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On the self-injectivity and CM-free of an Artin algebra with radical cubic zero

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Abstract. In this paper, we succeed in proving that a connected Artin algebra whose Jacobson cubic radical is zero with each simple module and each Gorenstein projective module having the square radical of the cover projective of its first syzygy to be zero and verifying the coincidence covers, is either self-injective or CM-free.

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Introduction

Let K be a commutative ring and A an Artin algebra over K of Jacobson radical with nilpotency index 3, $\mathrm{mod}(A)$ will be the category of finitely generated modules over A and all modules in this work are in left in the category $\mathrm{mod}(A)$, we know that the non-isomorphic simple A-modules are in finite number and denoted $(S_i)_{i\in I}$ allowing us to define the extension quiver denoted Q_A according to the condition $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$ by defining an arrow from S_i to S_j thus A is connected if and only if the algebra Q_A is connected.
We also recall that $\mathrm{Ext}^n_A(M, N) = H^n(\mathrm{Hom}_A(P_*, N))$ is the nth cohomology of the cochain complex of K-modules $\mathrm{Hom}_A(P_*, N)$ which is :

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(P_0, N) \longrightarrow \mathrm{Hom}_A(P_1, N) \longrightarrow \mathrm{Hom}_A(P_2, N) \longrightarrow \cdots$$

where $\mathrm{Hom}_A(P_n, N)$ is in degree n and P_* the complex deduced from the projective resolution of M, where A is a K-algebra with K is a commutative ring.

In 2012, Xiao-Wu Chen classified the algebras whose radical square is zero and showed

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that they are either self-injective or CM-free, see [5]. However, this result has never been proven in the case of cube radical 0 and the same author pointed out that it is false using a counter-example with a non-zero square radical algebra.

One year later, Luo Rong proved that all local algebras with radical cubic zero are PCM-free, see [2] .

An important invariant in our subject is the radical square of the projective cover of the first syzygy of each simple A-module and each Gorenstein-projective A-module that we consider here to be 0, so by a formula characterizing the calculation of the radical of syzygy of an indecomposable A-module of Lowey length 2, that is to say from the square radical 0, we will extend this calculation to a Gorenstein-projective A-module and by characterization of the latter, we construct a family $(S_i)_{i\in J}$ of simple A-modules non-projective and Gorenstein-projective and the family $(P_i)_{i\in J}$ of projective modules such that each P_i is associated to S_i. The quiver Q_A will be connected, which implies the self-injectivity for the algebra A with zero radical cubic.

Since that assumption is not always true, we succeed by using our condition, to establish that the property of auto-injectivity and CM-free remain valid. If M is an A-module, we note $\Omega(M)$ as the first syzygy of M and $P(M)$ as the projective cover of M with $\Omega(M)$ is the kernel of the essential epimorphism: $P(M) \longrightarrow M$ ie $P(M)/\mathrm{rad}(M) \simeq M/\mathrm{rad}(M)$ see [4], similarly we note $\Omega^2(M) = \Omega(\Omega(M))$.

In the upcoming parts of the paper, we start by discussing some properties of the Gorenstein-projective modules, then in the case of an Artin algebra A whose Jacobson cubic radical is zero while considering a non-projective and indecomposable Gorenstein-projective, we characterize the radical of syzygy of an A-module. Finally, we prove by using our condition, that the properties of self-injectivity and CM-free remain true.

1. Gorenstein-projective modules

1.1 Definition. Recall that the infinite sequence of A- modules

$$(C_n)_{n\in\mathbb{Z}} : \cdots \longrightarrow C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} \cdots$$

is called complex and denoted $C_\star$ if $Im(d^{n-1}) \subseteq Ker(d^n)$.
Denote $B^n(C_\star) = Im(d^{n-1})$, $Z^n(C_\star) = Ker(d^n)$ and $H^n(C_\star) = Z^n(C_\star)/B^n(C_\star)$ where $Z^n(C_\star)$ is called cocycle and the complex $C_\star$ is called acyclic if it is an exact sequence, i.e. $H^n(C_\star) = 0$ for all n.

