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σ-Prime Spectrum of Almost Distributive Lattices

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Abstract. For each α-ideal of an almost distributive lattice (ADL) to become a σ-ideal, a set of equivalent conditions is derived, which tends to result in a characterization of generalized Stone ADLs. On an ADL, a one-to-one correspondence is derived between the set of all prime σ-ideals of the ADL and the set of all prime σ-ideals of the quotient ADL. Finally, proved some properties of prime σ-ideals of a normal ADL topologically.

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1. Introduction

The concept of an almost distributive lattice (ADL) was introduced by Swamy and Rao, [13] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(R)$ of all principal ideals of R forms a distributive lattice. Also, the concepts of minimal prime ideal belonging to an ideal of an ADL in [7], normal ADL in [6], pseudo-complemented ADL in [14] and Stone ADL in [15] were introduced. The notions of α-ideals and σ-ideals of distributive lattices were introduced in [2] and [3] respectively. In this paper, for each α-ideal of an ADL to become a σ-ideal, a set of equivalent conditions is derived, which tends to result in a characterization of generalized Stone ADLs. Some necessary and sufficient conditions for the maximal ideal

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of an ADL are derived. For each ideal of an ADL that becomes a σ-ideal, a set of equivalent conditions is derived, resulting in a characterization of relatively complemented ADLs. On an ADL, a one-to-one correspondence is derived between the set of all prime σ-ideals of the ADL and the set of all prime σ-ideals of the quotient ADL. Finally, we proved some properties of the set of all σ-ideals of a normal ADL topologically. Throughout this paper, R stands for an ADL with 0.

2. Preliminaries

This section contains definitions and important results from [5] and [13], which will be required in the paper's text.

Definition 1. *[13] An algebra* $R = (R, \vee, \wedge, 0)$ *of type* $(2, 2, 0)$ *is called an almost distributive lattice (abbreviated as ADL) if it satisfies the following conditions:*

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(3) $(a \vee b) \wedge b = b$

(4) $(a \vee b) \wedge a = a$

(5) $a \vee (a \wedge b) = a$

(6) $0 \wedge a = 0$

(7) $a \vee 0 = a$ *for all* $a, b, c \in R$.

Theorem 1. *[13] If* $(R, \vee, \wedge, 0)$ *is an ADL, then*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* R

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(7) $a \wedge (a \vee b) = a, (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(8) $a \wedge a = a$ *and* $a \vee a = a$ *for all* $a, b, c \in R$.

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties makes an ADL R a distributive lattice.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.

As in distributive lattices [1, 4], a non-empty subset I of an ADL R is called an ideal of R if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$. Also, a non-empty subset F of R is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

The set $\mathcal{I}(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $I, J \in \mathcal{I}(R)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal (filter) P of R is called a prime ideal (filter) if for any $x, y \in R, x \wedge y \in P (x \vee y \in P) \Rightarrow x \in P$ or $y \in P$. A proper ideal (filter) M of R is said to be maximal if it is not properly contained in any proper ideal (filter) of R. It can be observed that every maximal ideal (filter) of R is a prime ideal (filter). Every proper ideal (filter) of R is contained in a maximal ideal (filter). For any subset S of R the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$ and such an ideal is called the principal ideal of R. Similarly, for any $S \subseteq R$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$ and such a filter is called the principal filter of R.

For any $a, b \in R$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \wedge (b] = (a \wedge b]$. Hence, the set $(PI(R), \vee, \cap)$ of all principal ideals of R is a sublattice of the distributive lattice $(\mathcal{I}(R), \vee, \cap)$ of all ideals of R. Also, we have that the set $(\mathcal{F}(R), \vee, \cap)$ of all filters of R is a bounded distributive lattice.

