EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 2, 2024, 1113-1128
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fermatean Neutrosophic INK-Algebras

Anas Al-Masarwah [1], M. Kaviyarasu[2,*], Kholood Alnefaie[3], M. Rajeshwari [4]

[1] *Department of Mathematics, Faculty of Science, Ajloun National University, P. O. Box 43, Ajloun 26810, Jordan*
[2] *Department of Mathematics, Vel Tech Rangarajan Dr Sagunthala R&D Institute of Science and Technology, Chennai, Tamil Nadu, India-600 062*
[3] *Department of Mathematics, College of Science, Taibah University, Madinah 42353, Saudi Arabia*
[4] *Department of Mathematics, Presidency University, Bangalore, 560064, India.*

Abstract. This paper introduces the concept of the direct product of sets that involve Fermatean neutrosophic (FN) elements in structures called INK-algebras. It defines terms like the direct product of Fermatean neutrosophic INK-ideals (FNINK-Is) in INK-algebras and Fermatean neutrosophic sets (FNSs), FNINK-Is, and Fermatean neutrosophic closed INK-ideals (FNCINK-Is). The proof of theorems illustrating the relationships between these ideas is included in the paper. It also defines the INK-subalgebra embedded in an INK-algebra and gives a theorem elucidating the connection between the direct product of FNINK-Is and the images of these subalgebras. In essence, the paper investigates and establishes connections between different mathematical ideas concerning INK-algebras and FNSs.

2020 Mathematics Subject Classifications: 03E72, 03G25, 28E10, 03B52

Key Words and Phrases: INK-algebra, Direct product, Fermatean neutrosophic set, Fermatean neutrosophic INK-ideal, Fermatean neutrosophic closed INK-ideal

---

1. Introduction

Zadeh [1] seminal work on fuzzy sets established the basis for fuzzy logic. Fuzzy sets provide a more flexible representation of uncertainty by assigning degrees of membership to elements. Atanassov [2] paper explores intuitionistic fuzzy sets, which include degrees of membership and degrees of non-membership. This gives a more complete picture of uncertainty in decision-making. Neutrosophic logic has been introduced by Smarandache which involves various disciplines of philosophy and mathematics that studies indeterminacy, uncertainty, and contradictions. Neutrosophic logic is a three-valued logic system

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i2.5090

Email addresses: anas.almasarwah@anu.edu.jo (A. Al-Masarwah), kavitamilm@gmail.com (M. Kaviyarasu), knefaie@taibahu.edu.sa (K. Alnefaie), rajeshwari@presidencyuniversity.in (M. Rajeshwari)

that includes the truth values "true" and "false," as well as a third value termed indeterminate, which represents uncertain or ambiguous information. This method is especially beneficial for dealing with difficulties involving inadequate or inconsistent data, which are widespread in domains such as artificial intelligence, decision making, philosophy, and cognitive science.

Abdel-Bassset et al. [3] present a decision-making framework for professional selection based on bipolar neutrosophic sets. The approach seeks to address uncertainty and imprecision in decision-making processes, his work expands on intuitionistic fuzzy sets to include neutrosophic sets, allowing for indeterminacy, uncertainty, and contradictory information [4]. Jun [5] research on neutrosophic subalgebras in BCK/BCI-algebras advances our understanding of algebraic structures with neutrosophic elements. Kaviyarasu et al. [6] investigates fuzzy subalgebras and fuzzy INK-ideals in INK-algebras, providing insights into the integration of fuzzy logic in algebraic structures. Additionally, Kaviyarasu and Indhira present a review of BCI/BCK-algebras and discuss their development, contributing to the understanding of these specific algebraic structures [7], investigate fuzzy p-ideals in INK-algebras [8], adding to the understanding of fuzzy ideals in the context of specific algebraic structures. Jun et al.[9] collaboration aims to integrate neutrosophic N-structures into BCK/BCI-algebras.

Jun et al.[10] investigate neutrosophic positive implicative N-ideals in the context of BCK-algebras, advancing our understanding of neutrosophic structures in algebraic systems. Ozturk and Jun [11] investigates neutrosophic ideals in BCK/BCI algebras based on neutrosophic points, broadening the application of neutrosophic concepts to algebraic structures. Songsaeng and Iampan [12] introduce neutrosophic set theory to UP-algebra and demonstrate its utility in a specific algebraic context.Kaviyarasu, Indhira and Chandrasekaran [[13], [14], [15]] investigate the direct product of intuitionistic fuzzy INK-ideals, providing insights into the interaction of different algebraic structures; discuss intuitionistic fuzzy translation in INK-Algebra, contributing to the understanding of translation operations in algebraic structures; and apply neutrosophic sets in INK-Algebra, extending the study of neutrosophic concepts to a specific algebraic context. As an extension of partial algebra, Smarandache [16] presents the theory of neutro algebra, which advances the development of algebraic structures in addition to neutro and anti-algebraic structures, which provide additional insights into mathematical structures. Abdel-Basset et al. proposed a novel plithogenic model for supply chain problem solving, which incorporates neutrosophic and plithogenic sets into optimization theory [17]. Making contributions to the fields of environmental technology and innovation, Mohamed and Abdel [[18], [19], [20]] introduce an integrated plithogenic MCDM approach for assessing the financial performance of manufacturing industries, utilizing a combination of mathematical decision-making methods, as well as a novel framework for assessing the innovation value proposition for smart product-service systems. Neutrosophic Vague Binary BCK/BCI-algebra, which explores the use of vague and neutrosophic notions in a binary algebraic framework, is covered by Remya and Francina Shalini [21]. Muralikrishna and Manokaran [22] introduce MBJ- neutrosophic B-ideals in B-algebras, which adds to the understanding of neutrosophic ideals in specific algebraic structures. For more results on algebraic structures with uncertainty (see works by the authors of [23–27].

