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Adomian Decomposition Method and Block by Block Method for Solving Nonlinear Functional Volterra Integral Equation in Two-Dimensions

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Abstract. This work proposes a new definition of the nonlinear functional Volterra integral equation in two-dimensions (2D)of the second kind with continuous kernel. Furthermore, the work is concerned with studying this new equation numerically. The existence of a unique solution preposition by the equation is proven. In addition, the approximate solutions are obtained by two powerful methods Adomian Decomposition method (ADM) and Block by block Method (BBM). The given numerical examples showed the efficiency and accuracy of the introduced methods.

2020 Mathematics Subject Classifications: 34A12, 65A05,65D15

Key Words and Phrases: Functional Integral Equation, Adomian Decomposition Method, Block by block Method

1. Introduction

The nonlinear functional integral equation is the result of numerous viscoelastic material issues in the theories of elasticity and hydrodynamics. The references edited by Tricomi[9], Hochastadt [11] and Green [10] contain many different methods to give us the way for solving functional integral equation analytically. In practice, approximate methods are needed. There are different methods available to guide us towards obtaining the numerical solution. The excellent expositions by Atkinson [5], Linz [16], Delves and Mohamed [7], Kumar [15], [14] are recommended reading for the interested reader. The authors examined a group of nonlinear functional integral equations in [18]. The adequate conditions for the existence of the Lp-solution of a Volterra functional-integral problem in a Banach space were examined by Aldona in [8]. The variational iteration method was utilized by the authors in [6] to derive the numerical solution for the one-dimensional functional integral equations. Certain conclusions for a Volterra–Hammerstein integral equation were established by the authors in [17].The authors used a numerical technique based on the radial basis function in [12] to get numerical solutions of the two-dimensional

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functional linear integral equations of Fredholm.AL-Bugami conducted a numerical study of the two-dimensional integral equations in [3], [2]. The mixed integral equations with continuous and singular kernels were examined by the authors in [1],[13], [4]. Using the ADM and BBM, we investigate the novel equation for the nonlinear functional integral equation in 2D in this work. Because it deals with the examination of numerical solutions for NT-DFVIE, this work is noteworthy. Consider the NT-DFVIE.

$$\mu u(x,y) = g(x,y) + f(x,y,\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u(t,s))dtds) \tag{1}$$

The functions$_{g(x,y),f(x,y,v(x,y))}$and $_{\theta(x,y,u(x,y))}$are given analytical functions defined, respectively, and$_{p(x,y,t,s)}$ is the kernel of (1), $_{p(x,y,t,s)\geq 0,}$ and $u\,(x,y)$is the solution to be determined, while the constant parameter μdefines the kind of the (1).

2. Existences and uniqueness of a solution

This section will examine and use the Picard method to demonstrate that, under certain circumstances, there is a unique solution to (1).

We assume the following conditions to be satisfied:

1- $f(x,y,v\,(x,y))$and $\theta(x,y,u\,(x,y))$ in $0 \leq x \leq X$, $0 \leq y \leq Y < \infty$, such that

$$\{\int_0^x \{\int_0^y |f(x,y,v(x,y))|^2\,dx\}^{1/2} dy\}^{1/2} \leq A_1 \|v\|,$$

and

$$\{\int_0^x \{\int_0^y |\theta(x,y,w(x,y))|^2\,dx\}^{1/2} dy\}^{1/2} \leq A_2 \|w\|,$$

2- The kernel $_{p(x,y,t,s)}$satisfies:

$$p(x,y,t,s) \leq C, \qquad (C \text{ is a constant})$$

3- The two continuous function $f(x,y,v\,(x,y))$and $\theta(x,y,u\,(x,y))$ satisfy the Lipschitz condition:

$$\begin{aligned}
&|f(x,y,v_1(x,y)) - f(x,y,v_2(x,y))| \leq B_1\,|v_1 - v_2|\,; \forall v_1, v_2 \in (-\infty,\infty),\\
&|\theta(x,y,w_1(x,y)) - \theta(x,y,w_2(x,y))| \leq B_2\,|w_1 - w_2|\,; \forall w_1, w_2 \in (-\infty,\infty),
\end{aligned}$$

where$_{B_1,B_2}$are Lipchitz's constants such that $_{B_1 B_2 C = M_2 < 1.}$

We consider the existence and uniqueness of the solution for the (1). Also, the continuity and the normality of the integral operator are proved. We define the nonlinear integral operators Sample theorem with citation:

$$(Wu)(x,y) = f(x,y,\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u(t,s))dtds) \tag{2}$$