Definition 2. *[9] An ideal I of an ADL R is said to be dense if $I^* = (0]$. Otherwise, I is called a non-dense ideal.*

Definition 3. *[10] For any non-empty subset A of an ADL R, define $A^* = \{x \in R \mid a \wedge x = 0$ for all $a \in A\}$. Here, A^* is called the annihilator of A in R.*

For any $a \in R$, we have $\{a\}^* = (a)^*$.

Annihilators have many important properties. We give some of them in the following:

Theorem 2. *[10] Let R be an ADL. For any $x, y \in R$, we have:*

(1) $x \leq y \Rightarrow (y)^* \subseteq (x)^*$

(2) $(x \wedge y)^* = (y \wedge x)^*$

(3) $(x \vee y)^* = (y \vee x)^*$

(4) $(x \vee y)^* = (x)^* \cap (y)^*$

(5) $(x)^* \vee (y)^* \subseteq (x \wedge y)^*$

(6) $x = 0 \Leftrightarrow (x)^* = R$.

Definition 4. *[7] A prime ideal of R is called a minimal prime ideal if it is a minimal element in the set of all prime ideals of R ordered by set inclusion.*

Theorem 3. *[7] Let R be an ADL. Then a prime ideal P is minimal if and only if for any $x \in P$, there exists an element $y \notin P$ such that $x \wedge y = 0$.*

Theorem 4. *[7] Let R be an ADL with maximal elements. Then P is a prime ideal of R if and only if $R \setminus P$ is a prime filter of R.*

Definition 5. *[9] An ideal I of an ADL R is said to be an α-ideal if $(a]^{**} \subseteq I$ for all $a \in I$.*

In [13], an ADL R is said to be relatively complemented if for any $a, b \in R$ with $a \leq b$, the interval $[a, b]$ is a complemented lattice.

Theorem 5. *[13] An ADL R is relatively complemented if and only if for any $a, b \in R$, there exists a unique $x \in R$ such that $a \vee x = a \vee b$ and $a \wedge b = 0$.*

Definition 6. *[12] For any ADL R with maximal elements, define $B = \{a \in L \mid$ there exists $b \in R$ such that $a \wedge b = 0$ and $a \vee b$ is maximal$\}$, this is called a Birkhoff centre of an ADL R. If $B = R$, then R is called a complemented ADL.*

Definition 7. *[14] Let $(R, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on R is called a pseudo-complementation on R if for any $a, b \in R$, it satisfies the following conditions:*

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$.

Then $(R, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Definition 8. *[15] Let R be an ADL and $*$ a pseudo-complementation on R. Then R is called a Stone ADL if for any $x \in R, x^* \vee x^{**} = 0^*$.*

Lemma 1. *[15] Let R be a Stone ADL and $a, b \in R$. Then the following conditions hold:*

(1) $0^* \wedge a = a$ *and* $0^* \vee a = 0^*$

(2) $(a \wedge b)^* = a^* \vee b^*$.

Definition 9. *[10] An ADL R is said to be a generalized Stone ADL if $(a]^* \vee (a]^{**} = R$ for all $a \in R$.*

3. On σ-ideals of ADLs

For each α-ideal of an ADL to become a σ-ideal, a set of equivalent conditions is derived, which tends to result in a characterization of generalized Stone ADLs. For each ideal of an ADL that becomes a σ-ideal, a set of equivalent conditions is derived, resulting in a characterization of relatively complemented ADLs. On an ADL, a one-to-one correspondence is derived between the set of all prime σ-ideals of the ADL and the set of all prime σ-ideals of the quotient ADL. The following definition is adopted from [11].