This paper presents a new concept based on two distinct sets, called FNSs, and investigates their direct product in the framework of INK-algebra. Specifically, it looks at the relation between FNINK-Ss and FNINK-Is, as well as the conditions that hold for this relation.

1.1. Motivation

- It aims to provide a new perspective on FN elements in mathematical structures.
- Ensures a systematic and coherent discussion of these mathematical concepts.
- Validating the proposed connections rigorously through theorems adds credibility and reliability.
- Exploration of Substructures contributes to a more comprehensive understanding of the intricate relationships within the algebraic framework.

1.2. Novelty

- The paper aims to present a novel mathematical framework by introducing the concept of the direct product, which involves sets with FN elements within the domain of INK-algebras.
- The goal is to create a precise mathematical language by defining terms like direct product of FNINK-Is, FN-Ss and FNCINK-Is, which will improve discourse clarity.
- The study illuminates complex mathematical relationships through rigorous theorem proofs, delving into interconnected ideas in INK-algebras and FN-Ss.

1.3. Structure of the paper

The paper begins with an introductory exploration of the novel concept of the direct product within structures known as INK-algebras, which include sets enriched with FN elements. It then defines key terms like the direct product of FNINKs, FN-Ss and FNCINK-Is. The narrative then proceeds to provide a comprehensive exposition, including proofs of theorems that intricately illustrate the relationships between these defined concepts. Furthermore, it broadens its scope to define an INK-subalgebra embedded within an INK-algebra, as well as a theorem that explains the relationship between the direct product of FNINKs and the images of these sub algebras. In essence, the paper culminates in a thorough investigation, establishing connections and interrelations between diverse mathematical ideas concerning both INK-algebras.

2. Basic Definitions

In the beginning the research, the definition and beneficial properties of INK-algebras will be explained.

Definition 1 ([15]). *An INK-algebra is a mathematical structure with specific rules; it is represented by the notation* $(\chi, \cdot, 0)$. *For any elements* $\vartheta, \eta, z \in \chi$

(1) $((\vartheta \cdot \eta) \cdot (\vartheta \cdot z)) \cdot (z \cdot \eta) = 0$,

(2) $((\vartheta \cdot z) \cdot (\eta \cdot z)) \cdot (\vartheta \cdot \eta) = 0$,

(3) $(\vartheta \cdot 0) = 0$,

(4) $(\vartheta \cdot \eta) = 0$ *and* $\eta \cdot \vartheta = 0$ *imply* $\eta = \vartheta$.

The operation $\cdot$ denotes a binary operation and 0 is a constant belonging to the set χ.

Definition 2 ([6]). *A non-empty subset* S *of a INK-algebra* $(\chi, \cdot, 0)$ *is considered as an INK-subalgebra of* χ, *if for every elements* ϑ *and* $\eta \in \chi$, *the result of the operation* $(\vartheta \cdot \eta)$ *is also an element of* S.

Definition 3 ([6]). *Let* $(\chi, \cdot, 0)$ *be an INK-algebra. An ideal of* χ *is defined as a nonempty subset* $\Im$ *of* χ *such that it satisfies the following conditions,* $\forall \vartheta, \eta \in \chi$

(1) $0 \in \Im$,

(2) $(\vartheta \cdot \eta) \in \Im$ *and* $\eta \in \Im$ *imply* $\vartheta \in \Im$.

Definition 4 ([6]). *Let an INK-algebra* χ *have a non-empty subset* $\Im$. *If all of the following hold for every* $\vartheta, \eta, z \in \chi$, *then* $\Im$ *is called an INK-ideal of* χ.

(1) $0 \in \Im$,

(2) $(z \cdot \vartheta) \cdot (z \cdot \eta) \in \Im$ *and* $z \in \Im$ *imply* $\vartheta \in \Im$.

Definition 5 ([22]). *The structure of a FNS* $\mathfrak{M}$ *defined on a nonempty set* χ *can be expressed as:* $\mathfrak{M} = \left\{ \left\langle \vartheta, \rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta) \right\rangle | \vartheta \in \chi \right\}$, *where* $\rho^{\mathfrak{T}} : \chi \to [0,1]$ *is a membership function* $\rho^{\mathfrak{I}} : \chi \to [0,1]$ *is a indeterminate membership function and* $\rho^{\mathfrak{F}} : \chi \to [0,1]$ *is a non-membership function and these three functions are satisfying the inequalities;* $0 \leq (\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta))^3 + (\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta))^3 \leq 1$, $0 \leq (\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta))^3 \leq 1$ *and* $0 \leq (\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta))^3 + (\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta))^3 + (\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta))^3 \leq 2$.

Here, $\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta)$ *and* $\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta)$ *are dependent components and* $\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta)$ *is an independent component.*

Throughout the current research article, we shall use $\mathfrak{M} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}}, \rho^{\mathfrak{I}}_{\mathfrak{M}}, \rho^{\mathfrak{F}}_{\mathfrak{M}} \right\rangle$ *for the FNS* $\mathfrak{M} = \left\{ \left\langle \vartheta, \rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta) \right\rangle | \vartheta \in \chi \right\}$.