1.2 Remark. If the C_n's are projective for all n and $\mathrm{Hom}(C_\star, A)$ is acylic, then $C_\star$ is called totally acyclic.

1.3 Definition. An A-module M in $\mathrm{mod}(A)$ is said Gorenstein-projective provided that there is a totally acyclic complex $(P_\star)$ of projective modules such that its 0-th cocycle $Z^0(P_\star)$ is isomorphic to M.

1.4 Example. - Any A-module P projective is a Gorenstein-projective module; just consider the exact sequence $P_\star$:

$$\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{Id_P} P \longrightarrow 0 \longrightarrow \cdots$$

and we have $Z^0(P_\star) \simeq P$.
- Over the algebra $\mathbb{Z}/4\mathbb{Z}$, the complex:

$$\cdots \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots$$

defined by multiplication by 2 is acyclic and it remains exact by applying the functor $\mathrm{Hom}(-, P)$ with P a projective module, then $2\mathbb{Z}/4\mathbb{Z}$ is a Gorenstein-projective but not projective as $\mathbb{Z}/4\mathbb{Z}$-module.

1.5 Remark. The full subcategory of the finite type A-module category constituted by the Gorenstein-projective modules is denoted A-Gproj and the subcategory of projective A-modules is included in A-Gproj.
Denote by ${}^\perp A$ the full subcategory of A-mod consisting by modules M such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \geq 1$.

Recall the following lemma, see [3].

1.6 Lemma. *Let $(P_\star)$ be a complex of projective A-modules. Then the following statements are equivalent:*
(1) the complex $(P_\star)$ is totally acyclic.
(2) the complex $(P_\star)$ is acyclic and each cocycle $Z^i(P_\star)$ lies in ${}^\perp A$;
(3) the complex $\mathrm{Hom}((P_\star), A)$ is totally acyclic.

We note we will need the corollary and proposition stated in [1] thereafter which says the following,

1.7 Corollary. *If M is a Gorenstein-projective module, then $\mathrm{Ext}^1_A(M, L) = 0$ for all module L of finite projective dimension.*

1.8 Proposition. *Let M in mod(A), then M is a Gorenstein-projective module if and only if there exists a long exact sequence:*

$$0 \longrightarrow M \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

with P_i are the projective modules and each cocycle in ${}^\perp A$.

Proof. By definition M is in A-Gproj then, there exists a totally acyclic complex $P_\star$ such that $M \simeq Z^0(P_\star)$ and we have:

$$0 \longrightarrow M \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

.
Now for the reciprocal, we consider

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

the projective resolution of M and by spiling the two resolutions, we get an acyclic complex P such that $M \simeq Z^0(P_\star)$.

1.9 PROPOSITION. *Let*

$$\varepsilon: \ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a short exact sequence of A-modules. Then we have the following statements:
(1) if X, Z are Gorenstein-projective, then so is Y.
(2) if Y, Z are Gorenstein-projective, then so is X.

Proof.

Since X and Z are Gorenstein-projective modules, then by Proposition 1.8, we have two monomorphisms:

$$0 \longrightarrow X \xrightarrow{i_X} X'$$

and

$$0 \longrightarrow Z \xrightarrow{i_Z} Z'$$

such that X' and Z' are projective modules and the cokernels X_1 and Z_1 of i_X and i_Z respectively are Gorenstein-projective.
Since by Corollary 1.7, $\mathrm{Ext}^1_A(Z, X') = 0$ and by applying the functor $\mathrm{Hom}(-, X')$ to ε , we infer that the induced map:

$$\mathrm{Hom}_A(Y, X') \xrightarrow{\mathrm{Hom}_A(f,X')} \mathrm{Hom}_A(X, X')$$

is epimorphism, then there exists a morphism $a : Y \longrightarrow X'$ such that: $a \circ f = Id_X$ therefore we have the following exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_X} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \downarrow \begin{pmatrix} a \\ i_Z \circ g \end{pmatrix} & & \downarrow{\scriptstyle i_Z} & & \\
0 & \longrightarrow & X' & \longrightarrow & X' \bigoplus Z' & \xrightarrow[\begin{pmatrix} 0 & 1 \end{pmatrix}]{} & Z' & \longrightarrow & 0
\end{array}$$

