u_1, u_2) &= \frac{dN(u_0)}{du_0}u_2 + \frac{1}{2}\frac{d^2N(u_0)}{du_0^2}u_1^2, \\ A_3(u_0, u_1, u_2, u_3) &= \frac{dN(u_0)}{du_0}u_3 + \frac{d^2N(u_0)}{du_0^2}u_1u_2 + \frac{1}{3!}\frac{d^3N(u_0)}{du_0^3}u_1^3, \\ &\vdots \end{aligned}$$

Remarkable, it is obvious that the Adomian polynomials A_n's depend on the solution components u_n. For instance, A_0 relies merely on u_0; A_1 relies merely on u_0 and u_1; A_2 relies merely on u_0, u_1 and u_2 , and so on.

Finally, a realistic solution is obtained by considering the following m-term approximations as

$$\Psi_n = \sum_{j=0}^{n-1} u_j, \tag{10}$$

where

$$u(x) = \lim_{n\to\infty} \Psi_n(x) = \sum_{j=0}^{\infty} u_j(x). \tag{11}$$

3. Adomian modification methods via orthogonal polynomials

The present section gives some important modifications of the standard ADM that are based on the application of orthogonal polynomials. One could easily recall the importance of orthogonal functions in approximation theory and numerical methods. Thus, these functions/polynomials are equally used in the present study to further optimize the exactness of the standard ADM. To begin with, let us make use of the Taylor's series expansion to expand the given source term $g(x)$ in Eq. (2) for an arbitrary positive integer, say m as follows

$$g(x) = \sum_{n=0}^{m} \frac{g^n(0)}{n!} x^n. \tag{12}$$

Therefore, in what follows, we have obtained series of Adomian modification methods via orthogonal polynomials. More specifically, we have utilized the following orthogonal polynomials including the Legendre's, Chebyshev's, Laguerre's, Hermit, Gegenbauer's and Jacobi's polynomials [5].

3.1. Adomian modification via Legendre's polynomials

To present a modification method based on the application of the Legendre's polynomials, we express the source term $g(x)$ given in Eq. (2) as a series of Legendre's polynomial as follows [8, 10]

$$g(x) = \sum_{n=0}^{m} c_n P_n(x), \tag{13}$$

where $P_n(x)$ are the orthogonal Legendre's polynomials, and the coefficients of Legendre's expansion c_i are determined through

$$c_i = \frac{2i+1}{2} \int_{-1}^{1} g(x) P_i(x) dx, \quad i = 0, 1, \cdots$$

Thus, substituting Eq. (13) into Eq. (9), we get the following recursive solution

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \cdots + c_m P_m(x)], \\ u_{n+1} = -L^{-1} R u_n - L^{-1} A_n, \quad n \geq 0. \end{cases} \tag{14}$$

Finally, a realistic solution via the application of Legendre's polynomial is thus obtained in this regard by considering the following m-term approximations using $u(x) = \sum_{n=0}^{m} u_n$, where m is the order of the solution.

3.2. Adomian modification method via Chebyshev's polynomials

(i) **First kind Chebyshev's polynomials** In Hosseini [7], the source term $g(x)$ is suggested to be decomposed using Chebyshev's series as follows

$$g(x) = \sum_{n=0}^{m} c_n T_n(x), \tag{15}$$

where $T_n(x)$ are the orthogonal Chebyshev's polynomial of the first kind; while the coefficient of Chebyshev's expansion c_i are expressed as follows

$$\begin{aligned} c_0 &= \tfrac{1}{\pi}\int_{-1}^{1}\frac{g(x)T_0(x)}{\sqrt{1-x^2}}dx, \\ c_i &= \tfrac{2}{\pi}\int_{-1}^{1}\frac{g(x)T_i(x)}{\sqrt{1-x^2}}dx, i = 1, 2, \cdots \end{aligned} \tag{16}$$

Thus, upon using Eqs. (9) and (15), we get the following recursive solution

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0T_0(x) + c_1T_1(x) + c_2T_2(x) + \cdots + c_mT_m(x)], \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \end{cases} \tag{17}$$

(ii) **Second kind Chebyshev's polynomials** In the same fashion, we make use of the second kind Chebyshev's polynomials in approximating the source term $g(x)$ instead of the first kind Chebyshev's polynomials [9, 14] as follows

$$g(x) = \sum_{n=0}^{m} c_nU_n(x), \tag{18}$$

where $U_n(x)$ are the orthogonal Chebyshev's polynomial of the second kind; while the coefficients of the Chebyshev's expansion c_i are given by

$$c_i = \tfrac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}g(x)U_i(x)dx, \qquad i = 0, 1, 2, \cdots \tag{19}$$

Now, on using Eqs. (9) and (18), we get the following recursive solution

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0U_0(x) + c_1U_1(x) + c_2U_2(x) + \cdots + c_mU_m(x)], \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0. \end{cases} \tag{20}$$

Thus, realistic solutions via the application of the Chebyshev's polynomials of the first and second kinds are thus obtained in this regard by considering the following m-term approximations using $u(x) = \sum_{n=0}^{m} u_n$, where m is the order of the solution.