Definition 6 ([22]). *If* $\mathfrak{M} = \left\{ \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta) \right\rangle \right\}$ *and* $\mathfrak{N} = \left\{ \left\langle \rho^{\mathfrak{T}}_{\mathfrak{N}}(\vartheta), \rho^{\mathfrak{I}}_{\mathfrak{N}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{N}}(\vartheta) \right\rangle \right\}$ *be two FNSs, then* $\forall \vartheta \in \chi$

(i) $\overline{\mathfrak{M}} = \left\{ \left\langle 1 - \rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta), 1 - \rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta), 1 - \rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta) \right\rangle \right\}$

(ii) $\mathfrak{M} \cap \mathfrak{N} = \left\{ \left\langle min\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{T}}_{\mathfrak{N}}(\vartheta)\}, max\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{I}}_{\mathfrak{N}}(\vartheta)\}, max\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{N}}(\vartheta)\} \right\rangle \right\}.$

Definition 7. *A FNS $\mathfrak{M}$ of χ obtains the title of FNINK-Ss by satisfying the requirements,* $\forall\ \vartheta, \eta \in \chi$

(1) $\rho^{\mathfrak{T}}(\vartheta \cdot \eta) \leq min\left\{\rho^{\mathfrak{T}}(\vartheta), \rho^{\mathfrak{T}}(\eta)\right\},$

(2) $\rho^{\mathfrak{I}}(\vartheta \cdot \eta) \geq max\left\{\rho^{\mathfrak{I}}(\vartheta), \rho^{\mathfrak{I}}(\eta)\right\},$

(3) $\rho^{\mathfrak{F}}(\vartheta \cdot \eta) \geq max\left\{\rho^{\mathfrak{F}}(\vartheta), \rho^{\mathfrak{F}}(\eta)\right\}.$

Definition 8. *A FNS $\mathfrak{M}$ of χ is considered as a FN-I if it meets the described conditions,* $\forall\ \vartheta, \eta \in \chi$

(1) $\rho^{\mathfrak{T}}(0) \leq \rho^{\mathfrak{T}}(\vartheta), \rho^{\mathfrak{I}}(0) \geq \rho^{\mathfrak{I}}(\vartheta), \rho^{\mathfrak{F}}(0) \geq \rho^{\mathfrak{F}}(\vartheta),$

(2) $\rho^{\mathfrak{T}}(\vartheta) \leq min\left\{\rho^{\mathfrak{T}}(\vartheta \cdot \eta), \rho^{\mathfrak{T}}(\eta)\right\},$

(3) $\rho^{\mathfrak{I}}(\vartheta) \geq max\left\{\rho^{\mathfrak{I}}(\vartheta \cdot \eta), \rho^{\mathfrak{I}}(\eta)\right\},$

(4) $\rho^{\mathfrak{F}}(\vartheta) \geq max\left\{\rho^{\mathfrak{F}}(\vartheta \cdot \eta), \rho^{\mathfrak{F}}(\eta)\right\}.$

Example 1. *If $\chi = \{0, x, y, z\}$ is a set with a binary operation $\cdot$ given by the following Table:*

Table 1: The operation $\cdot$

$\cdot$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Thus, $(\chi.\cdot, 0)$ is an INK-algebra. Consider a FNS $\mathfrak{M}$ in χ, where $\rho^{\mathfrak{T}}_{\mathfrak{M}}(0) = 0.8, \rho^{\mathfrak{T}}_{\mathfrak{M}}(x) = 0.4, \rho^{\mathfrak{T}}_{\mathfrak{M}}(y) = \rho^{\mathfrak{T}}_{\mathfrak{M}}(z) = 0.2, \rho^{\mathfrak{I}}_{\mathfrak{M}}(0) = 0.7, \rho^{\mathfrak{I}}_{\mathfrak{M}}(x) = 0.5, \rho^{\mathfrak{I}}_{\mathfrak{M}}(y) = \rho^{\mathfrak{I}}_{\mathfrak{M}}(z) = 0.3$ *and* $\rho^{\mathfrak{F}}_{\mathfrak{M}}(0) = 0.1, \rho^{\mathfrak{F}}_{\mathfrak{M}}(x) = \rho^{\mathfrak{F}}_{\mathfrak{M}}(y) = 0.4, \rho^{\mathfrak{F}}_{\mathfrak{M}}(z) = 0.3$. *Then, $\mathfrak{M}$ is a FN-I of χ, which is easily verified.*

Definition 9. *A FNs $\mathfrak{M}$ of χ is considered as a FNINK-I of χ if it meets the described conditions,* $\forall\ \vartheta, \eta, z \in \chi$

(1) $\rho^{\mathfrak{T}}(0) \leq \rho^{\mathfrak{T}}(\vartheta), \rho^{\mathfrak{I}}(0) \geq \rho^{\mathfrak{I}}(\vartheta), \rho^{\mathfrak{F}}(0) \geq \rho^{\mathfrak{F}}(\vartheta),$

(2) $\rho^{\mathfrak{T}}(\vartheta) \leq min\left\{\rho^{\mathfrak{T}}((z \cdot \vartheta) \cdot (z \cdot \eta)), \rho^{\mathfrak{T}}(\eta)\right\},$

(3) $\rho^{\mathfrak{I}}(\vartheta) \geq max\left\{\rho^{\mathfrak{I}}((z \cdot \vartheta) \cdot (z \cdot \eta)), \rho^{\mathfrak{I}}(\eta)\right\},$

(4) $\rho^{\mathfrak{F}}(\vartheta) \geq max\left\{\rho^{\mathfrak{F}}((z \cdot \vartheta) \cdot (z \cdot \eta)), \rho^{\mathfrak{F}}(\eta)\right\}.$

3. Formation of Direct Product: FNINK-Ss and FNINK-Is

Definition 10. *INK-algebras χ_1 and χ_2 contain two FNSs, $\mathfrak{M}$ and $\mathfrak{N}$. The structure*

$$\mathfrak{M} \times \mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} \right\rangle$$

is defined as the direct product of FNSs $\mathfrak{M}$ and $\mathfrak{N}$, specified by, $\forall(\vartheta, \eta) \in \chi_1 \times \chi_2$

(1) $\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta, \eta) = min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta)\right\}$,

(2) $\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta, \eta) = max\left\{\rho^{\mathfrak{J}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{J}}_{\mathfrak{N}}(\eta)\right\}$,

(3) $\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta, \eta) = max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta)\right\}$.