By the Snake lemma, we have a short exact sequence:
$0 \longrightarrow coker(i_X) \longrightarrow Y_1 \longrightarrow coker(i_Z) \longrightarrow 0$

Therefore $0 \longrightarrow X_1 \longrightarrow Y_1 \longrightarrow Z_1 \longrightarrow 0$ where $Y_1 = \begin{pmatrix} a \\ i_Z \circ g \end{pmatrix}$ and since X_1 and Z_1 are in ${}^\perp A$ we infer that Y_1 in ${}^\perp A$ and by iterating this process and using Proposition 1.8, we show that Y is Gorenstein-projective.
For a second statement, we take the exact sequence:

$$0 \longrightarrow Z' \longrightarrow P \longrightarrow Z \longrightarrow 0$$

such that P is projective and Z' is Gorenstein-projective, and by the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z' & = & Z' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & U & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We have U a Gorenstein projective module by using (1) in a middle column and by the first row,$U \simeq X \bigoplus P$ since P is projective we infer that X is Gorenstein projective.

Recall for a left A-module ${}_AM$ over an Artin algebra, there exists a projective module denoted $P(M)$ such that the kernel of the epimorphism $f : P(M) \longrightarrow M$ is superfluous in $P(M)$ that to say $\ker(f) \subseteq \operatorname{rad}(P(M))$. Also, this epimorphism is called the projective cover of M and we have a short exact sequence:

$$0 \longrightarrow \Omega(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$$

1.10 Corollary. *Let M be an A-module.*
If M is Gorenstein-projective, then so are $\Omega^i(M)$ for $i \geq 1$.

Proof. Take a short exact sequence :

$$0 \longrightarrow \Omega(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$$

by applying Proposition 1.9 and since $P(M)$ is projective, then $\Omega(M)$ are likewise, and by iterating this process in the short exact sequence:

$$0 \longrightarrow \Omega^{i+1}(M) \longrightarrow P(\Omega^i(M)) \longrightarrow \Omega^i(M) \longrightarrow 0$$

we have the result.

For the following work, we need the following lemma, see [1, 2, 6].

1.11 Lemma. *Let M be a non-projective and indecomposable Gorenstein-projective A-module, the A-module $\Omega(M)$ is also non-projective indecomposable, and Gorenstein-projective.*

2. Radical of syzygy of an A-module

In what follows, A is an Artin algebra whose Jacobson cubic radical is zero.

2.1 Lemma. *Let M be a non-projective module in* $\text{mod}A$ *with* $\text{rad}^2(M) = 0$ *and* ΩM *indecompasable. If* $f : P(M) \to M$ *is a projective cover of* M*, then:*
$\text{rad}(\Omega M) = \text{rad}^2(P(M))$.

Proof.
Let $f : P \to M$ be a projective cover of M. Then, $\Omega M \subseteq \text{rad}(P)$, and hence, $\text{rad}(\Omega M) \subseteq \text{rad}^2(P)$. Since $\text{rad}^2(M) = 0$, $\text{rad}^2(P) \subseteq \Omega M$.

Suppose that $\text{rad}^2(P) \nsubseteq \text{rad}(\Omega M)$. Then; there exists a maximal submodule L of ΩM such that $\text{rad}^2(P) \nsubseteq L$.
Since $\text{rad}^3(A) = 0$, $\text{rad}^2(P)$ is semi-simple. Thus, $S \nsubseteq L$ where S is some simple submodule of $\text{rad}^2(P)$, and consequently, $\Omega M = S \oplus L$.
Since now ΩM is indecomposable, there is a contradiction and the proof of the lemma is complete.