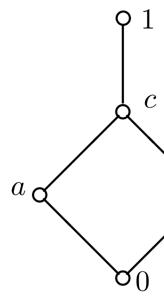
Definition 10. *An ideal K of an ADL R is said to be a σ-ideal of R if $K^\sigma = K$, where $K^\sigma = \{x \in R \mid K \vee (x)^* = R\}$.*

Proposition 1. *Every prime σ-ideal of an ADL R with maximal elements is a minimal prime ideal.*

Proof. Let M be any prime σ-ideal of an ADL R. Then $M^\sigma = M$. Let $t \in M = M^\sigma$. Then $(t)^* \vee M = R$. That implies $x \vee y$ is a maximal element of R for some $x \in (t)^*$ and $y \in M$. Since M is prime, we get $x \notin M$. Hence for any $t \in M$, there exists an element $x \notin M$ such that $x \in (t)^*$ (i.e., $x \wedge t = 0$). Hence, M is a minimal prime ideal of R.

In general, every minimal prime ideal need not be a σ-ideal.

Example 1. *Consider a distributive lattice $L = \{0, a, b, c, 1\}$ and discrete ADL $D = \{0', a'\}$.*



Clearly, $R = D \times L = \{(0',0), (0',a), (0',b), (0',c), (0',1), (a',0), (a',a), (a',b), (a',c), (a',1)\}$ is an ADL with zero element $(0, 0')$. Consider the minimal prime ideal $P = \{(0',0), (0',a)\}$. Clearly, $(0',a)^* \vee P \neq R$. That implies $(0',a) \notin P^\sigma$. Hence, P is not a σ-ideal of R.

Theorem 6. *In an ADL R with maximal elements, the following conditions are equivalent:*

(1) R is a generalized Stone ADL

(2) every α-ideal is a σ-ideal

(3) every prime α-ideal is a σ-ideal

(4) every minimal prime ideal is a σ-ideal.

Proof. (1) $\Rightarrow$ (2): Assume (1). Let K be an α-ideal of an ADL R. Then $(a)^{**} \subseteq K$ for all $a \in K$. Since R is a generalized Stone ADL, we have $(a)^* \vee (a)^{**} = R$ and hence $(a)^* \vee K = R$. That implies $a \in K^\sigma$ for all $a \in K$. Therefore, $K \subseteq K^\sigma$. Clearly, we have $K^\sigma \subseteq K$. Thus, $K^\sigma = K$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (4): Assume (3). Clearly, we have that every minimal prime ideal is a prime α-ideal. By our assumption, we get that every minimal prime ideal is a σ-ideal.

(4) $\Rightarrow$ (1): Assume (4). Let $a \in R$. We prove that $(a)^* \vee (a)^{**} = R$. Suppose $(a)^* \vee (a)^{**} \neq R$. Then there exists a prime filter Q such that $((a)^* \vee (a)^{**}) \cap Q = \emptyset$. Since Q is a proper filter, there exists a maximal filter N such that $Q \subseteq N$. Clearly, $R \setminus N$ is a minimal prime ideal. By our assumption, we get $R \setminus N$ is a σ-ideal of R. If $a \in N$, there exists an element $b \notin N$ such that $a \vee b$ is a maximal element of R. Since $a \vee b \in Q \subseteq N$ and $b \notin N$, we get $a \in Q$. Since $a \in (a)^{**} \subseteq (a)^* \vee (a)^{**}$, we get $((a)^* \vee (a)^{**}) \cap Q \neq \emptyset$, it gives a contradiction. Therefore, $a \notin N$ and hence $a \in R \setminus N = (R \setminus N)^\sigma$. That implies $(a)^* \vee (R \setminus N) = R$. So that there exist $x \in (a)^*$ and $y \in R \setminus N$ such that $x \vee y$ is maximal. Since $y \in R \setminus N$, we get $y \notin Q$. Since $x \vee y$ is maximal, we get $x \in Q$. Since $x \in (a)^* \subseteq (a)^* \vee (a)^{**}$, we get $((a)^* \vee (a)^{**}) \cap Q \neq \emptyset$, which is a contradiction. Hence, $(a)^* \vee (a)^{**} = R$.

We denote Spec$R$ and Max$R$ as the sets of all prime ideals and maximal ideals of R, respectively. For any $N \in \text{Max}R$, define $N^{\mathcal{O}} = \{a \in R \mid (a)^* \nsubseteq N\}$.