Definition 11. *The direct product of FNINK-Ss of $\chi_1 \times \chi_2$ is a FNSs*

$$\mathfrak{M} \times \mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} \right\rangle$$

of χ_1 and χ_2 if, $\forall(\vartheta_1, \eta_1), (\vartheta_2, \eta_2) \in \chi_1 \times \chi_2$

(1) $\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}((\vartheta_1, \eta_1) \cdot (\vartheta_2, \eta_2)) \le min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1, \eta_1), \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2, \eta_2)\right\}$,

(2) $\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}((\vartheta_1, \eta_1) \cdot (\vartheta_2, \eta_2)) \ge max\left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1, \eta_1), \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2, \eta_2)\right\}$,

(3) $\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}((\vartheta_1, \eta_1) \cdot (\vartheta_2, \eta_2)) \ge max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1, \eta_1), \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2, \eta_2)\right\}$.

Definition 12. *The direct product of FNINK-I of $\chi_1 \times \chi_2$ is a FNSs*

$$\mathfrak{M} \times \mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} \right\rangle$$

of χ_1 and χ_2 if , $\forall(\vartheta_1, \eta_1), (\vartheta_2, \eta_2), (\vartheta_3, \eta_3) \in \chi_1 \times \chi_2$

(1) $\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0, 0) \le \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta, \eta), \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(0, 0) \ge \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta, \eta), \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(0, 0) \ge \rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta, \eta)$,

(2) $\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_1, \eta_1) \le min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2, \eta_2)\right\}$,

(3) $\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1, \eta_1) \ge max\left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2, \eta_2)\right\}$,

(4) $\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1, \eta_1) \ge max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3, \eta_3) \cdot (\vartheta_1, \eta_1)) \cdot ((\vartheta_3, \eta_3) \cdot (\vartheta_2, \eta_2))), \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2, \eta_2)\right\}$.

Definition 13. *The direct product of FNCINK-I of $\chi_1 \times \chi_2$ is a FNSs*

$$\mathfrak{M} \times \mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} \right\rangle$$

of χ_1 and χ_2 if it meets $((2)$, (3) *and* (4) $of\ Definition$ $12)$ *and the following inequalities,* $\forall(\vartheta, \eta) \in \chi_1 \times \chi_2$

(1) $\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}((0,0)\cdot(\vartheta,\eta)) \leq \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)$,

(2) $\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}((0,0)\cdot(\vartheta,\eta)) \geq \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)$,

(3) $\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}((0,0)\cdot(\vartheta,\eta)) \geq \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)$.

Theorem 1. *Let* $\mathfrak{M} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}}, \rho^{\mathfrak{J}}_{\mathfrak{M}}, \rho^{\mathfrak{F}}_{\mathfrak{M}} \right\rangle$ *and* $\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{N}}, \rho^{\mathfrak{J}}_{\mathfrak{N}}, \rho^{\mathfrak{F}}_{\mathfrak{N}} \right\rangle$ *be two FNINK-Ss of* χ_1 *and* χ_2*, respectively. Then, the direct product* $\mathfrak{M}\times\mathfrak{N}$*, defined by*

$$\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} \right\rangle,$$

is a FNINK-S of $\chi_1\times\chi_2$.

Proof. Assume that $\mathfrak{M}$ and $\mathfrak{N}$ are two FNINK-Ss. Let $(\vartheta_1,\eta_1),(\vartheta_2,\eta_2) \in \chi_1\times\chi_2$. Then,

$$\begin{aligned}
\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}((\vartheta_1,\eta_1)\cdot(\vartheta_2,\eta_2)) &= \left\{\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_1\cdot\vartheta_2),(\eta_1\cdot\eta_2)\right\}\\
&= min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_1\cdot\vartheta_2),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_1\cdot\eta_2)\right\}\\
&\leq min\left\{min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_2)\right\}, min\left\{\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_1),\rho^{\mathfrak{T}}_{\mathfrak{M}}(\eta_2)\right\}\right\}\\
&= min\left\{min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_1)\right\}, min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_2),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_2)\right\}\right\}\\
&= min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1),\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\},
\end{aligned}$$

$$\begin{aligned}
\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}((\vartheta_1,\eta_1)\cdot(\vartheta_2,\eta_2)) &= \left\{\rho^{\mathfrak{J}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_1\cdot\vartheta_2),(\eta_1\cdot\eta_2)\right\}\\
&= max\left\{\rho^{\mathfrak{J}}_{\mathfrak{M}}(\vartheta_1\cdot\vartheta_2),\rho^{\mathfrak{J}}_{\mathfrak{N}}(\eta_1\cdot\eta_2)\right\}\\
&\geq max\left\{max\left\{\rho^{\mathfrak{J}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{J}}_{\mathfrak{M}}(\vartheta_2)\right\}, max\left\{\rho^{\mathfrak{J}}_{\mathfrak{N}}(\eta_1),\rho^{\mathfrak{J}}_{\mathfrak{M}}(\eta_2)\right\}\right\}\\
&= max\left\{max\left\{\rho^{\mathfrak{J}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{J}}_{\mathfrak{N}}(\eta_1)\right\}, max\left\{\rho^{\mathfrak{J}}_{\mathfrak{M}}(\vartheta_2),\rho^{\mathfrak{J}}_{\mathfrak{N}}(\eta_2)\right\}\right\}\\
&= max\left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1),\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}((\vartheta_1\eta_1)\cdot(\vartheta_2,\eta_2)) &= \left\{\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_1\cdot\vartheta_2),(\eta_1\cdot\eta_2)\right\}\\
&= max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_1\cdot\vartheta_2),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_1\cdot\eta_2)\right\}\\
&\geq max\left\{max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_2)\right\}, max\left\{\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_1),\rho^{\mathfrak{F}}_{\mathfrak{M}}(\eta_2)\right\}\right\}\\
&= max\left\{max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_1)\right\}, max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_2),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_2)\right\}\right\}
\end{aligned}$$

$$= max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}.$$

Hence, $\mathfrak{M}\times\mathfrak{N}=\left\langle\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-S of $\chi_1\times\chi_2$.