2.2 Corollary. *Let M be a non-projective and indecomposable Gorenstein-projective A-module in* $\text{mod}A$ *with* $\text{rad}^2(M) = 0$*, then:* $\text{rad}(\Omega M) = \text{rad}^2(P(M))$

3. The self-injectivity and CM-free algebras

Recall that an Artin algebra A is said to be CM-finite if, up to isomorphism, there are only a finite number of indecomposable modules in A-Gproj, and this algebra is said to be CM-free if each indecomposable Gorenstein-projective module is projective.
The following theorem is the extension of Theorem 2.3.9 quoted by [2] in the case $\text{rad}^3(A) = 0$ under some condition and the generalization is false as the author has given this counterexample:
Let $A = \begin{bmatrix} K[X]/(X^2) & K[X]/(X^2) \\ 0 & K[X]/(X^2) \end{bmatrix}$ be the Artin algebra, it is easy to show that $\text{rad}^3(A) = 0$ and $\text{rad}^2(A) \neq 0$. Also, it is not self-injective .

Our work shows that the result is true in a particular case for a specific algebra that we will introduce its definition,

3.1 Definition. Let A be an artin algebra, we say that A verifies the coincidence covers if,
$P(\Omega(S)) \simeq P(\Omega(T))$ implies $S \simeq T$ for all S and T simples modules.

3.2 Theorem. *Let A be a connected Artin algebra with radical cubed zero, and* $\text{rad}^2(P(\Omega M)) = 0$ *for any A-module M either simple or Gorenstein-projective and verifying the coincidence covers, then A is self-injective or CM-free.*

Proof. Suppose that A is not CM-free, then there exists a non-projective indecomposable A-module M and Gorenstein-projective, and we have a short exact not-split sequence:

$$0 \longrightarrow M \xrightarrow{f} P(M) \xrightarrow{p} M' \longrightarrow 0$$

and we have $M' = coker(f)$ is a Gorenstein-projective A-module and p is the projective cover of M', then $\Omega(M') = M$ and since $\Omega(M') \subseteq \text{rad}(P(M))$, then $\text{rad}^2(\Omega(M')) = 0$, i.e. $\text{rad}^2(M) = 0$ and since M is non-projective and indecomposable, then by Corollary 2, we have

$$\text{rad}(\Omega(M)) = \text{rad}^2(P(M)) = \text{rad}^2(P(\Omega(M'))) = 0$$

, then $\Omega(M)$ is semisimple and since $\Omega(M)$ is indecomposable, the A-module $\Omega(M)$ is simple.
Let $S_1 = \Omega(M)$ and take the short not-split exact sequence:

$$0 \longrightarrow \Omega(S_1) \xrightarrow{i_2} P_0 \xrightarrow{p} S_1 \longrightarrow 0$$

We consider $S_2 = \Omega(S_1)$, then $\text{Ext}^1_A(S_1, S_2) \neq 0$ and we are finding an arrow in S_1 to S_2 because S_2 is non-projective and indecomposable, i.e. S_2 is a simple module.
In fact, we show that the only arrow having the target S_2 has S_1 for vertices in the contrast case, assume that there exists a simple module S such that $\text{Ext}^1_A(S, S_2) \neq 0$ and consider the projective resolution of S is as follows:

$$0 \longrightarrow \Omega^2(S) \longrightarrow P_1(S) \longrightarrow P_0(S) \longrightarrow S \longrightarrow 0$$

We know that,

$$0 \longrightarrow \Omega(S) \xrightarrow{i_0} P_0(S) \longrightarrow S \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^2(S) \longrightarrow P_1(S) \longrightarrow \Omega(S) \longrightarrow 0$$

are the short exact sequences characterezing the first and second syzygy of S, then $\Omega^2(S) \subseteq \text{rad}(P_1(S))$ and since $\text{rad}^2(P_1(S)) = 0$, then $\text{rad}(\Omega^2(S)) = 0$, then $\Omega^2(S)$ is a semisimple module.
Since $\text{Ext}^1_A(S, S_2) \neq 0$ and $\text{Hom}(\Omega(S), S_2) \neq 0$ so, there exists an epimorphism $f \neq 0$ in $\text{Hom}(\Omega(S), \Omega(S_1))$ and we get the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^2(S) & \xrightarrow{i} & P & \xrightarrow{\pi} & \Omega(S) & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow h & \nearrow k & \downarrow f & & \\ 0 & \longrightarrow & \Omega^2(S_1) & \xrightarrow{i_1} & P_1 & \xrightarrow{\pi_1} & \Omega(S_1) & \longrightarrow & 0 \end{array}$$