Proposition 2. *For any $N \in \text{Max}R$, $N^{\mathcal{O}}$ is an ideal of R contained in N.*

Proof. Clearly, $(0)^* = R \nsubseteq N$ and hence $N^{\mathcal{O}}$ is non-empty. Let $a, b \in N^{\mathcal{O}}$. Then $(a)^* \nsubseteq N$ and $(b)^* \nsubseteq N$. That implies $(a \vee b)^* \subseteq (a)^* \nsubseteq N$. Hence, $a \vee b \in N^{\mathcal{O}}$. Let $a \in N^{\mathcal{O}}$. Then $(a)^* \nsubseteq N$. Let r be any element of R. Since $r \wedge a \leq a$, we get $(a)^* \subseteq (r \wedge a)^* = (a \wedge r)^*$. Since $(a)^* \nsubseteq N$, we get $(a \wedge r)^* \nsubseteq N$. That implies $a \wedge r \in N^{\mathcal{O}}$. Therefore, $N^{\mathcal{O}}$ is an ideal of R. Let $a \in N^{\mathcal{O}}$. Then $(a)^* \nsubseteq N$. There exists $t \in (a)^*$ such that $t \notin N$. Since $t \in (a)^*$, we have $a \wedge t = 0$. We have that every maximal ideal is prime and hence N is prime. Since $a \wedge t = 0 \in N$ and $t \notin N$, we get $a \in N$. Therefore, $N^{\mathcal{O}} \subseteq N$.

For any ideal K of an ADL R, define $\mathcal{W}(K) = \{N \in \text{Max}R \mid K \subseteq N\}$.

Proposition 3. *Let K be an ideal of R. Then we have the following:*

(1) $K^\sigma = \bigcap_{N \in \mathcal{W}(K)} N^{\mathcal{O}}$

(2) if $K \in \text{Spec}R$, then $K^\sigma \subseteq K^{\mathcal{O}}$

(3) if $K \in \text{Max}R$, then $K^\sigma = K^{\mathcal{O}}$.

Proof. (1) Let $a \in K^\sigma$. Then $(a)^* \vee K = R$. Let $N \in \mathcal{W}(K)$. Then $K \subseteq N$. That implies $(a)^* \vee N = R$ and hence $(a)^* \nsubseteq N$. That implies $a \in N^{\mathcal{O}}$ for $N \in \text{Max}R$. Therefore, $K^\sigma \subseteq \bigcap_{N \in \mathcal{W}(K)} N^{\mathcal{O}}$. Conversely, let $a \in \bigcap_{N \in \mathcal{W}(K)} N^{\mathcal{O}}$. Then $a \in \mathcal{O}(N)$ for all

$N \in \mathcal{W}(K)$. That implies $(a)^* \nsubseteq N$ for all $N \in \text{Max}R, K \subseteq N$. Suppose $(a)^* \vee K \neq R$. Then $N' \in \text{Max}R$ such that $(a)^* \vee K \subseteq N'$. That implies $(a)^* \subseteq N'$ and $K \subseteq N'$, which is a contradiction. That implies $(a)^* \vee K = R$. Therefore, $a \in K^\sigma$ and hence $\bigcap_{N \in \mathcal{W}(K)} N^{\mathcal{O}} \subseteq K^\sigma$. Thus, $K^\sigma = \bigcap_{N \in \mathcal{W}(K)} N^{\mathcal{O}}$.

(2) Let $K \in \text{Spec}R$ and $a \in K^\sigma$. Then $(a)^* \vee K = R$. Since K is a proper ideal of R, we get $(a)^* \nsubseteq K$ and hence $a \in K^{\mathcal{O}}$. Therefore, $K^\sigma \subseteq K^{\mathcal{O}}$.

