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J-Open Independent Sets in Graphs

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Abstract. Let G be a graph with vertex and edge-sets $V(G)$ and $E(G)$, respectively. Then $O \subseteq V(G)$ is called a J-open independent set of G if O is a singleton set or O is an independent set of G and for every $a, b \in V(G)$, $N_G(a)\backslash N_G(b) \neq \varnothing$ and $N_G(b)\backslash N_G(a) \neq \varnothing$. The maximum cardinality of a J-open independent set of G, denoted by $\alpha_J(G)$, is called the J-open independence number of G. In this paper, we introduce this parameter and we show that it is always less than or equal to the standard independence (resp. J-total domination) parameter of a graph. In fact, their differences can be made arbitrarily large. In addition, we show that J-open independence parameter is incomparable with hop independence parameter. Moreover, we derive some formulas and bounds of the parameter for some classes of graphs and the join of two graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: J-open set, J-open independent set, J-open independence number

1. Introduction

An independent set in a graph is a subset of a vertex-set of a graph where each pair of distinct vertices are not of distance one. In other words, it is a set of vertices that are not connected by an edge. This concept is fundamental in graph theory and has a wide range of applications in various field. Some studies on independent sets in graphs can be found in [2–4, 12, 17, 18].

In 2022, hop independent set in a graph and its parameter was introduced by Hassan et al. [8]. They defined a set $S \subseteq V(G)$ is a hop independet set of G if any two distinct vertices in S are not at a distance two from each other, that is, $d_G(u, w) \neq 2$ for any distinct vertices $u, w \in S$. The maximum cardinality of a hop independent set of G,

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denoted by $\alpha_h(G)$, is called the hop independence number of G. They have shown that any maximum hop independent set S of G is always a hop dominating, that is, the hop independence number of a graph is always greater than or equal to the hop domination parameter. Moreover, they derived some bounds and formulas for some special graphs and graphs under some binary operations. Some studies on variants of hop independent sets and other hop-related concepts can be found in [1, 5–7, 9, 11, 13–16]

In this paper, we introduce new independence parameter called J-open independence. We investigate this concept on some families of graphs and on the join of two graphs. We believe, the results of this study could led to other interesting research directions in the future.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. The greatest distance between any two vertices in G, denoted by $diam(G)$, is called the *diameter* of G.

Two vertices x, y of G are *adjacent*, or *neighbors*, if xy is an edge of G. The *open neighborhood* of x in G is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of X in G is the set $N_G[X] = N_G(X) \cup X$.

A subset $D = \{d_1, d_2, \cdots, d_m\}$ of vertices of G is called a J-open set if $N_G(d_i) \setminus N_G(d_j) \neq \varnothing$ for every $i \neq j$, where $i, j \in \{1, 2, \ldots, m\}$. A J-open set is called a J-total dominating set of G if $D = \{d_1, d_2, \ldots, d_m\}$ is a total dominating set of G. The J-total domination number of G, denoted by $\gamma_{Jt}(G)$, is the maximum cardinality of a J-total dominating set of G [10].

A *path graph* is a non-empty graph with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$, where the $x_i^{'}s$ are all distinct. The path of order n is denoted by P_n. If G is a graph and u and v are vertices of G, then a path from vertex u to vertex v is sometimes called a *u-v path.* The *cycle graph* C_n is the graph of order $n \geq 3$ with vertex-set $\{x_1, x_2, \ldots, x_n\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. Let G and H be any two graphs. The *join* of G and H, denoted by $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

A subset S of $V(G)$ is called a *independent* if for every pair of distinct vertices $x, y \in S$, $d_G(x, y) \neq 1$. The maximum cardinality of a independent set in G, denoted by $\alpha(G)$, is called the *independence* number of G. Any independent set S with cardinality equal to $\alpha(G)$ is called an α-set of G.

A subset S of $V(G)$ is called a *hop independent* set of G if any two distinct vertices in S are not at distance two from each other, that is, $d_G(v, w) \neq 2$ for any two distinct vertices $v, w \in S$. The *hop independence number* of G, denoted by $\alpha_h(G)$, is the maximum cardinality of a hop independent set of G [8].

3. Results

We begin this section by introducing the concept of J-open independence in graphs.

Definition 1. Let G be a graph with vertex and edge-sets $V(G)$ and $E(G)$, respectively. Then $O \subseteq V(G)$ is called a J-open independent set of G if O is a singleton set or O is an independent set of G and for every $a, b \in V(G)$, $N_G(a)\backslash N_G(b) \neq \varnothing$ and $N_G(b)\backslash N_G(a) \neq \varnothing$. The maximum cardinality of a J-open independent set of G, denoted by $\alpha_J(G)$, is called the J-open independence number of G.