3.3. Adomian modification method via Laguerre's polynomials

In the same way, it is suggested that the source term $g(x)$ to be decomposed using Laguerre's series [11] as follows

$$g(x) = \sum_{n=0}^{m} c_nL_n(x) \tag{21}$$

where $L_n(x)$ are orthogonal Laguerre's polynomials, and c_i are given by

$$c_i = \int_{0}^{\infty} e^{-x}L_i(x)g(x)dx, \qquad i = 0, 1, \cdots \tag{22}$$

Accordingly Eqs.(9) and (21), we get the recursive solution as in the preceding method as follows

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0 L_0(x) + c_1 L_1(x) + c_2 L_2(x) + \cdots + c_m L_m(x)], \\ u_{n+1} = -L^{-1} R u_n - L^{-1} A_n, \qquad n \geq 0, \end{cases} \tag{23}$$

where the closed-form solution $u(x)$ is obtained upon summing the individual components as suggested by ADM.

3.4. Adomian modification method via Hermite's polynomials

In the same way, it is suggested that the source term $g(x)$ to be decomposed using Hermite's series [12] as follows

$$g(x) = \sum_{n=0}^{m} c_n H_n(x), \tag{24}$$

where $H_n(x)$ are orthogonal Hermite's polynomials, and the coefficient c_i are determined using

$$c_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_i(x) g(x) dx, \qquad i = 0, 1, \cdots \tag{25}$$

What's more from Eqs.(9) and (21), we get the following recursive solution as explained earlier as follows

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0 H_0(x) + c_1 H_1(x) + c_2 H_2(x) + \cdots + c_m H_m(x)], \\ u_{n+1} = -L^{-1} R u_n - L^{-1} A_n, \qquad n \geq 0, \end{cases} \tag{26}$$

where the closed-form solution $u(x)$ is obtained upon summing the individual components as suggested by ADM.

3.5. Adomian modification methods via Gegenbauer's and Jacobi's polynomials

(i) **Gegenbauer's polynomials**

Firstly, we express the source term $g(x)$ via the Gegenbauer's series [6] as follows

$$g(x) = \sum_{n=0}^{m} c_n C_n^{\alpha}(x), \tag{27}$$

where $C_n^{\alpha}(x)$ are the orthogonal Gegenbauer's polynomials, and the coefficients of Gegenbauer's expansion c_i are defined as follows

$$c_i = \frac{\int_{-1}^{1} g(x) C_i^{\alpha}(x)(1-x^2)^{\alpha-1/2} dx}{\int_{-1}^{1} [C_i^{\alpha}(x)]^2 (1-x^2)^{\alpha-1/2} dx}, \quad i = 0, 1, 2, \cdots \tag{28}$$

where the normalization of the functions are done using as $(1-x^2)^{\alpha-1/2}$ as the weight function.

Lastly, substituting Eq. (27) into Eq. (9), we get the recursive solution without further delay as follows

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0 C_0^{\alpha}(x) + c_1 C_1^{\alpha}(x) + c_2 C_2^{\alpha}(x) + \cdots + c_m C_m^{\alpha}(x)], \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \qquad n \geq 0. \end{cases} \tag{29}$$

(ii) **Jacobi's polynomials**

Considering Jacobi's orthogonal polynomials over [−1, 1], we in the same way decompose the source term $g(x)$ as follows

$$g(x) = \sum_{n=0}^{m} c_n P_n^{(\alpha,\beta)}(x), \tag{30}$$

where $\alpha, \beta > -1$ and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi's polynomials that are orthogonal, and the coefficients c_i of Jacobi expansion are defined as follows

$$c_i = \frac{\int_{-1}^{1} g(x) P_i^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx}{\int_{-1}^{1}[P_i^{(\alpha,\beta)}(x)]^2(1-x)^{\alpha}(1+x)^{\beta}dx}, \qquad i = 0, 1, 2, \cdots \tag{31}$$

where the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ is used for the normalization in this equation.

Thus, the following recursive solution is obtained from Eqs. (30) and (9) as follows

$$\begin{cases} u_0 = \phi(x) + L^{-1}[c_0 P_0^{(\alpha,\beta)}(x) + c_1 P_1^{(\alpha,\beta)}(x) + c_2 P_2^{(\alpha,\beta)}(x) + \cdots + c_m P_m^{(\alpha,\beta)}(x)], \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, \qquad n \geq 0. \end{cases} \tag{32}$$

Moreover, realistic solutions via the application of the above polynomials could be obtained in the same manner by considering the following m-term approximations using $u(x) = \sum_{n=0}^{m} u_n$.