(3) Let $K \in \text{Max}R$. Clearly, we have that K is a prime ideal of R. By (2), we get $K^\sigma \subseteq K^{\mathcal{O}}$. Let $a \in K^{\mathcal{O}}$. Then $(a)^* \nsubseteq K$. That implies $(a)^* \vee K = R$. Therefore, $a \in K^\sigma$ and hence $K^{\mathcal{O}} \subseteq K^\sigma$. Thus, $K^\sigma = K^{\mathcal{O}}$.

Theorem 7. *In an ADL R with maximal elements, the following are equivalent:*

(1) R is complemented ADL

(2) for any $N \in \text{Max}R$, $N^{\mathcal{O}} \in \text{Max}R$

(3) for any ideals N, N' of R, $N \vee N' = R \Rightarrow N^\sigma \vee N'^\sigma = R$

(4) for any ideals N, N' of R, $N \vee N' = R \Rightarrow N^\sigma \vee N'^\sigma = (N \vee N')^\sigma$

(5) for any $N, N' \in \text{Max}R$ with $N \neq N'$, $N^{\mathcal{O}} \vee N'^{\mathcal{O}} = R$

(6) for any $N \in \text{Max}R, N$ is the unique member of $\text{Max}R$ such that $N^{\mathcal{O}} \subseteq N$.

Proof. (1) $\Rightarrow$ (2): Assume (1). Let $N \in \text{Max}R$ and $a \in N$. By our assumption, there exists an element $a' \in R$ such that $a \wedge a' = 0$ and $a \vee a'$ is maximal. Clearly, we get that $a' \in (a)^*$ and $a' \notin N$. That implies $(a)^* \nsubseteq N$ and hence $a \in N^{\mathcal{O}}$. Therefore, $N \subseteq N^{\mathcal{O}}$. Since $N \subseteq N^{\mathcal{O}}$, we get $N = N^{\mathcal{O}}$. Since $N \in \text{Max}R$, we get that $N^{\mathcal{O}} \in \text{Max}R$.

(2) $\Rightarrow$ (3): Assume (2). Let N, N' be any two ideals of R such that $N \vee N' = R$. Suppose $N^\sigma \vee N'^\sigma \neq R$. Then there exists $Q \in \text{Max}R$ such that $N^\sigma \vee N'^\sigma \subseteq Q$. That implies $N^\sigma \subseteq Q$ and $N'^\sigma \subseteq Q$. By Proposition 3, $\bigcap_{N_i \in \mathcal{W}(N)} N_i^{\mathcal{O}} \subseteq Q$ and $\bigcap_{N'_i \in \mathcal{W}(N')} N_i'^{\mathcal{O}} \subseteq Q$. That implies $N_i^{\mathcal{O}} \subseteq Q$ and $N_i'^{\mathcal{O}} \subseteq Q$, for some $N_i \in \mathcal{W}(N)$ and $N'_i \in \mathcal{W}(N')$. By our assumption, we get that $N_i \subseteq Q$ and $N'_i \subseteq Q$. Since $N \subseteq N_i, N' \subseteq N'_i$, we get $N \subseteq Q$ and $N' \subseteq Q$. That implies $L = N \vee N' \subseteq Q$ and hence $Q = R$, we get a contradiction. Thus, $N^\sigma \vee N'^\sigma = R$.

(3) $\Rightarrow$ (4): Assume (3). Let N, N' be two ideals of R. Then $N^\sigma \vee N'^\sigma \subseteq (N \vee N')^\sigma$. Let $a \in (N \vee N')^\sigma$. Then $(a)^* \vee (N \vee N') = R$. That implies $((a^*) \vee N) \vee ((a)^* \vee N') = R$. By our assumption, we have that $((a^*) \vee N)^\sigma \vee ((a)^* \vee N')^\sigma = R$. That implies $a \in ((a^*) \vee N)^\sigma \vee ((a)^* \vee N')^\sigma$. Then there exist $b \in ((a^*) \vee N)^\sigma$ and $c \in ((a)^* \vee N')^\sigma$ such that $a = b \vee c$. Since $b \in ((a^*) \vee N)^\sigma$ and $c \in ((a)^* \vee N')^\sigma$, we get that $(b)^* \vee ((a^*) \vee N) = R$ and $(c)^* \vee ((a)^* \vee N') = R$. That implies $(b \wedge a)^* \vee N = R$ and $(c \wedge a)^* \vee N' = R$. Therefore, $b \wedge a \in N^\sigma$ and $c \wedge a \in N'^\sigma$. Hence, $a = a \wedge a = (b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a) \in N^\sigma \vee N'^\sigma$. Thus, $(N \vee N')^\sigma \subseteq N^\sigma \vee N'^\sigma$.