Theorem 2. *Let* $\mathfrak{M}=\left\langle\rho^{\mathfrak{T}}_{\mathfrak{M}},\rho^{\mathfrak{I}}_{\mathfrak{M}},\rho^{\mathfrak{F}}_{\mathfrak{M}}\right\rangle$ *and* $\mathfrak{N}=\left\langle\rho^{\mathfrak{T}}_{\mathfrak{N}},\rho^{\mathfrak{I}}_{\mathfrak{N}},\rho^{\mathfrak{F}}_{\mathfrak{N}}\right\rangle$ *be two FNINK-Is of* χ_1 *and* χ_2*, respectively. Then, the direct product* $\mathfrak{M}\times\mathfrak{N}$*, defined by*

$$\mathfrak{M}\times\mathfrak{N}=\left\langle\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle,$$

is a FNINK-I of $\chi_1\times\chi_2$.

Proof. For any $(\vartheta,\eta)\in\chi_1\times\chi_2$. We have

$$\begin{aligned}\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)&=min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(0),\rho^{\mathfrak{T}}_{\mathfrak{N}}(0)\right\}\\&\le min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta)\right\}\\&=\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta),\end{aligned}$$

$$\begin{aligned}\rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}(0,0)&=max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(0),\rho^{\mathfrak{I}}_{\mathfrak{N}}(0)\right\}\\&\ge max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta),\rho^{\mathfrak{I}}_{\mathfrak{N}}(\eta)\right\}\\&=\rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta,\eta)\end{aligned}$$

and

$$\begin{aligned}\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(0,0)&=max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(0),\rho^{\mathfrak{F}}_{\mathfrak{N}}(0)\right\}\\&\ge max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta)\right\}\\&=\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta,\eta).\end{aligned}$$

Also, for any $(\vartheta_1,\eta_1),(\vartheta_2,\eta_2),(\vartheta_3,\eta_3)\in\chi_1\times\chi_2$. We have

$$\begin{aligned}\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_1,\eta_1)&=min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_1)\right\}\\&\le min\left\{min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}((\vartheta_3\cdot\vartheta_1)\cdot(\vartheta_3\cdot\vartheta_2)),\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_2)\right\},min\left\{\rho^{\mathfrak{T}}_{\mathfrak{N}}((\eta_3\cdot\eta_1)\cdot(\eta_3\cdot\eta_2)),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_2)\right\}\right\}\\&=min\left\{min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}((\vartheta_3\cdot\vartheta_1)\cdot(\vartheta_3\cdot\vartheta_2)),\rho^{\mathfrak{T}}_{\mathfrak{N}}((\eta_3\cdot\eta_1)\cdot(\eta_3\cdot\eta_2))\right\},min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta_2),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta_2)\right\}\right\}\\&=min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1)),((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_2,\eta_2)\right\},\end{aligned}$$

$$\rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_1,\eta_1)=max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{I}}_{\mathfrak{N}}(\eta_1)\right\}$$

$$\geq max\left\{max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}((\vartheta_3\cdot\vartheta_1)\cdot(\vartheta_3\cdot\vartheta_2)),\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta_2)\right\},max\left\{\rho^{\mathfrak{I}}_{\mathfrak{N}}((\eta_3\cdot\eta_1)\cdot(\eta_3\cdot\eta_2)),\rho^{\mathfrak{I}}_{\mathfrak{N}}(\eta_2)\right\}\right\}$$
$$= max\left\{max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}((\vartheta_3\cdot\vartheta_1)\cdot(\vartheta_3\cdot\vartheta_2)),\rho^{\mathfrak{I}}_{\mathfrak{N}}((\eta_3\cdot\eta_1)\cdot(\eta_3\cdot\eta_2))\right\},min\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta_2),\rho^{\mathfrak{I}}_{\mathfrak{N}}(\eta_2)\right\}\right\}$$
$$= max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1)),((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_2,\eta_2)\right\}$$

and

$$\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_1,\eta_1) = max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_1),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_1)\right\}$$
$$\geq max\left\{max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}((\vartheta_3\cdot\vartheta_1)\cdot(\vartheta_3\cdot\vartheta_2)),\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_2)\right\},max\left\{\rho^{\mathfrak{F}}_{\mathfrak{N}}((\eta_3\cdot\eta_1)\cdot(\eta_3\cdot\eta_2)),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_2)\right\}\right\}$$
$$= max\left\{max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}((\vartheta_3\cdot\vartheta_1)\cdot(\vartheta_3\cdot\vartheta_2)),\rho^{\mathfrak{F}}_{\mathfrak{N}}((\eta_3\cdot\eta_1)\cdot(\eta_3\cdot\eta_2))\right\},min\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta_2),\rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta_2)\right\}\right\}$$
$$= max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1)),((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta_2,\eta_2)\right\}$$

Hence, $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$.

Theorem 3. *Let* $\mathfrak{M} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}},\rho^{\mathfrak{I}}_{\mathfrak{M}},\rho^{\mathfrak{F}}_{\mathfrak{M}}\right\rangle$ *and* $\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{N}},\rho^{\mathfrak{I}}_{\mathfrak{N}},\rho^{\mathfrak{F}}_{\mathfrak{N}}\right\rangle$ *be two FNCINK-Is of* χ_1 *and* χ_2*, respectively. Then, the direct product* $\mathfrak{M}\times\mathfrak{N}$*, defined by*

$$\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle,$$

is a FNCINK-I of $\chi_1\times\chi_2$.