4. Illustrative examples

The current section demonstrates the application of the Adomian modification methods via orthogonal polynomials to comparatively examine different forms of IVPs of ODEs as test examples. Moreover, we shall utilize seven-term approximations via the Maple 18 package programmer for the computational simulation.

Example 1. *Consider the IVP of Duffing's equation [6]*

$$\begin{aligned} u'' + 3u - 2u^3 = \sin(2x)\cos(x), \quad 0 \leq x \leq 1, \\ u(0) = 0, \quad u'(0) = 1, \end{aligned} \tag{33}$$

that admits the following exact solution $u(x) = \sin(x)$.

Firstly, we express the given equation in operator form as follows

$$u = x + L^{-1}(\sin(2x)\cos(x)) - 3L^{-1}(u) + 2L^{-1}(u^3), \tag{34}$$

where $L^{-1}(.)$ is the inverse operator defined by $L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$ and $N(u) = u^3$ Substituting Eqs.(5) and (6) into Eqs. (34) , we get the following recursive solution

$$\begin{aligned} &u_0 = x + L^{-1}(sin(2x)cos(x)),\\ &u_{n+1} = -3L^{-1}(u_n) + 2L^{-1}A_n(u_0, u_1, \cdots), n \geq 0, \end{aligned}$$

From Eq.(7), the nonlinear tearm $N(u) = u^3$ requires the following Adomian polynomials

$$\begin{aligned} &A_0 = u_0^3,\\ &A_1 = (3u_0^2 u_1),\\ &A_2 = (3u_0^2 u_2 + 3u_0 u_1^2),\\ &A_3 = (3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3),\\ &\vdots \end{aligned} \tag{35}$$

In what follows, we shall be utilizing the proposed modification methods to treat the governing Duffing's equation. More so, we shall be starting with the classical Taylor's series before the proposed schemes. Additionally, we denote the solution $u(x)$ based on the respective modifications as follows: $u_t(x)$ via the Taylor's series expansion; $u_P(x)$ via the Legendre's series expansion; $u_T(x)$ via the Chebyshev's series expansion; $u_L(x)$ via the Laguerre's series expansion; $u_H(x)$ via the Hermite's series expansion; $u_g^1(x)$ via the Gegenbauer's series expansion ($\alpha = 1$); and lastly $u_j^{(1,1)}(x)$ via the Jacobi's series expansion ($\alpha = 1, \beta = 1$).

Modification method via Taylor's series will be used for the expansion of the source term $g(x)$ for $m = 6$ as follows

$$g(x) = 2x - \frac{7}{3}x^3 + \frac{61}{60}x^5 + O(x^7), \tag{36}$$

Then, we get the following iterative components

$$\begin{aligned} &u_0 = u(0) + xu'(0) + L^{-1}(2x - \tfrac{7}{3}x^3 + \tfrac{61}{60}x^5) = x + \tfrac{1}{3}x^3 - \tfrac{7}{60}x^5 + \tfrac{61}{2520}x^7,\\ &u_1 = -3L^{-1}(u_0) + 2L^{-1}A_0 = -\tfrac{1}{2}x^3 + \tfrac{1}{20}x^5 + \tfrac{47}{840}x^7 - \tfrac{89}{60480}x^9 + \cdots,\\ &u_2 = -3L^{-1}(u_1) + 2L^{-1}A_1 = \tfrac{3}{40}x^5 - \tfrac{3}{40}x^7 - \tfrac{523}{20160}x^9 + \cdots,\\ &u_3 = -3L^{-1}(u_2) + 2L^{-1}A_2 = -\tfrac{3}{560}x^7 + \tfrac{29}{960}x^9 + \cdots,\\ &u_4 = -3L^{-1}(u_3) + 2L^{-1}A_3 = \tfrac{1}{4480}x^9 + \cdots,\\ &\vdots \end{aligned}$$

such that upon summing the above components yields the following series solution

$$u_t(x) = \sum_{n=0}^{6} u_n(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{73}{24192}x^9 + \cdots \tag{37}$$

Modification method via Legendre's polynomials is up now. Expanding of the source term $g(x)$ via Legendre's polynomials for $m = 6$ gives

$$g(x) = \sum_{n=0}^{6} c_n P_n(2x-1), \quad 0 \leq x \leq 1, \tag{38}$$

where $P_n(.)$ are orthogonal Legendre's polynomials, and c_i are given by

$$c_i = \frac{2i+1}{2} \int_{-1}^{1} g(0.5x+0.5) P_i(x) dx, \qquad i = 0, 1, \cdots \tag{39}$$

This means that

$$g(x) \approx -0.00001047 + 2.000674384x - 0.106599082x^2 + \cdots - 0.4690686000x^6. \tag{40}$$