And

$$(\bar{W}u)(x,y) = g(x,y) + (Wf)(x,y) \tag{3}$$

As for the normality of the nonlinear integral operator Wu, we see that

$$\|Wu| = \{\int_0^x \{\int_0^y \left|\left|f(x,y,\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u(t,s))dtds)\right|^2 dx\right|^2 \}^{\frac{1}{2}} dy\}^{\frac{1}{2}} \tag{4}$$

Applying condition (1), we obtain

$$\|Wu| = A_1\{\int_0^x \{\int_0^y \left|\left|\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u(t,s))dtds)\right|^2 dx\right|^2 \}^{\frac{1}{2}} dy\}^{\frac{1}{2}} \tag{5}$$

Then, we get

$$\|Wu| = A_1\{\int_0^x \{\int_0^y | \int_0^x \int_0^y |p(x,y,t,s)|^2.|\theta(t,s,u(t,s))|^2 dtds)|^2 dtdx\}^{\frac{1}{2}} dsdy\}^{\frac{1}{2}} \tag{6}$$

Which can be adapted in the form

$$\|Wu\| = A_1\{\int_0^x \int_0^y \{\int_0^x \int_0^y |p(x,y,t,s)|^2 dxdt\}^{\frac{1}{2}} dyds\}^{\frac{1}{2}}.\{\int_0^x \{\int_0^y |\theta(t,s,u(t,s))|^2 dtds\}^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{7}$$

If we use the boundary conditions (1), (2), we will have

$$\|Wu\| = A_1 A_2 C \|u\| = M_1 \|u\| \tag{8}$$

To demonstrate the integral operator's continuation, take$_{u_1(x,y),u_2(x,y)\in L_2[0,x]\times L_2[0,y],}$to have:

$$\begin{aligned} &\left\|\bar{W}u_1 - \bar{W}u_2\right\| \\ &= \{\textstyle\int_0^x \{\int_0^y \left|\left|f(x,y,\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u_1(t,s))dtds)\right.\right. \\ &-f(x,y,\textstyle\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u_2(t,s))dtds)^2 dx^2\}^{\frac{1}{2}} dy\}^{\frac{1}{2}} \end{aligned} \tag{9}$$

Using the condition (3), we obtain

$$\left\|\bar{W}u_1 - \bar{W}u_2\right\| \le B_1\{\{\int_0^x \int_0^y \left|\int_0^x \int_0^y p(x,y,t,s)[\theta(t,s,u_1(t,s)) - \theta(t,s,u_2(t,s))]\right|^2 dx\}^{\frac{1}{2}} dy\}^{\frac{1}{2}} \tag{10}$$

Then,

$$\left\|\bar{W}u_1 - \bar{W}u_2\right\| \le B_1\{\{\int_0^x \int_0^y \int_0^x \int_0^y |p(x,y,t,s)\}^2.|\theta(t,s,u_1(t,s)) - \theta(t,s,u_2(t,s))|^2 dx\}^{\frac{1}{2}} dy\}^{\frac{1}{2}} \tag{11}$$

From the conditions (1), (3), we have

$$\left\|\bar{W}u_1 - \bar{W}u_2\right\| \le B_1 B_2 C \|u_1 - u_2\| = M_2 \|u_1 - u_2\| \qquad (M_2 < 1). \tag{12}$$

Hence, $_{\bar{W}}$is a contraction operator and $_{\bar{W}}$ has a unique fixed point, which is the unique solution to (1), of course.