Proof. By applying Theorem 2, the FNS $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$. Now, $\forall(\vartheta,\eta)\in\chi_1\times\chi_2$, we have

$$\begin{aligned}\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}((0,0)\cdot(\vartheta,\eta)) &\leq \rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}((0\cdot\vartheta),(0\cdot\eta))\\ &= min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(0\cdot\vartheta),\rho^{\mathfrak{T}}_{\mathfrak{N}}(0\cdot\eta)\right\}\\ &\leq min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}}(\vartheta),\rho^{\mathfrak{T}}_{\mathfrak{N}}(\eta)\right\}\\ &= \rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta,\eta),\end{aligned}$$

$$\begin{aligned}\rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}((0,0)\cdot(\vartheta,\eta)) &\geq \rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}((0\cdot\vartheta),(0\cdot\eta))\\ &= max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(0\cdot\vartheta),\rho^{\mathfrak{I}}_{\mathfrak{N}}(0\cdot\eta)\right\}\\ &\geq max\left\{\rho^{\mathfrak{I}}_{\mathfrak{M}}(\vartheta),\rho^{\mathfrak{I}}_{\mathfrak{N}}(\eta)\right\}\\ &= \rho^{\mathfrak{I}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta,\eta)\end{aligned}$$

and

$$\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}((0,0)\cdot(\vartheta,\eta)) \geq \rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}((0\cdot\vartheta),(0\cdot\eta))$$

$$
\begin{aligned}
&= max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(0\cdot\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{N}}(0\cdot\eta)\right\}\\
&\geq max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}}(\vartheta), \rho^{\mathfrak{F}}_{\mathfrak{N}}(\eta)\right\}\\
&= \rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(\vartheta,\eta).
\end{aligned}
$$

Hence, $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNCINK-I of $\chi_1\times\chi_2$.

Theorem 4. *Let* $\mathfrak{M} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}}, \rho^{\mathfrak{I}}_{\mathfrak{M}}, \rho^{\mathfrak{F}}_{\mathfrak{M}}\right\rangle$ *and* $\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{N}}, \rho^{\mathfrak{I}}_{\mathfrak{N}}, \rho^{\mathfrak{F}}_{\mathfrak{N}}\right\rangle$ *be two FNINK-Is of* χ_1 *and* χ_2, *respectively. Then,* $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}, \overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ *is a FNINK-I of* $\chi_1\times\chi_2$, *where* $\overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})} = 1-\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}$.

Proof. According to Theorem 2. $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$. Then,

$$
\begin{aligned}
\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0) &\leq \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)\\
1-\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(0,0) &\geq 1-\rho^{\mathfrak{T}}_{(\mathfrak{M})\times\mathfrak{N})}(\vartheta,\eta)\\
\overline{\rho}^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(0,0) &\geq \overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta).
\end{aligned}
$$

Now, for any $(\vartheta_1,\eta_1),(\vartheta_2,\eta_2),(\vartheta_3,\eta_3)\in\chi_1\times\chi_2$. We have

$$
\begin{aligned}
\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) &\leq min\left\{\rho^{\mathfrak{T}}_{\mathfrak{M}\times\mathfrak{N}}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))), \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}\\
1-\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) &\geq 1-min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))), \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}\\
\overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) &\geq max\left\{1-\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))), 1-\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}\\
\overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) &\geq max\left\{\overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))), \overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}.
\end{aligned}
$$

Hence, $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}, \overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$.

Theorem 5. *Let* $\mathfrak{M} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}}, \rho^{\mathfrak{I}}_{\mathfrak{M}}, \rho^{\mathfrak{F}}_{\mathfrak{M}}\right\rangle$ *and* $\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{N}}, \rho^{\mathfrak{I}}_{\mathfrak{N}}, \rho^{\mathfrak{F}}_{\mathfrak{N}}\right\rangle$ *be two FNINK-Is of* χ_1 *and* χ_2, *respectively. Then,* $\mathfrak{M}\times\mathfrak{N} = \left\langle \overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ *is a FNINK-I of* $\chi_1\times\chi_2$, *where* $\overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} = 1-\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}$.

Proof. According to Theorem 2. $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}, \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$. Then,

$$
\begin{aligned}
\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0) &\geq \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)\\
1-\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(0,0) &\leq 1-\rho^{\mathfrak{F}}_{(\mathfrak{M})\times\mathfrak{N})}(\vartheta,\eta)
\end{aligned}
$$

$$\overline{\rho}^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(0,0) \leq \overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta).$$

Now, for any $(\vartheta_1,\eta_1),(\vartheta_2,\eta_2),(\vartheta_3,\eta_3) \in \chi_1\times\chi_2$. We have

$$\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) \geq max\left\{\rho^{\mathfrak{F}}_{\mathfrak{M}\times\mathfrak{N}}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}$$
$$1-\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) \leq 1-max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}$$
$$\overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) \leq min\left\{1-\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),1-\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}$$
$$\overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1) \leq min\left\{\overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}.$$

Hence, $\mathfrak{M}\times\mathfrak{N} = \left\langle \overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$.

Theorem 6. *Let $\mathfrak{M} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{M}},\rho^{\mathfrak{J}}_{\mathfrak{M}},\rho^{\mathfrak{F}}_{\mathfrak{M}}\right\rangle$ and $\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{\mathfrak{N}},\rho^{\mathfrak{J}}_{\mathfrak{N}},\rho^{\mathfrak{F}}_{\mathfrak{N}}\right\rangle$ be two FNINK-Is of χ_1 and χ_2, respectively. Then, $\mathfrak{M}\times\mathfrak{N} = \left\langle \overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})},\overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$, where $\overline{\rho}^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})} = 1-\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}$ and $\overline{\rho}^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})} = 1-\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}$.*

Proof. The proof is produced by using Theorem 4 and Theorem 5 together.