Thus, we get the following solution components

$$\begin{aligned}
u_0 &= u(0) + xu'(0) + L^{-1}(-0.00001047 + 2.000674384x + \cdots - 0.4690686000x^6), \\
&= x - 0.000005237x^2 + 0.3334454093x^3 + \cdots - 0.008375467814x^8, \\
u_1 &= -0.5x^3 + 0.00000130940075x^4 + \cdots, \\
u_2 &= 0.075x^5 - 1.307582000 \times 10^{-7} x^6 + \cdots, \\
u_3 &= -0.005357142858x^7 + \cdots, \\
&\vdots
\end{aligned}$$

such that their summation yields

$$u_P(x) = \sum_{n=0}^{6} u_n(x) = x - 0.000005237603000x^2 - 0.1665545908x^3 - 0.00088541019923x^4 + \cdots \tag{41}$$

Modification method via Chebyshev's polynomials goes off by expanding the source term $g(x)$ as follows

$$g(x) = \sum_{n=0}^{6} c_n T_n(2x-1), \quad 0 \leq x \leq 1, \tag{42}$$

where $T_n(.)$ are orthogonal Chebyshev's polynomials, and c_i are given by where

$$\begin{aligned}
c_0 &= \tfrac{1}{\pi} \int_{-1}^{1} \frac{g(0.5x+0.5)T_0(x)}{\sqrt{1-x^2}} dx, \\
c_i &= \tfrac{2}{\pi} \int_{-1}^{1} \frac{g(0.5x+0.5)T_i(x)}{\sqrt{1-x^2}} dx, \quad i = 1, 2, \cdots
\end{aligned} \tag{43}$$

such that

$$g(x) \approx -0.000004054169 + 2.000464751x - 0.0088519738x^2 + \cdots - 0.4661302071x^6. \tag{44}$$

Therefore, the solution components are as follows

$$
\begin{aligned}
u_0 &= u(0) + xu'(0) + L^{-1}(-0.000004054169 + 2.000464751x - 0.0088519738x^2 + \cdots - 0.4661302071x^6),\\
&= x - 0.0000020270845x^2 + 0.3334107920x^3 + \cdots - 0.008323753699x^8,\\
u_1 &= -0.5x^3 + 5.06771124910^{-7}x^4 + \cdots,\\
u_2 &= 0.075x^5 - 5.06771124910^{-8}x^6 + \cdots,\\
u_3 &= -0.005357142858x^7 + \cdots,\\
&\vdots
\end{aligned}
$$

that leads to the following series solution

$$
u_T(x) = \sum_{n=0}^{6} u_n(x) = x - 0.0000020270845x^2 - 0.1665892081x^3 - 0.0007371577121x^4 + \cdots \tag{45}
$$

Modification method via Laguerre's polynomials starts off by expanding the source term $g(x)$ in the following form

$$
g(x) = \sum_{n=0}^{6} c_n L_n(x), \quad 0 \leq x \leq 1, \tag{46}
$$

where $L_n(.)$ are orthogonal Laguerre's polynomials, and c_i are given by

$$
c_i = \int_0^{\infty} e^{-x} L_i(x) g(x) dx, \quad i = 0, 1, \cdots \tag{47}
$$

such that

$$
g(x) \approx \frac{148321}{625000} + \frac{150161}{156250}x - \frac{219727}{250000}x^2 + \cdots - \frac{25849}{450000000}x^6. \tag{48}
$$

This gives the following iterative solutions

$$
\begin{aligned}
u_0 &= u(0) + xu'(0) + L^{-1}\left(\frac{148321}{625000} + \frac{150161}{156250}x - \frac{219727}{250000}x^2 + \cdots - \frac{25849}{450000000}x^6\right),\\
&= x + 0.118656800x^2 + 0.160171733x^3 - 0.07324233332x^4 + \cdots - 0.000001025753968x^8,\\
u_1 &= -0.5x^3 - 0.02966420000x^4 - 0.07597424000x^5 + \cdots,\\
u_2 &= 0.075x^5 + 0.002966420000x^6 - 0.07685530286x^7 + \cdots,\\
u_3 &= -0.005357142857x^7 + \cdots,\\
&\vdots
\end{aligned}
$$

that sums to the following

$$
u_L(x) = \sum_{n=0}^{6} u_n(x) = x + 0.1186568000x^2 - 0.3398282667x^3 - 0.1029065333x^4 + \cdots \tag{49}
$$