3. Numerical methods for solving NT-DFVIE.

3.1. The ADM

This section uses ADM to handle NT-DFVIE of the second kind:

$$\mu u(x,y) = g(x,y) + f(x,y,\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u(t,s))dtds), \tag{13}$$

The unknown solution is an infinite series of the following form:

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \tag{14}$$

In addition, the term $\theta(t,s,u(t,s))$in (13) is decomposed into an infinite series

$$\theta(t,s,u) = \sum_{n=0}^{\infty} A_n \tag{15}$$

Moreover, the following equation defines Adomian's polynomial, denoted asA_n

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[\theta(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, \qquad n = 0, 1, 2, ... \tag{16}$$

Substituting from (15), (16) into(13), we get

$$u_0(x,y) = g(x,y) \tag{17}$$

$$u_i(x,y) = f(x,y,\int_0^x \int_0^y p(x,y,t,s)A_{i-1}(u_0(t,s),....,u_{i-1}(t,s))dtds), \qquad i \geq 1 \tag{18}$$

3.2. The BBM

Consider

$$\mu u(x,y) = g(x,y) + f(x,y,\int_0^x \int_0^y p(x,y,t,s)\theta(t,s,u(t,s))dtds) \tag{19}$$

Then, we get

$$U_2(x,y) \approx u(x_2,y_2) = g(x,y) + f(x,y,\int_0^x \int_0^y p(x_2,y_2,t,s)\theta(t,s,u_2(t,s))dtds) \tag{20}$$

or in the form

$$U_2(x,y) \approx u(x_2,y_2) = g(x,y) + f(x,y,\int_0^x \int_0^y p(x_2,y_2,t,s)dtds) \tag{21}$$

Using Simpson's rule to approximate the integrals

$$U_2 = g(x_2, y_2) + f(x, y, \frac{h}{3}\{p(x_2, y_2, x_0, y_0, U_0) + 4p(x_2, y_2, x_1, y_1, U_1) + p(x_2, y_2, x_2, y_2, U_2)\}) \tag{22}$$

where

$$U_0 = g(x_0, y_0). \tag{23}$$

Also, we get

$$U_1(x, y) \approx u(x_1, y_1) = g(x_1, y_1) + f(x, y, \int_0^x \int_0^y p(x_1, y_1, t, s, u_1(t, s))dtds) \tag{24}$$

$$U_1 = g(x_1, y_1) + f(x, y, \frac{h}{3}\{p(x_1, y_1, x_0, y_0, U_0) + 4p(x_1, y_1, x_{1/2}, y_{1/2}, U_{1/2}) + p(x_1, y_1, x_1, y_1, U_1)\} \tag{25}$$

Therefore, we obtain:

$$U_{1/2} = \frac{3}{8}U_0 + \frac{3}{4}U_1 - \frac{1}{8}U_2 \tag{26}$$

Substituting (26) into (25), we obtain:

$$U_1 = g(x_1, y_1) + f(x, y, \frac{h}{3}\{p(x_1, y_1, x_0, y_0, U_0) + 4p(x_1, y_1, x_{1/2}, y_{1/2}, \frac{3}{8}U_0 + \frac{3}{4}U_1 - \frac{1}{8}U_2)\}) \tag{27}$$

In general, consider (20), where$0 \le x \le a$, $0 \le y \le b$. Let$0 = x_0 < x_1 < \cdots < x_N = a$, $0 = y_0 < y_1 < \cdots < y_N = b$,

$$U_{2m+1}(x, y) \approx u(x_{2m+1}, y_{2m+1}) = g(x_{2m+1}, y_{2m+1}) + f(x, y, \int_0^{x_{2m+1}} \int_0^{y_{2m+1}} p(x_{2m+1}, y_{2m+1}, t, s, u(t, s))dtds) \tag{28}$$

Or equivalently

$$\begin{aligned} U_{2m+1}(x, y) = g(x_{2m+1}, y_{2m+1}) + f(x, y, \textstyle\int_0^{x_{2m}} \int_0^{y_{2m}} p(x_{2m+1}, y_{2m+1}, t, s, u(t, s))dtds) + \\ f(x, y, \textstyle\int_0^{x_{2m+1}} \int_0^{y_{2m+1}} p(x_{2m+1}, y_{2m+1}, t, s, u(t, s))dtds) \end{aligned} \tag{29}$$