(4) $\Rightarrow$ (5): Assume (4). Let $N, N' \in \text{Max}R$ such that $N \neq N'$. Then there exist $a, b \in R$ such that $a \in N \setminus N'$ and $b \in N' \setminus N$. That implies $N' \vee (a] = R$ and $N \vee (b] = R$. So that $N' \vee N \vee (a] \vee (b] = R$. Therefore, $N' \vee N \vee (a \vee b] = R$. Since $a \vee b \in N \vee N'$, we get $N \vee N' = R$. That implies $(N \vee N')^{\sigma} = R$. By our assumption, we get $N^{\sigma} \vee N'^{\sigma} = R$. Since $N^{\sigma} \vee N'^{\sigma} \subseteq N^{\mathcal{O}} \vee N'^{\mathcal{O}}$, we get $N^{\mathcal{O}} \vee N'^{\mathcal{O}} = R$.

(5) $\Rightarrow$ (6): Let $N, N' \in \text{Max}R$ with $N^{\mathcal{O}} \subseteq N$ and $N'^{\mathcal{O}} \subseteq N$. Suppose $N \neq N'$. Then by our assumption, we have that $N^{\mathcal{O}} \vee N'^{\mathcal{O}} = R$. That implies $N = R$, which is a contradiction. Hence $N = N'$.

(6) $\Rightarrow$ (1): Let $a \in R$ and m be any maximal element of R. Suppose $m \notin (a] \vee (a)^*$. Then $(a] \vee (a)^* \subseteq N$ for some maximal ideal N of R. That implies $(a] \subseteq N$ and $(a)^* \subseteq N$. Therefore, $a \in N$ and $a \notin N^{\mathcal{O}}$. Since $a \notin N^{\mathcal{O}}$, there is a maximal ideal N_1 of R such that $a \notin N_1$ and $N^{\mathcal{O}} \subseteq N_1$. By our assumption, we get that $N = N_1$ and hence $a \notin N$, which is a contradiction. Therefore, $m \in (a] \vee (a)^*$. That implies there exists an element $t \in (a)^*$ such that $a \vee t = m$. Therefore, $a \wedge t = 0$ and $a \vee t = m$. Thus, L is a complemented ADL.

In general, every maximal ideal of R need not be a σ-ideal.

Example 2. *Let $R = \{0, a, b, c\}$. Define two binary operations $\vee$ and $\wedge$ on R as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Consider the maximal ideal $I = \{0, c\}$. Now, $(c)^* \vee I = \{0\} \vee I = I \neq R$. Hence, every maximal ideal of R is not a σ-ideal.

Theorem 8. *In an ADL R, the following are equivalent:*

(1) R is a complemented ADL

(2) every maximal ideal is a σ-ideal

(3) every maximal ideal is a minimal prime ideal.

Theorem 9. *In an ADL R, the following conditions are equivalent:*

(1) R is relatively complemented

(2) every principal ideal is a σ-ideal

(3) every ideal is a σ-ideal

(4) every prime ideal is a σ-ideal

(5) every prime ideal is minimal.