Example 1. Consider the graph $G = P_4$ in Figure 1 below. Let $O = \{a, d\}$. Then $d_G(a, d) = 3$. Thus, O is an independent set of G. Observe that $N_G(a) = \{b\}$ and $N_G(d) = \{c\}$. Hence, $N_G(a)\backslash N_G(d) = \{b\}\backslash\{c\} = \{b\}$, and $N_G(d)\backslash N_G(a) = \{c\}\backslash\{b\} = \{c\}$. Therefore, O is a J-open independent set of G. Moreover, it can be verified that $\alpha_J(G) = 2$.

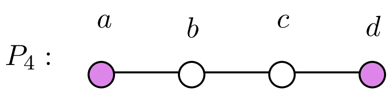


Figure 1: Graph $G = P_4$ with $\alpha_J(P_4) = 2$

Remark 1. *Let G be a graph. Then each of the following holds:*

(i) A J-open set O of G may not be an independent set of G.

(ii) An independent set of G may not be a J-open set of G.

The remark above says that the definition of a J-open independence makes sense.

4. Relationships of J-Open Independence and Independence Parameters

Theorem 1. *Let G be a graph. Then*

(i) $\alpha_J(G) \leq \alpha(G)$*; and*

(ii) $1 \leq \alpha_J(G) \leq |V(G)| - 1$.

Proof. (i) Let G be a graph and let O be a maximum J-open independent set of G. Then O is an independent set of G and $\alpha_J(G)=|O|$. Since $\alpha(G)$ is the maximum cardinality among all independent sets in G, if follows that $\alpha(G) \geq |O|=\alpha_J(G)$.

(ii) Let G be a graph and let $V(G) = \{a_1, a_2, ..., a_n\}$. Then $\{a_1\}$ is a J-open independent set of G. Thus, $\alpha_J(G) \geq |\{a_1\}|=1$. To show that $\alpha_J(G) \leq |V(G)| - 1$. Suppose that

G is connected. Let $a_i, a_j \in V(G)$ such that $d_G(a_i, a_j) = 1$ for some $i, j \in \{1, 2, \ldots, n\}$. Then a_i and a_j cannot be both in an independent set S of G. Hence, if $a_i \in S$, then $a_j \notin S$ or if $a_j \in S$, then $a_i \notin S$. Thus, $\alpha(G) \leq |V(G)| - 1$. By (i), $\alpha_J(G) \leq |V(G)| - 1$.

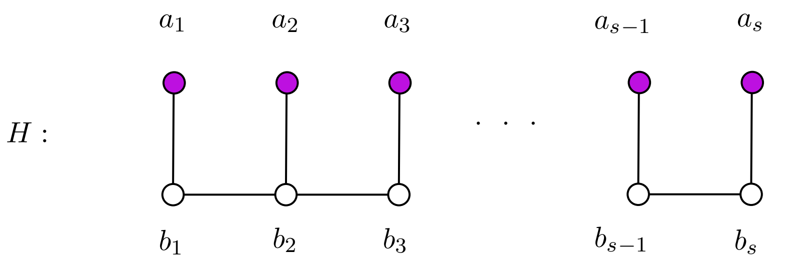
Now, suppose that G is disconnected. Let $Q_1, ..., Q_k, k \geq 2$ be components of G. Since $G \neq \overline{K}_n$, it follows that Q_i is non-trivial for each $i \in \{1, ..., k\}$. Thus, $\alpha(Q_i) \leq |V(Q_i)| - 1$ for each $i \in \{1, ..., k\}$. Therefore,

$$\alpha(G) = \alpha(Q_1) + ... + \alpha(Q_k) = |V(Q_i)| - 1 + ... + |V(Q_k)| - 1 = |V(G)| - k \leq |V(G)| - 1.$$

Consequently, $\alpha_J(G) \leq |V(G)| - 1$ by (i). □

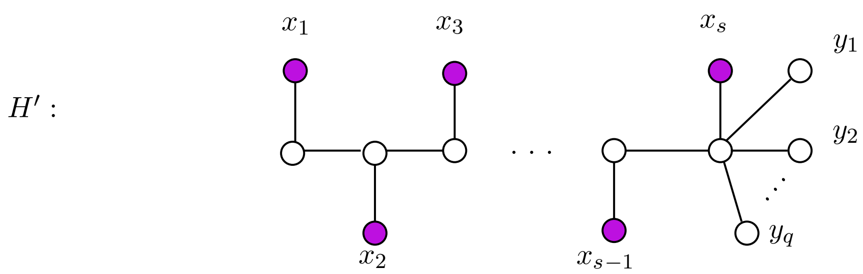
Theorem 2. *Let s, t be positive integers such that $1 \leq s \leq t$. Then there exists a connected graph H such that $\alpha_J(H)$=s and $\alpha(H)$=t. In other words, $\alpha(H) - \alpha_J(H)$ can be made arbitrarily large.*

Proof. Consider the following two cases.
Case 1. $s = t$
Consider the graph H below.