Then,

$$\begin{aligned} &U_{2m+1}(x, y) = g(x_{2m+1}, y_{2m+1}) + f(x, y, \tfrac{h}{3}[p(x_{2m+1}, y_{2m+1}, x_0, y_0, U_0) + \\ &4p(x_{2m+1}, y_{2m+1}, x_1, y_1, U_1) + ... + \\ &p(x_{2m+1}, y_{2m+1}, x_{2m}, y_{2m}, U_{2m})] + \tfrac{h}{6}p(x_{2m+1}, y_{2m+1}, x_{2m}, y_{2m}, U_{2m}) \\ &+ \tfrac{2h}{3}p(x_{2m+1}, y_{2m+1}, x_{2m+\frac{1}{2}}, y_{2m+\frac{1}{2}}, \tfrac{3}{8}U_{2m} \\ &+ \tfrac{3}{4}U_{2m+1} - \tfrac{1}{8}U_{2m+2}) + \tfrac{h}{6}p(x_{2m+1}, y_{2m+1}, x_{2m+1}, y_{2m+1}, U_{2m+1})) \end{aligned} \tag{30}$$

Also,

$$U_{2m+2}(x, y) = g(x_{2m+2}, y_{2m+2}) + f(x, y, \int_0^{x_{2m+2}} \int_0^{y_{2m+2}} p(x_{2m+2}, y_{2m+2}, t, s, u(t, s))dtds) \tag{31}$$

$$U_{2m+2} = g(x_{2m+2}, y_{2m+2}) + f(x, y, \tfrac{h}{3}\{p(x_{2m+2}, y_{2m+2}, x_0, y_0, U_0)+ 4p(x_{2m+2}, y_{2m+2}, x_1, y_1, U_1) + ... + p(x_{2m+2}, y_{2m+2}, x_{2m+2}, y_{2m+2}, U_{2m+2})\}) \quad (32)$$

4. Numerical experiments and discussions

1.Consider

$$u(x,y) = g(x,y) + f(x,y, \int_0^x \int_0^y (xys)(u(t,s))^k dtds \quad (33)$$

$u(x,y) = xy$ is the exact solution. If we set k=1, in (33), one has

$$u(x,y) = g(x,y) + f(x,y, \int_0^x \int_0^y (xys)u(t,s)dtds \quad (34)$$

which is refereed to as**LT-DFVIE**, and if we set $\phi(x)$in (33), we obtained the integral equation, may consider the suggestion is called the **NT-DFVIE** of the second kind. Additionally, the associated errors are calculated for both the linear and nonlinear cases. We solve (33) using **ADM** and **BBM.** In the following Tables (1)-(2) we present the exact solution ($\boldsymbol{u}_{Exac}$**)** and the approximate solutions ($\boldsymbol{u}_{ADM}$, $\boldsymbol{u}_{BBM}$), and the corresponding errors ($\boldsymbol{Error}_{ADM}$,$\boldsymbol{Error}_{BBM}$) at N= 10.

Block-by-block method		***ADM***		$\boldsymbol{u}_{Exact}$	$\boldsymbol{y}$	$\boldsymbol{x}$
$\boldsymbol{Error}_{BBM}$	$\boldsymbol{u}_{BBM}$	$\boldsymbol{Error}_{ADM}$	$\boldsymbol{u}_{ADM}$			
0.00000	0.000000	0.00000000	0.0000000	0.0000000	0.0	0.0
0.6950×10^{-8}	0.0100000	0.277777×10^{-8}	0.0100000	0.0100000	0.1	0.1
0.44804×10^{-5}	0.0400044	0.355560×10^{-6}	0.0400003	0.0400000	0.2	0.2
0.00004059	0.09004059	0.607559×10^{-5}	0.0900006	0.0900000	0.3	0.3
0.00024690	0.16024690	0.000045529	0.1600455	0.1600000	0.4	0.4
0.00179211	0.25179211	0.000217285	0.2502172	0.2500000	0.5	0.5
0.01059498	0.34940502	0.000780024	0.3607800	0.3600000	0.6	0.6
0.01366165	0.50366165	0.002303060	0.4923030	0.4900000	0.7	0.7
0.03285243	0.67285243	0.005902298	0.6459022	0.6400000	0.8	0.8
0.04750594	0.85750594	0.013603387	0.8236033	0.8100000	0.9	0.9
0.08797644	1.08797645	0.028908955	1.0289089	1.0000000	1.0	1.0

Table (1) Numerical results by using **BBM** and **ADM**, N=10,k=1.