Modification method via Hermite's polynomials expands the source term $g(x)$ as follows

$$g(x) = \sum_{n=0}^{6} c_n H_n(x), \quad 0 \le x \le 1, \tag{50}$$

where $H_n(.)$ are orthogonal Hermite's polynomials, and c_i are given by

$$c_i = \frac{1}{2^i i! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_i(x) g(x) dx, \quad i = 0, 1, \cdots \tag{51}$$

such that

$$g(x) \approx 1.412928152x - 0.8518569111x^3 + 0.1099617181x^5. \tag{52}$$

We, therefore, obtain

$$\begin{aligned}
u_0 &= u(0) + xu'(0) + L^{-1}(1.412928152x - 0.8518569111x^3 + 0.1099617181x^5), \\
&= x + 0.2354880253x^3 - 0.04259284556x^5 + 0.002618136146x^7, \\
u_1 &= -0.5x^3 + 0.06467679622x^5 + \cdots, \\
u_2 &= 0.075x^5 - 0.07604834258x^7 + \cdots, \\
u_3 &= -0.005357142858x^7 + \cdots, \\
&\vdots
\end{aligned}$$

and leading to the following series solution

$$u_H(x) = \sum_{n=0}^{6} u_n(x) = x - 0.2645119748x^3 + 0.09708395066x^5 + \cdots \tag{53}$$

Modification method via Gegenbauer's polynomials equally starts off by expanding the function $g(x)$ as follows

$$g(x) = \sum_{n=0}^{6} c_n C_n^{\alpha}(2x - 1), \quad 0 \le x \le 1, \tag{54}$$

where $C_n^{\alpha}(.)$ are orthogonal Gegenbauer's polynomials, and c_i are given by

$$c_i = \frac{\int_{-1}^{1} g(0.5x + 0.5) C_i^{\alpha}(x)(1 - x^2)^{\alpha - 1/2} dx}{\int_{-1}^{1} [C_i^{\alpha}(x)]^2 (1 - x^2)^{\alpha - 1/2} dx}, \quad i = 0, 1, 2, \cdots \tag{55}$$

such that for $\alpha = 1$, we get

$$g(x) \approx -0.000017465407 + 2.000848250x - 0.0119838208x^2 + \cdots - 0.4708946817x^6. \tag{56}$$

Accodingly, we get

$$\begin{aligned}
u_0 &= u(0) + xu'(0) + L^{-1}(-0.000017465407 + 2.000848250x - 0.0119838208x^2 + \cdots - 0.4708946817x^6),\\
&= x - 0.0000087327035x^2 + 0.3334747083x^3 + \cdots - 0.008408833601x^8,\\
u_1 &= -0.5x^3 + 0.000002183175875x^4 + \cdots,\\
u_2 &= 0.075x^5 - 2.183175875 \times 10^{-7}x^6 + \cdots,\\
u_3 &= -0.005357142858x^7 + \cdots,\\
&\vdots
\end{aligned}$$

that leads to the following series solution

$$u_g(x) = \sum_{n=0}^{6} u_n(x) = x - 0.0000087327035x^2 - 0.1665252918x^3 - 0.0009964685573x^4 + \cdots \tag{57}$$

Modification method via Jacobi's polynomials also goes as explained by expanding $g(x)$ as

$$g(x) = \sum_{n=0}^{6} c_n P_n^{(\alpha,\beta)}(2x-1), 0 \leq x \leq 1 \tag{58}$$

where $P_n(.)$ are orthogonal Jacobi's polynomials, and c_i are given by

$$c_i = \frac{\int_{-1}^{1} g(0.5x+0.5)P_i^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx}{\int_{-1}^{1}[P_i^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta dx}, \quad i = 0, 1, 2, \cdots \tag{59}$$

such that for $\alpha = \beta = 1$, one gets

$$g(x) \approx 0.0001041764 + 1.997243414x + 0.020167913x^2 + \cdots - 0.3992660100x^6. \tag{60}$$

In the same manner, we get the following solution components

$$\begin{aligned}
u_0 &= u(0) + xu'(0) + L^{-1}(0.0001041764 + 1.997243414x + 0.020167913x^2 + \cdots - 0.3992660100x^6),\\
&= x + 0.0000520882000x^2 + 0.332873902x^3 + \cdots - 0.007129750179x^8,\\
u_1 &= -0.5x^3 - 0.00001302205000x^4 + \cdots,\\
u_2 &= 0.075x^5 + 1.30220500 \times 10^{-6}x^6 + \cdots,\\
u_3 &= -0.00535714285x^7 + \cdots,\\
&\vdots
\end{aligned}$$

that sums to the following

$$u_j(x) = \sum_{n=0}^{6} u_n(x) = x + 0.00005208820000x^2 - 0.1671260977x^3 + 0.00166763736x^4 + \cdots \tag{61}$$

Finally, we report the absolute error differences between the exact solution $u(x)$ and the respective modification solutions in Table 1. In this table, $u_t(x)$ stands for the modification via the Taylor series; $u_P(x)$ via the Legendre's series; $u_T(x)$ via the Chebyshev's series; $u_L(x)$ via the Laguerre's series; $u_H(x)$ via the Hermite's series; $u_g(x)$ via the Gegenbauer's series ($\alpha = 1$); and finally $u_j(x)$ via the Jacobi series ($\alpha = 1, \beta = 1$).