Let $O = \{a_1, a_2, \cdots, a_s\}$. Then O is both a maximum J-open independent and maximum independet set of H. Thus, $\alpha_J(H) = G = \alpha(H)$.

Case 2: $s < t$ Let $q = t - s$ and consider the graph H' below.



Let $O_1 = \{x_1, x_2, \cdots, x_s\}$ and $O_2 = O_1 \cup \{y_1, y_2 \cdots, y_q\}$ Then O_1 and O_2 are maximum J-open independent and maximum independent sets of H', respectively. Therefore, $\alpha_J(H') = s$ and $\alpha(H') = s + q = t$. Consequently, $\alpha_J(H') = s < t = \alpha(H')$. □

5. Relationships of J-Open Independence and J-Total Domination Parameters

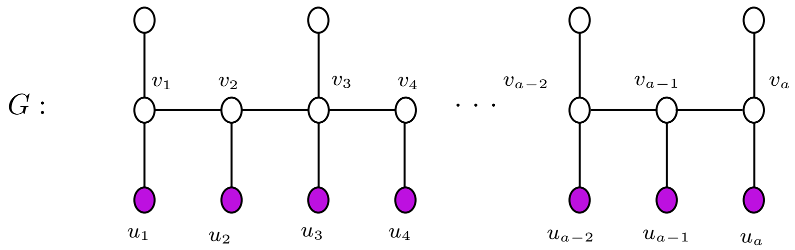
Proposition 1. *Let G be a graph such that $G \neq \overline{K}_n$. Then $\alpha_J(G) \leq \gamma_{Jt}(G)$, and its bound is tight.*

Proof. Let G be a graph such that $G \neq \overline{K}_n$. and let Q be a maximum J-open independent set of G. Then Q is a J-open set in G. Since $\gamma_{Jt}(G)$ is a maximum cardinality of a J-open set of G, it follows that $\gamma_{Jt}(G) \geq |Q| = \alpha_J(G)$.

For the tightness, consider P_4. Then $\alpha_J(P_4) = 2 = \gamma_{Jt}(P_4)$. □

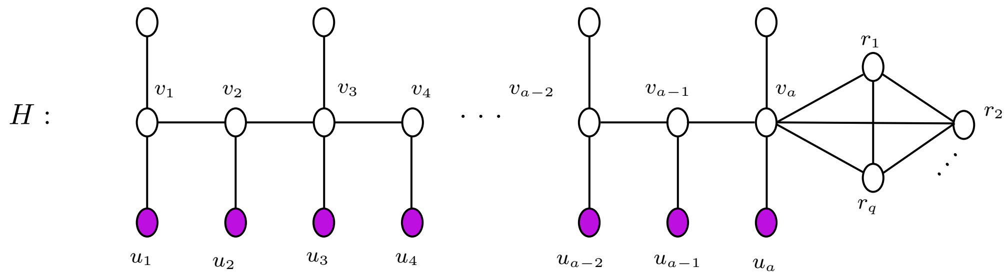
Theorem 3. *Let a and b be positive integers such that $1 \leq a \leq b$. Then there exists a connected graph G such that $\alpha_J(G) = a$ and $\gamma_{Jt}(G) = b$.*

Proof. Suppose that $a = b$. Consider the graph G below.



Let $O_1 = \{u_1, u_2, \ldots, u_a\}$ and $O_2 = \{v_1, v_2, \ldots, v_a\}$, Then O_1 and O_2 are maximum J-open independent and maximumu J-total dominating sets of G, respectively. Therefore, $\alpha_J(G) = a = \gamma_{Jt}(G)$.

Now suppose that $a < b$. Let $q = b - a$ and consider the graph H below, where $\{v_a, r_1, r_2, \ldots, r_q\}$ is a clique.