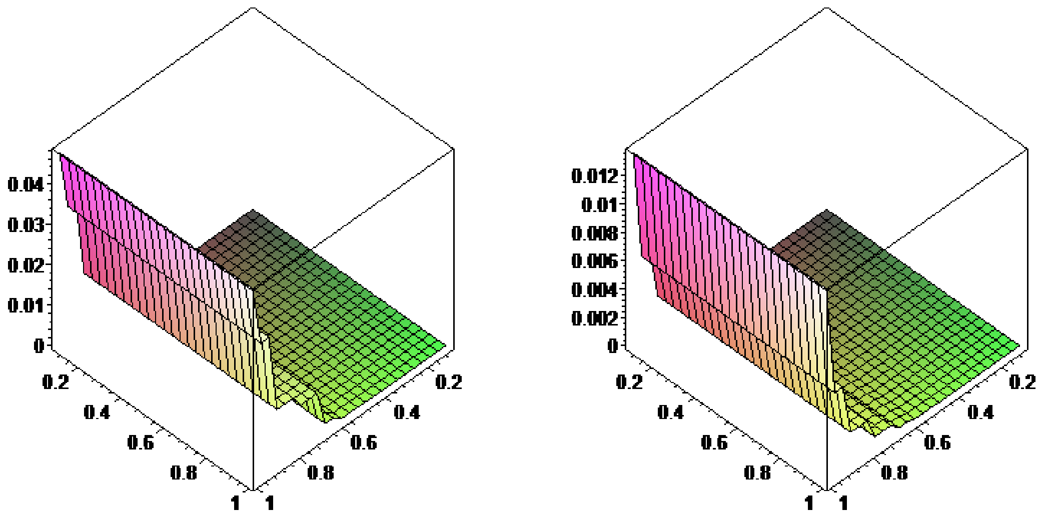

Fig. 1, the plot 3D errors for the value of errors by using **BBM** and **ADM**, N=10, k=1.

Block-by-block method		***ADM***		u_{Exact}	y	x
$Error_{BBM}$	u_{BBM}	$Error_{ADM}$	u_{ADM}			
0.000000	0.0000000	0.000000	0.0000000	0.0000000	0.0	0.0
0.20760×10^{-7}	0.01000002	0.16590×10^{-7}	0.0100000	0.0100000	0.1	0.1
0.44804×10^{-5}	0.04000448	0.20942×10^{-5}	0.0400020	0.0400000	0.2	0.2
0.000040593	0.09004059	0.000034949	0.0900349	0.0900000	0.3	0.3
0.000246903	0.16024690	0.000253147	0.1602531	0.1600000	0.4	0.4
0.001792115	0.25179211	0.001154385	0.2511543	0.2500000	0.5	0.5
0.010594983	0.34940501	0.003908646	0.3639086	0.3600000	0.6	0.6
0.013661652	0.50366165	0.010721293	0.5007212	0.4900000	0.7	0.7
0.032852435	0.67285243	0.025066934	0.6650669	0.6400000	0.8	0.8
0.047505948	0.85750594	0.051535161	0.8615351	0.8100000	0.9	0.9
0.087976455	1.08797645	0.094913086	1.0949130	1.0000000	1.0	1.0

Table(2) Numerical results by using **BBM** and **ADM**, N=10,k=2.

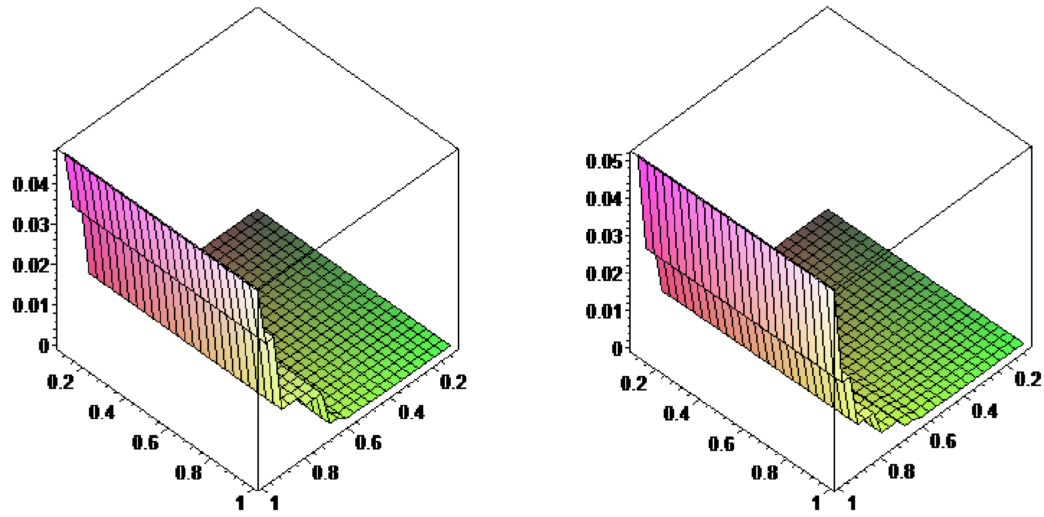

Fig. 2, the plot 3D errors for the value of errors by using **BBM** and **ADM**, N=10, k=2.