Table 1: Absolute error comparisons via the proposed modifications for Example 1

x	$\lvert u(x)-u_t(x)\rvert$	$\lvert u(x)-u_P(x)\rvert$	$\lvert u(x)-u_T(x)\rvert$	$\lvert u(x)-u_g(x)\rvert$	$\lvert u(x)-u_j(x)\rvert$	$\lvert u(x)-u_L(x)\rvert$	$\lvert u(x)-u_H(x)\rvert$
0	0	0	0	0	0	0	0
0.25	1.130×10^{-8}	1.50×10^{-9}	6.80×10^{-9}	2.750×10^{-8}	3.852×10^{-7}	4.465×10^{-3}	1.444×10^{-3}
0.50	5.743×10^{-6}	5.10×10^{-9}	2.680×10^{-8}	5.050×10^{-8}	8.532×10^{-7}	6.491×10^{-3}	9.756×10^{-3}
0.75	2.154×10^{-4}	1.508×10^{-7}	1.710×10^{-7}	7.830×10^{-8}	1.199×10^{-6}	3.965×10^{-3}	2.484×10^{-2}
1	2.790×10^{-3}	1.365×10^{-6}	1.394×10^{-6}	1.272×10^{-6}	2.724×10^{-6}	2.387×10^{-2}	3.959×10^{-2}

Example 2. *Consider the following nonlinear IVP of ODE [7]*

$$\begin{gathered} u'' + uu' = 2\cos(x^2) + x\sin(2x^2) - 4x^2\sin(x^2), \quad 0 \le x \le 1, \\ u(0) = 0, \quad u'(0) = 0, \end{gathered} \tag{62}$$

admiting the exact solution $u(x) = \sin(x^2)$.

As in the preceding example 1, we start off by expressing the governing model in the ADM operator form

$$u = L^{-1}(2\cos(x^2) + x\sin(2x^2) - 4x^2\sin(x^2)) - L^{-1}(uu'), \tag{63}$$

where the inverse operator takes the expression $L^{-1}(.) = \int_0^x \int_0^x (.)dxdx$ and $N(u) = uu'$. Substituting Eqs. (5) and (6) into Eq. (63), we get the following recursive solution

$$\begin{aligned} &u_0 = L^{-1}(2cos(x^2) + xsin(2x^2) - 4x^2\sin(x^2)), \\ &u_{n+1} = -L^{-1}A_n(u_0, u_1, \cdots), n \ge 0, \end{aligned}$$

From Eq. (7), the nonlinear term $N(u) = uu'$ is also expressed through the following Adomian polynomials

$$\begin{aligned}
A_0 &= u_0 u_0', \\
A_1 &= u_1 u_0' + u_0 u_1', \\
A_2 &= u_2 u_0' + u_1 u_1' + u_0 u_2', \\
A_3 &= u_3 u_0' + u_2 u_1' + u_1 u_2' + u_0 u_3', \\
&\vdots
\end{aligned}$$

Thus, without further delay, by the same procedure as in Example 1, we present the respective solution via the application of the proposed modification methods for m = 6 as follows

$$\begin{aligned}
u_t(x) &= \sum_{n=0}^{6} u_n(x) = x^2 - \frac{1}{6}x^6 + \frac{1}{54}x^9 - \frac{1}{648}x^{12} + \cdots, \\
u_P(x) &= \sum_{n=0}^{6} u_n(x) = 1.000731859x^2 - 0.01306146233x^3 + 0.08294338582x^4 + \cdots, \\
u_T(x) &= \sum_{n=0}^{6} u_n(x) = 1.000309776x^2 - 0.009680124167x^3 + 0.07289912332x^4 + \cdots, \\
u_L(x) &= \sum_{n=0}^{6} u_n(x) = 1.964424074x^2 - 1.747651757x^3 + 0.5839657700x^4 + \cdots \qquad (64) \\
u_H(x) &= \sum_{n=0}^{6} u_n(x) = 1.501201152x^2 + 0.1619524511x^3 - 0.7135433355x^4 + \cdots, \\
u_g(x) &= \sum_{n=0}^{6} u_n(x) = 1.001214606x^2 - 0.01625100983x^3 + 0.0916858975x^4 + \cdots \\
u_j(x) &= \sum_{n=0}^{6} u_n(x) = 1.001718846x^2 - 0.01916817000x^3 + 0.09913448668x^4 + \cdots,
\end{aligned}$$

Similarly, we report in Table 2 the absolute error differences between the exact solution $u(x)$ and the respective solutions by the proposed modification methods.