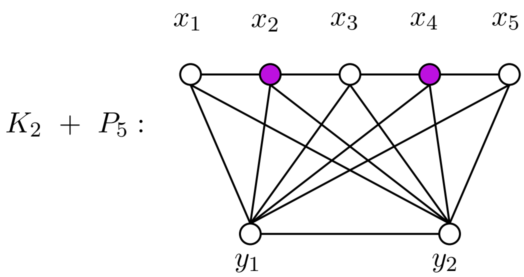


Let $Q_1 = \{u_1, u_2, \ldots, u_a\}$ and $Q_2 = \{v_1, v_2, \ldots, v_a, r_1, \ldots, r_q\}$. Then Q_1 and Q_2 are maximum J-open independent and maximum J-total dominating sets of H, respectively. Hence, $\alpha_J(H) = a$ and $\gamma_{Jt}(H) = a + q = b$. □

6. The Incomparability of J-Open Independence and Hop Independence Parameters

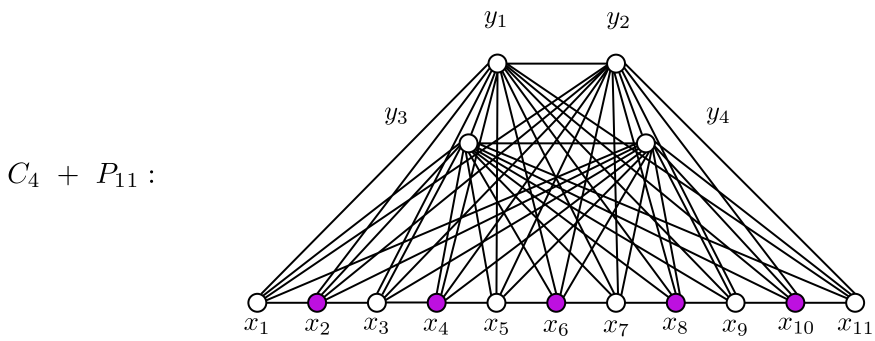
Remark 2. *The hop independence and J-open independence parameters of a graph are incomparable.*

To see this, consider the graph $K_2 + P_5$ below.



Let $O_1 = \{x_2, x_4\}$ and $O_2 = \{x_1, x_2, y_1, y_2\}$. Then O_1 and O_2 are maximum J-open independent and maximum hop independent sets of $K_2 + P_5$, respectively. Hence, $\alpha_J(K_2 + P_{11}) = 2$ and $\alpha_h(K_2 + P_5) = 4$.

Next, consider the graph $C_4 + P_{11}$ below.



Let $O' = \{y_1, y_2, x_{10}, x_{11}\}$ and $O'' = \{x_2, x_4, x_6, x_8, x_{10}\}$. Then O' and O'' are maximum hop independent and J-open independent sets of $C_4 + P_{11}$ respectively. Thus, $\alpha_h(C_4 + P_{11}) = 4$ and $\alpha_J(C_4 + P_{11}) = 5$. □

7. J-Open Independence in the Join of Two Graphs

Theorem 4. *Let G and H be a graphs. A subset O of a vertex-set $V(G+H)$ of $G+H$ is a J-open independent set of $G + H$ if and only if O satisfies one of the following conditions:*

(i) O is a J-open independent set of G.

(ii) O is a J-open independent set of H.

Proof. Let O be a J-open independent set of $G + H$. Then either $O \subseteq V$(H) or $O \subseteq V(G)$. If $O \subseteq V$(G), then O is a J-open independent set of G. Thus, (i) holds. If $O \subseteq V$(H), then O is a J-open indepedent set of H. Hence, (ii) holds.

Conversely, suppose that (i) holds. Since $V(G) \subseteq V$(G+H), O is a J-open independent set of $G+H$. Assume that(ii) holds. Since $V(H) \subseteq V$(G+H), it follows that O is a J-open independent set of $G + H$. □

Corollary 1. *Let G and H be graphs. Then*

$$\alpha_J(G + H) = max\{\alpha_J(G), \alpha_J(H)\}.$$

Proof. Let O be a maximum J-open independent set of $G + H$. Then by Theorem 4, O is either a J-open independent set of G or H. If O is a J-open independent set of G, then $\alpha_J(G + H) = |O| \leq \alpha_J(G)$. If O is a J-open independent set of H, then $\alpha_J(G + H) = |O| \leq \alpha_J(H)$.

On the other hand, suppose that O is a maximum J-open independent set of G. Then by Theorem 4, O is a J-open independent set of $G + H$. Thus,

$$\alpha_J(G) = |O| \leq \alpha_J(G + H).$$

Similarly, If O is a maximum J-open independent set of H, then O is a J-open independent set of $G + H$. Hence, $\alpha_J(H) = |O| \leq \alpha_J(G + H)$. Consequently,

$$\alpha_J(G + H) = max\{\alpha_J(G), \alpha_J(H)\}.$$

□

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