2.Consider

$$u(x,y) = g(x,y) + f(x,y,\int_0^x \int_0^y (xy)(u(t,s))^k dtds \tag{35}$$

$u(x,y) = xy/2$ is the exact solution, if we set k=1, in (35), one has

$$u(x,y) = g(x,y) + f(x,y,\int_0^x \int_0^y (xy)u(t,s)dtds \tag{36}$$

Block-by-block method		***ADM***		u_{Exact}	y	x
$Error_{BBM}$	U_{BBM}	$Error_{ADM}$	u_{ADM}			
0.0000000	0.0000000	0.0000000	0.0000000	0.00000000	0.0	0.0
0.9793×10^{-8}	0.00500000	0.625003×10^{-7}	0.0050000	0.00500000	0.1	0.1
0.224024×10^{-5}	0.02000224	0.400040×10^{-5}	0.0200040	0.02000000	0.2	0.2
0.000020296	0.04502029	0.000045585	0.0450455	0.04500000	0.3	0.3
0.000123451	0.08012345	0.000256410	0.0802564	0.08000000	0.4	0.4
0.000896057	0.12589605	0.000980387	0.1259803	0.12500000	0.5	0.5
0.0052974915	0.17470250	0.002939725	0.1829397	0.18000000	0.6	0.6
0.006830862	0.25183082	0.00746423	0.2524642	0.24500000	0.7	0.7
0.016426217	0.33642621	0.01680854	0.3368085	0.32000000	0.8	0.8
0.023752974	0.42875297	0.03460287	0.4396028	0.40500000	0.9	0.9
0.043988227	0.54398822	0.06651660	0.5665166	0.50000000	1.0	1.0

Table (3) Numerical results by using **BBM** and **ADM**, N=10,k=1.

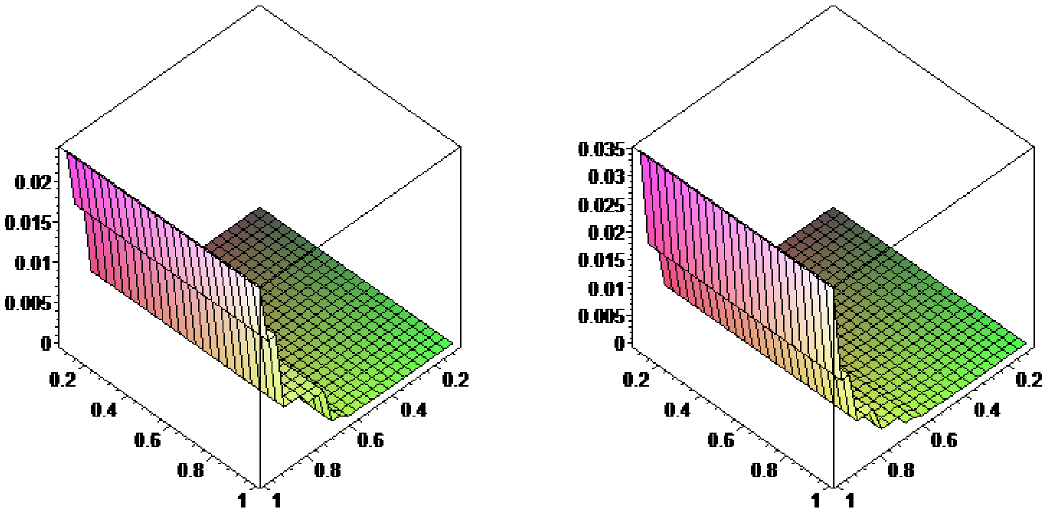

Fig. 3, the plot 3D errors for the value of errors by using **BBM** and **ADM**, N=10, k=1.