Table 2: Absolute error comparisons via the proposed modifications for Example 2

x	$\lvert u(x) - u_t(x)\rvert$	$\lvert u(x) - u_P(x)\rvert$	$\lvert u(x) - u_T(x)\rvert$	$\lvert u(x) - u_g(x)\rvert$	$\lvert u(x) - u_j(x)\rvert$
0	0	0	0	0	0
0.25	6.259×10^{-8}	1.174×10^{-6}	2.881×10^{-6}	4.032×10^{-6}	1.150×10^{-5}
0.50	2.754×10^{-5}	2×10^{-7}	3.136×10^{-6}	9.808×10^{-6}	2.402×10^{-5}
0.75	8.542×10^{-4}	1.10×10^{-6}	3.980×10^{-6}	1.448×10^{-5}	3.430×10^{-5}
1	8.087×10^{-3}	3.201×10^{-7}	5.459×10^{-6}	1.631×10^{-5}	3.972×10^{-5}

x	$\|u(x)-u_L(x)\|$	$\|u(x)-u_H(x)\|$
0	0	0
0.25	3.494×10^{-2}	3.088×10^{-2}
0.50	5.352×10^{-2}	9.762×10^{-2}
0.75	2.793×10^{-2}	1.235×10^{-1}
1	1.948×10^{-1}	5.025×10^{-2}

Example 3. *Let us consider the following nonlinear IVP of ODE [11]*

$$\begin{aligned} u''+u'+2xu^3 &= 2xe^{-3x}, \quad 0\leq x\leq 1,\\ u(0)=1, &\quad u'(0)=-1, \end{aligned} \tag{65}$$

having the exact solution $u(x)=e^{-x}$.

Firstly, we express the equation in the following operator form

$$u=1-x+L^{-1}(2xe^{-3x})-L^{-1}(u')-2L^{-1}(xu^3) \tag{66}$$

where $L^{-1}(.)=\int_0^x\int_0^x(.)dxdx$ and $N(u)=u^3$. Substituting Eqs. (5) and (6) into Eq. (66), we get the following recursive solution

$$\begin{aligned} &u_0=1-x+L^{-1}(2xe^{-3x}),\\ &u_{n+1}=-L^{-1}(u_n^{'})-2L^{-1}xA_n(u_0,u_1,\cdots),n\geq 0, \end{aligned}$$

The nonlinear term $N(u)=u^3$ is given as in Example 1.

Thus, by the same procedure as in Example 1, we present the respective solution via the application of the proposed modification methods for $m=6$ as follows

$$\begin{aligned} u_t(x) &= \sum_{n=0}^{6}u_n(x)=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\frac{1}{720}x^6-\frac{1}{5040}x^7+\frac{81}{1120}x^8+\cdots,\\ u_P(x) &= \sum_{n=0}^{6}u_n(x)=1-x+0.5000639003x^2-0.1679247192x^3+0.05074785700x^4+\cdots,\\ u_T(x) &= \sum_{n=0}^{6}u_n(x)=1-x+0.5000267045x^2-0.1675800359x^3+0.04949342753x^4+\cdots,\\ u_L(x) &= \sum_{n=0}^{6}u_n(x)=1-x+0.5778656006x^2-0.4966684977x^3+0.5997034709x^4+\cdots,\\ u_g(x) &= \sum_{n=0}^{6}u_n(x)=1-x+0.5001073089x^2-0.1682601554x^3+0.05188023933x^4+\cdots,\\ u_j(x) &= \sum_{n=0}^{6}u_n(x)=1-x+0.500154036x^2-0.1685812115x^3+0.0529062453x^4+\cdots, \end{aligned} \tag{67}$$

Accordingly, we report in Table 3 the absolute error differences between the exact solution $u(x)$ and the respective solutions by the proposed modification methods.

Table 3: Absolute error comparisons via the proposed modifications for Example 3

x	$\lvert u(x)-u_t(x)\rvert$	$\lvert u(x)-u_P(x)\rvert$	$\lvert u(x)-u_T(x)\rvert$	$\lvert u(x)-u_g(x)\rvert$	$\lvert u(x)-u_j(x)\rvert$	$\lvert u(x)-u_L(x)\rvert$
0	0	0	0	0	0	0
0.25	9.764×10^{-7}	6.150×10^{-8}	1.585×10^{-7}	3.051×10^{-7}	8.615×10^{-7}	1.424×10^{-3}
0.50	2.223×10^{-4}	9.90×10^{-8}	3.338×10^{-7}	5.373×10^{-7}	1.497×10^{-6}	4.071×10^{-4}
0.75	5.086×10^{-3}	6.693×10^{-6}	6.967×10^{-6}	5.916×10^{-6}	4.718×10^{-6}	2.418×10^{-3}
1	4.545×10^{-2}	2.439×10^{-4}	2.443×10^{-4}	2.431×10^{-4}	2.417×10^{-4}	4.348×10^{-3}

Example 4. *Consider the following nonlinear IVP [11]*

$$\begin{gathered} u''+u'-uu'=(-2+4x^2-2x)e^{-x^2}+2xe^{-2x^2}, \quad 0\le x\le 1, \\ u(0)=1, \quad u'(0)=0, \end{gathered}$$

that admits the exact solution $u(x)=e^{-x^2}$.