, then $f \neq 0$ implies $h \neq 0$ if not we get $f \circ \pi = \pi_1 \circ h = 0$ absurd, so because $f \neq 0$ and $\Omega(S_1)$ is a simple module, then f is an epimorphism, and since P_1 is a projective module, then there exists a non-zero morphism $K : P_1 \longrightarrow \Omega(S)$ such that $f \circ k = \pi_1$, if one could prove that $i_0 \circ k$ is an epimorphism, that would be a great proof, however, this is not that easy since the only information that we have, is that i_0 is injective. This point will be true through the Gorenstein projective, but this is very difficult to prove even if the existence of such epimorphism looks right by using this property, as in the end $i_0 \circ k$ being an epimorphism will imply a contradiction that $S = 0$ Thus, we choose to proceed as follows.

As $\Omega^2(S)$ is semisimple and $\Omega^2(S_1)$ is simple and $g \neq 0$, then $\Omega^2(S_1)$ is a direct summand of $\Omega^2(S)$, so we can inject $\Omega^2(S_1)$ into $\Omega^2(S)$ which in turn is injected into P, let $l : \Omega^2(S_1) \hookrightarrow P$ be the injection composed of these last injections and with the previous notations and in short we have $l \in \mathrm{Hom}(S_3, P)$ with $S_3 = \Omega^2(S_1)$ and by application of the functor $\mathrm{Hom}(-, P)$ to the short exact sequence,

$$0 \longrightarrow S_3 \xrightarrow{i_1} P_1 \xrightarrow{\pi_1} S_2 \longrightarrow 0$$

we will have,

$$0 \longrightarrow \mathrm{Hom}(S_2, P) \longrightarrow \mathrm{Hom}(P_1, P) \xrightarrow{\mathrm{Hom}(i_1,P)} \mathrm{Hom}(S_3, P) \longrightarrow \mathrm{Ext}^1_A(S_2, P) = 0$$

and this because S_2 is a simple and Gorenstein projective A-module,
so $l = \mathrm{Hom}(i_1, P)(a) = a \circ i_1$ with $a \in \mathrm{Hom}(P_1, P)$ and note that S_3 is the socle of P_1 on which a is non-zero so a is a monomorphism and $length(P_1) \leq length(P)$ and as f and g are epimorphisms, then h is also and $length(P) \leq length(P_1)$, therefore, $P \simeq P_1$ that implies that $\Omega(S_1) \simeq \Omega(S)$, and like A verifying the coincidence covers see Definition 3.1, therefore $S_1 \simeq S$.

Finally, our construction affirms the existence of the set $\{S_1, S_2, ..., S_{n-1}\}$ of pairwise non-isomorphic simple A-modules; moreover, each S_i satisfies that any simple A-module S with $\mathrm{Ext}^1_A(S, S_i) \neq 0$ is isomorphic to S_{i-1} and we considered $S_{i+1} = \Omega(S_i)$, then we have any simple A-module S with $\mathrm{Ext}^1_A(S_i, S) \neq 0$ is isomorphic to S_{i+1} and by identification S_0 with S_{n-1} and S_n with S_1 it follows that the full subquiver Q_A with vertices $\{S_1, S_2, ..., S_{n-1}\}$ is a connected . Since the algebra A is connected and since there are simple A-modules associated with all the indecomposable projective A-modules given by $\{P_1, P_2, ..., P_{n-1}\}$ it follows by Theorem 9.3.7 [4] that the algebra A is self-injective.

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