Block-by-block method		***ADM***		u_{Exact}	y	x
$Error_{BBM}$	U_{BMM}	$Error_{ADM}$	u_{ADM}			
0.00000000	0.00000000	0.0000000	0.0000000	0.00000000	0.0	0.0
0.9793×10^{-7}	0.004999902	0.12456×10^{-6}	0.00500012	0.00500000	0.1	0.1
0.747110×10^{-5}	0.020007471	0.789017×10^{-5}	0.02000789	0.02000000	0.2	0.2
0.0000357169	0.045035716	0.000088335	0.04508833	0.04500000	0.3	0.3
0.0001830350	0.080183035	0.000484473	0.08048447	0.08000000	0.4	0.4
0.0011159424	0.126115942	0.001791559	0.12679155	0.12500000	0.5	0.5
0.0149879842	0.165012015	0.005149897	0.18514989	0.18000000	0.6	0.6
0.012010961	0.257010961	0.012413373	0.25741337	0.24500000	0.7	0.7
0.027208265	0.347208265	0.026247589	0.34624759	0.32000000	0.8	0.8
0.048275031	0.453275031	0.050109867	0.45510986	0.40500000	0.9	0.9
0.097513263	0.597513263	0.088059093	0.58805909	0.50000000	1.0	1.0

Table (4) Numerical results by using **BBM** and **ADM**, N=10,k=2.

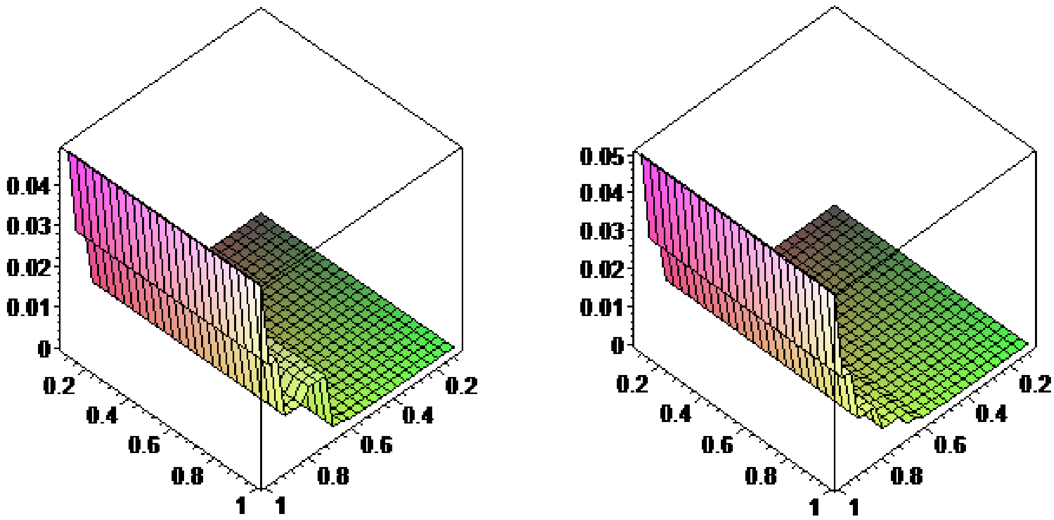

Fig. 4, the plot 3D errors for the value of errors by using **BBM** and **ADM**, N=10, k=2.

5. The conclusion

he new definition of the Functional Volterra Integral Equation in Two-Dimensional (NT-DFVIE) was presented in this study. Effective numerical techniques are also suggested in order to solve this equation. Error analysis and some numerical examples show the accuracy and performance of the methods. Based on the preceding examples and tables (1)-(4), we observe that:

1- As x andy are increasing in each interval [0, 1], the errors values for BBM, and ADM are also increasing.

2- The error results by using ADM is smaller than the error results by using BBM. So, the ADM is better than theBBM for solving nonlinear NT-DFVIE.

3- The error result by using ADM for linear case is larger than the nonlinear case, and by using BBM the error results for nonlinear case is larger than the linear case.

Additional Points

Future Work. Other methods, such as the homotopyperturbation approach, homotopy analysis method, Runge-Kutta method and the variational iterationapproach, will be used to solve the the nonlinear functional Volterra integral equation in two-dimension.

Data Availability

All the data are available within the article and as the references that were cited.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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