We start by expressing the model in an operator notation as follows

$$u=1+L^{-1}((-2+4x^2-2x)e^{-x^2}+2xe^{-2x^2})-L^{-1}(u')+L^{-1}(uu^{'}), \tag{68}$$

where $L^{-1}(.)=\int_0^x\int_0^x(.)dxdx$ and $N(u)=uu'$. Substituting Eqs. (5) and (6) into Eq. (68), we get the following recursive solution

$$\begin{aligned} u_0&=1+L^{-1}(-2+4x^2-2x)e^{-x^2}+2xe^{-2x^2}), \\ u_{n+1}&=-L^{-1}(u_n^{'})+L^{-1}A_n(u_0,u_1,\cdots),n\ge 0, \end{aligned}$$

The nonlinear term $N(u)=uu'$ is given as in Example 2.

Also, without lost of generality, by the same procedure as in Example 1, we present the respective solutions via the application of the proposed modification methods for $m=6$

as follows

$$
\begin{aligned}
u_t(x) &= \sum_{n=0}^{6} u_n(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{7}{216}x^9 + \cdots, \\
u_P(x) &= \sum_{n=0}^{6} u_n(x) = 1 - 0.9999424490x^2 - 0.0009236505467x^3 + 0.5047677208x^4, \\
u_T(x) &= \sum_{n=0}^{6} u_n(x) = 1 - 0.9999762780x^2 - 0.0006668175867x^3 + 0.504082822x^4 + \cdots, \\
u_L(x) &= \sum_{n=0}^{6} u_n(x) = 1 - 1.224806499x^2 + 0.7301686677x^3 - 0.1966821240x^4 + \cdots, \\
u_H(x) &= \sum_{n=0}^{6} u_n(x) = 1 - 0.8811213405x^2 - 0.07972377693x^3 + 0.2955485375x^4 + \cdots, \\
u_g(x) &= \sum_{n=0}^{6} u_n(x) = 1 - 0.9999022015x^2 - 0.001176415607x^3 + 0.505394132x^4 + \cdots, \\
u_j(x) &= \sum_{n=0}^{6} u_n(x) = 1 - 0.9998581770x^2 - 0.001420987833x^3 + 0.5059695492x^4 + \cdots,
\end{aligned}
\tag{69}
$$

Therefore, we report in Table 4 the absolute error differences between the exact solution $u(x)$ and the respective solutions by the proposed modification methods.

Table 4: Absolute error comparisons via the proposed modifications for Example 4

x	$\lvert u(x) - u_t(x)\rvert$	$\lvert u(x) - u_P(x)\rvert$	$\lvert u(x) - u_T(x)\rvert$	$\lvert u(x) - u_g(x)\rvert$	$\lvert u(x) - u_j(x)\rvert$
0	0	0	0	0	0
0.25	5.058×10^{-7}	1.208×10^{-7}	2.654×10^{-7}	3.305×10^{-7}	1.001×10^{-6}
0.50	9.578×10^{-5}	2.950×10^{-8}	2.344×10^{-7}	8.493×10^{-7}	2.121×10^{-6}
0.75	1.663×10^{-3}	1.045×10^{-7}	3.259×10^{-7}	1.272×10^{-6}	3.070×10^{-6}
1	1.011×10^{-2}	6.420×10^{-8}	5.574×10^{-7}	1.371×10^{-6}	3.558×10^{-6}

x	$\lvert u(x) - u_L(x)\rvert$	$\lvert u(x) - u_H(x)\rvert$
0	0	0
0.25	5.183×10^{-3}	5.497×10^{-3}
0.50	2.399×10^{-3}	1.126×10^{-2}
0.75	9.282×10^{-3}	3.958×10^{-3}
1	1.642×10^{-2}	1.438×10^{-2}

5. Conclusion

The present study proposed different modification methods for the standard Adomian Decomposition Method (ADM) to tackle a variety of problems of mathematical physics. These modification methods were based on the application of orthogonal polynomials that play vital parts in numerical methods, as well as in approximation theories. Furthermore, the study also scrutinized four different test nonlinear inhomogeneous IVPs, and distinctively examined their respective absolute error differences. Notably, the proposed modification methods based on the application of Legendre's orthogonal polynomials $u_P(x)$ was noted to have the least error among its contending companions in three of the test problems; it also performed outstandingly in the remaining problem. Finally, as certain computational benefits of the proposed modification are realized with regards high-level of accuracy and fewer computational steps in the study, it is therefore recommended to implement these methods on high-order IVPs arising in the general science and engineering applications.

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