Theorem 7. *Let $\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$ be a FNINK-I of $\chi_1\times\chi_2$. Then, $(\mathfrak{M}\times\mathfrak{N})^s = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s},\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})^s},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$.*

Proof. For any $(\vartheta,\eta)\in\chi_1\times\chi_2$. Then,

$$\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0) \leq \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)$$
$$\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)\right\}^s \leq \left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)\right\}^s$$
$$\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)^s\right\} \leq \left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)^s\right\}$$
$$\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s}(0,0)\right\} \leq \left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta,\eta)\right\},$$

$$\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0) \geq \rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)$$
$$\left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)\right\}^s \geq \left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)\right\}^s$$
$$\left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)^s\right\} \geq \left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)^s\right\}$$
$$\left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})^s}(0,0)\right\} \geq \left\{\rho^{\mathfrak{J}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta,\eta)\right\}$$

and

$$\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0) \geq \rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)$$

$$\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)\right\}^s \geq \left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)\right\}^s$$
$$\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0)^s\right\} \geq \left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta)^s\right\}$$
$$\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}(0,0)\right\} \geq \left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta,\eta)\right\}.$$

If $(\vartheta_1,\eta_1),(\vartheta_2,\eta_2)$ *and* $(\vartheta_3,\eta_3)\in\chi_1\times\chi_2$, then

$$\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1)\right\}^s \leq min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}^s$$
$$\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1)^s\right\} \leq min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2)))^s,\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)^s\right\}$$
$$\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta_1,\eta_1)\right\} \leq min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta_2,\eta_2)\right\},$$

$$\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1)\right\}^s \geq max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}^s$$
$$\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1)^s\right\} \geq max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2)))^s,\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)^s\right\}$$
$$\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta_1,\eta_1)\right\} \geq max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})^s}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta_2,\eta_2)\right\}$$

and

$$\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1)\right\}^s \geq max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\}^s$$
$$\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1)^s\right\} \geq max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2)))^s,\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)^s\right\}$$
$$\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta_1,\eta_1)\right\} \geq max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}(\vartheta_2,\eta_2)\right\}.$$

Hence, $(\mathfrak{M}\times\mathfrak{N})^s = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})^s},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})^s},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})^s}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$.

Theorem 8. *Let*

$$\mathfrak{M}\times\mathfrak{N} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}\right\rangle$$

and

$$\mathfrak{D}\times\mathfrak{E} = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})},\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})},\rho^{\mathfrak{F}}_{(\mathfrak{D}\times\mathfrak{E})}\right\rangle$$

be two FNINK-Is of $\chi_1\times\chi_2$. *Then,*

$$(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E}) = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}\right\rangle$$

is a FNINK-I of $\chi_1\times\chi_2$.

Proof. Since $\mathfrak{M}\times\mathfrak{N}$ and $\mathfrak{D}\times\mathfrak{E}$ are two FNINK-Is of $\chi_1\times\chi_2$. Then, $\forall(\vartheta,\eta)\in\chi_1\times\chi_2$, we have

$$\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(0,0) = min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0),\rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(0,0)\right\}$$

$$\begin{aligned}
&\leq min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta),\rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta,\eta)\right\}\\
&=\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta,\eta),
\end{aligned}$$

$$\begin{aligned}
\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(0,0) &= max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(0,0)\right\}\\
&\geq max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta,\eta)\right\}\\
&=\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta,\eta)
\end{aligned}$$

and

$$\begin{aligned}
\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(0,0) &= max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(0,0),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(0,0)\right\}\\
&\geq max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta,\eta),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta,\eta)\right\}\\
&=\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta,\eta).
\end{aligned}$$

Now, for any $(\vartheta_1,\eta_1),(\vartheta_2,\eta_2)$ *and* $(\vartheta_3,\eta_3)\in\chi_1\times\chi_2$, we have

$$\begin{aligned}
\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta_1,\eta_1) &= min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1),\rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_1,\eta_1)\right\}\\
&\leq min\Bigg\{min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\},\\
&\quad min\left\{\rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\right\}\Bigg\}\\
&= min\Bigg\{min\Big\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\\
&\quad \rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2)))\Big\},min\Big\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2),\rho^{\mathfrak{T}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\Big\}\Bigg\}\\
&= min\Bigg\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta_1,\eta_1) &= max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_1,\eta_1)\right\}\\
&\geq max\Bigg\{max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\},\\
&\quad max\left\{\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\right\}\Bigg\}\\
&= max\Bigg\{max\Big\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),
\end{aligned}$$

$$\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2)))\Big\}, max\Big\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2),\rho^{\mathfrak{I}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\Big\}\Bigg\}$$

$$= max\Bigg\{\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\Bigg\},$$

and

$$\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta_1,\eta_1) = max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_1,\eta_1),\rho^{\mathfrak{F}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_1,\eta_1)\right\}$$

$$\geq max\Bigg\{max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2)\right\},$$

$$max\left\{\rho^{\mathfrak{F}}_{(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\right\}\Bigg\}$$

$$= max\Bigg\{max\Big\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),$$

$$\rho^{\mathfrak{F}}_{(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2)))\Big\}, max\Big\{\rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})}(\vartheta_2,\eta_2),\rho^{\mathfrak{F}}_{(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\Big\}\Bigg\}$$

$$= max\Bigg\{\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(((\vartheta_3,\eta_3)\cdot(\vartheta_1,\eta_1))\cdot((\vartheta_3,\eta_3)\cdot(\vartheta_2,\eta_2))),\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}(\vartheta_2,\eta_2)\Bigg\}.$$

Hence, $(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E}) = \left\langle \rho^{\mathfrak{T}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})},\rho^{\mathfrak{I}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})},\rho^{\mathfrak{F}}_{(\mathfrak{M}\times\mathfrak{N})\cap(\mathfrak{D}\times\mathfrak{E})}\right\rangle$ is a FNINK-I of $\chi_1\times\chi_2$.