Proof. (1) $\Rightarrow$ (2): Assume (1). Clearly, we have that $(a]^\sigma \subseteq (a]$ for all $a \in R$. Suppose $(a]^\sigma \nsubseteq (a]$. Then there exists an element $b \in (a]$ such that $b \notin (a]^\sigma$. Since $a, b \in R$ and R is relatively complemented, there exists an element $x \in R$ such that $a \vee x = a \vee b$ and $a \wedge x = 0$. Since $a \wedge x = 0$, we get that $a \in (x)^*$ and hence $b \in (x)^*$. That implies $a \vee b \in (x)^*$, which gives $x \vee a \in (x)^*$. So that $x \in (x)^*$ and hence $x = 0$, which is a contraction. Hence, $(a]^\sigma \subseteq (a]$. Thus, every principal ideal is a σ-ideal.

(2) $\Rightarrow$ (3): Assume (2). Let K be any ideal of R and $a \in K$. Then $(a] \subseteq K$. That implies $(a]^\sigma \subseteq K^\sigma$. By our assumption, we get $(a] \subseteq K^\sigma$ and hence $a \in K^\sigma$. Therefore, $K \subseteq K^\sigma$. Since $K^\sigma \subseteq K$, we get $K^\sigma = K$. Thus, K is a σ-ideal of R.

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (5): Assume (4). Let M be any prime ideal of R. By our assumption, we have that σ-ideal of R. Let $a \in M$. Then $a \in M^\sigma$. That implies $(a)^* \vee M = R$. There exists $s \in (a)^*, t \in M$ such that $s \vee t$ is a maximal element of R. Clearly, we get $s \notin M$ and $s \wedge a = 0$. Therefore, for any $a \in M$, there exists an element $s \notin M$ such that $s \wedge a = 0$. Thus, M is a minimal prime ideal.

(5) $\Rightarrow$ (1): Assume (5). Let $x, y \in R$ such that $x \in [0, y]$. If $y \notin (x] \vee (x)^*$. Then there exists a prime ideal Q of R such that $(x] \vee (x)^* \subseteq Q$. That implies $(x)^* \subseteq Q$ and $(x] \subseteq Q$. So that $(x)^* \subseteq Q$ and $x \in Q$. By our assumption, Q is a minimal prime ideal. Since $(x)^* \subseteq Q$, we get $x \notin Q$, which is a contradiction. Therefore, $y \in (x)^* \vee (x]$. Hence, there exist $s \in (x], t \in (x)^*$ such that $y = t \vee s$. Since $t \in (x)^*$ and $s \in (x]$, we get $t \wedge x = 0$ and $x \wedge s = s$. Now, $x = x \wedge y = x \wedge (t \vee s) = (x \wedge t) \vee (x \wedge s) = x \wedge s = s$. Since $y = t \vee s = t \vee x$, we get $t \in [0, y]$. Therefore, R is relatively complemented.

Define a binary relation ϕ on R as $(a, b) \in \phi$ if and only if $(a)^* = (b)^*$ for all $a, b \in R$. Clearly, ϕ is a congruence relation on R with $ker\phi$ as the smallest congruence class modulo ϕ. Also, we have that R/ϕ is a quotient ADL by defining $[a]_\phi \wedge [b]_\phi = [a \wedge b]_\phi$ and $[a]_\phi \vee [b]_\phi = [a \vee b]_\phi$, where $[a]_\phi$ is the congruence class of x modulo ϕ. It can be easily verified that $f : R \to R/\phi$ is a homomorphism by defining $f(a) = [a]_\phi$.

Lemma 2. *Let ϕ be a congruence relation on an ADL R with a maximal element m and $a, b \in R$. Then*

(1) $a \leq b \Rightarrow [a]_\phi \subseteq [b]_\phi$

(2) $[a]_\phi = [0]_\phi \Leftrightarrow a = 0$

(3) $[a]_\phi = [m]_\phi \Leftrightarrow (a)^* = \{0\}$.