4. Comparison Analysis

A common ground between FNINK-Algebras and Neutrosophic INK-Algebras is INK-algebra, which emphasizes the integration of non-membership, indeterminacy, and uncertainty in algebraic structures. Both approaches are intended for complex system modelling and analysis, where a high prevalence of imprecise and incomplete information exists. Although both approaches provide useful tools for managing uncertainties in algebraic structures, FNINK-Algebras are a better method because of their improved specificity and precision. FNINK-algebras are a more sophisticated and elegant mathematical framework because the incorporation of Fermatean features enables a more detailed representation of indeterminacies and non-memberships. Specialized conditions for FNCINK-Is and direct products add to the robustness and generalizability of the method in different fields. As a result, FNINK-Algebras become the method of choice for accurately and completely representing uncertainties in complex systems.

5. Conclusion

The notion of the direct product of FNSs is used in this paper to discuss an INK-ideal inside an INK-algebra. We present the direct product for FNCINK-Is and Fermatean neutrosophic INK-algebras, examining a number of properties. The direct product of FNSs is shown to satisfy certain requirements in order to be considered a direct product of FNINK-Is inside an INK-algebra.

References

[1] L. A. Zadeh, Fuzzy sets, Inf. Control 8: 338-353, 1965.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst., 20(1): 87-96, 1986.

[3] M. Abdel Basset, A. Gamal, L. H. Son, F. Smarandache, A bipolar neutrosophic multi criteria decision making framework for professional selection, Appl. Sci., 10(4): 1202, 2000.

[4] F. Smarandache, Neutrosophic set-a generalization of the Intuitionistic fuzzy set, Journal of Pure Applied Mathematics, 24(3): 287-297. 2005.

[5] Y. B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. 117(11): 377-383, 2017.

[6] M. Kaviyarasu, K. Indhira, V. M. Chandrasekaran, Fuzzy subalgebras and fuzzy INK-ideals in INK-algebras, Int. J. Pure Appl. Math., 113(6): 47-55, 2017.

[7] M. Kaviyarasu, K. Indhira, Review on BCI/BCK-algebras and development, Int. J. Pure Appl. Math., 117(11): 377-383, 2017.

[8] M. Kaviyarasu, K. Indhira, V. M. Chandrasekaran, Fuzzy p-ideal in INK-algebra, Journal of XI'an University of Architecture & Technology, 12(3): 4746- 4752, 2018.

[9] Y. B. Jun., F. Smarandache, H. Bordbar, Neutrosophic N-structures applied to BCK/BCI-algebras, Information 8(4): 128, 2017.

[10] Y. B. Jun., F. Smarandache, S. Z. Song, M. Khan, Neutrosophic positive Implicative N-ideals in BCK-algebras, Axioms, 7(1): 2018.

[11] M. A. Ozturk, Y. B. Jun, Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points, Journal of The International Mathematical Virtual Institute, 8: 1-17, 2018.

[12] M. Songsaeng, A. Iampan, Neutrosophic set theory applied to UP-algebra, Eur. J. Pure Appl. Math., 12(4): 1382- 1409, 2019.

[13] M. Kaviyarasu, K. Indhira, V. M. Chandrasekaran, Direct product of intuitionistic fuzzy INK-ideals of INK-algebras, Material Today: Proceedings, 16(2): 449-455, 2019.

[14] M. Kaviyarasu, K. Indhira., V. M. Chandrasekaran, Intuitionistic fuzzy translation on INK-Algebra, Adv. Math., Sci. J., 9(1): 295-303, 2020.

[15] M. Kaviyarasu, K. Indhira., V. M. Chandrasekaran, Neutrosophic set in INK-Algebra, Adv. Math., Sci. J., 9(7): 4345-4352, 2020.

[16] F. Smarandache, Neutro algebra is a generalization of partial algebra, Int. J. Neutrosophic Sci., 2(1): 8-17, 2020.

[17] F. Smarandache, Introduction to neutro algebraic structures and anti algebraic structures (revisited), Fuzzy Sets Syst., 31: 1-16, 2020.

[18] M. Abdel-Basset, R. Mohamed, A. H. Zaied , A. Gamal, F. Smarandache, Solving the supply chain problem using the best-worst method based on a novel Plithogenic model, Optimization Theory Based on Neutrosophic and Plithogenic Sets, 1-19, 2020.

[19] M. Abdel Basset, W. Ding, R. Mohamed, N. Metawa, An integrated plithogenic MCDM approach for financial performance evaluation of manufacturing industries, Risk Manag., 1-27. 2020.

[20] M. Abdel Basset, Mohamed., M. Elhoseny, A novel framework to evaluate innovation value proposition for smart product-service systemsl, Environ. Technol. Innov., 101036, 2020.

[21] P. B. Remya, A. Francina Shalini, Neutrosophic vague binary BCK/BCI-algebra, Fuzzy Sets Syst., 35: 45-67, 2020.

[22] P. Muralikrishna, S. Manokaran, MBJ-neutrosophic B-ideal of B-algebra, Fuzzy Sets Syst., 35: 99-118, 2020.

[23] A. Al-Masarwah, M. Alqahtani, Operational algebraic properties and subsemigroups of semigroups in view of N-folded -structures, AIMS Math., 8(9): 22081-22096, 2023.

[24] A. Fallatah, M. Massa'deh, A. Alkouri, Normal and cosets of (γ, δ)-fuzzy HX-subgroups. J. Appl. Math. Inform. 40: 719–727, 2022.

[25] A. Al-Masarwah, A.G. Ahmad, m-polar fuzzy ideals of BCK/BCI-algebras, J. King Saud Univ. Sci. 31: 1220–1226, 2019.

[26] E.A. Abuhijleh, M. Massa'deh, A. Sheimat, A. Alkouri, Complex fuzzy groups based on Rosenfeld's approach, WSEAS Trans. Math. 20: 368–377, 2021.

[27] A. Al-Masarwah, A.G. Ahmad, m-polar (α, β)-fuzzy ideals in BCK/BCI-algebras. Symmetry 11: 44, 2019.