For any ideal K of R, define $\widetilde{K} = \{[a]_\phi \in R/\phi \mid a \in K\}$. Clearly, we have that $K \subseteq \widetilde{K}$.

Theorem 10. *For any α-ideal K of an ADL R, we have the following:*

(1) $[x]_\phi \in \widetilde{K} \Leftrightarrow x \in K$

(2) $\widetilde{K}$ *is an ideal of* R/ϕ

(3) if $\widetilde{K}$ *is a prime ideal of* R*, then* $\widetilde{K}$ *is a prime ideal of* R/ϕ*.*

Proof. (1) Assume that $[x]_\phi \in \widetilde{K}$. Then there exists an element $b \in K$ such that $[a]_\phi = [b]_\phi$. That implies $(a, b) \in \phi$ and hence $(a)^* = (b)^*$. Since K is an α-ideal of R and $b \in K$, we get that $a \in K$. Assume that $a \in K$. Then $[a]_\phi \in \widetilde{K}$.

(2) By (1), it is clear.

(3) By (1) and (2), it can be verified easily.

From the above result, we get the following result:

Corollary 1. *For any α-ideals K_1, K_2 of an ADL R, $K_1 \subseteq K_2 \Leftrightarrow \widetilde{K_1} \subseteq \widetilde{K_2}$.*

For any $s \in R$, define $([s]_\phi)^\circ = \{[t]_\phi \in R/\phi \mid [s]_\phi \wedge [t]_\phi = [0]_\phi\}$.

Lemma 3. *For any $s, t \in R$, we have the following:*

(1) $([s]_\phi)^\circ = \{[t]_\phi \in R/\phi \mid s \wedge t = 0\}$

(2) $([s]_\phi)^\circ$ *is an ideal of* R/ϕ

(3) $([0]_\phi)^\circ = R/\phi$ *and* $([m]_\phi)^\circ = ([0]_\phi)^\circ$, *where m is any maximal element of R*

(4) $t \in (s)^* \Leftrightarrow [t]_\phi \in ([s]_\phi)^\circ$

(5) $(s)^* = (t)^* \Leftrightarrow ([s]_\phi)^\circ = ([t]_\phi)^\circ$

(6) $[s]_\phi \subseteq [t]_\phi \Rightarrow ([t]_\phi)^\circ \subseteq ([s]_\phi)^\circ$

(7) $([s]_\phi)^\circ \cap ([t]_\phi)^\circ = ([s \vee t]_\phi)^\circ$.

Definition 11. *Let K be an ideal of R/ϕ. Define $\rho(K) = \{[a]_\phi \in R/\phi \mid ([a]_\phi)^\circ \vee K = R/\phi\}$, where $([a]_\phi)^\circ \vee K$ is the supremum of $([a]_\phi)^\circ$ and K in R/ϕ.*

Theorem 11. *Let K be an ideal of R with maximal elements and ϕ be the congruence relation on R. Then we have the following:*

(1) $\rho(K)$ is an ideal of R/ϕ contained in K

(2) if K is an α-ideal of R, then so is $\widetilde{K}$ in R/ϕ

(3) if K is a σ-ideal of R, then so is $\widetilde{K}$ in R/ϕ.

Proof. (1) Clear.

(2) For any α-ideal K, we have that $\widetilde{K}$ is an ideal of R/ϕ. Let $a, b \in R$ such that $([a]_\phi)^\circ = ([b]_\phi)^\circ$ and $[a]_\phi \in \widetilde{K}$. Then we have that $(a)^* = (b)^*$ and $a \in K$. Since K is an α-ideal, we get $b \in K$. That implies $[b]_\phi \in \widetilde{K}$. Therefore, $\widetilde{K}$ is an α-ideal of R/ϕ.

(3) Let K be a σ-ideal of R. Clearly, $\widetilde{K}$ is an ideal of R/ϕ and $\rho(\widetilde{K}) \subseteq \widetilde{K}$. Let $[a]_\phi \in \widetilde{K}$. Then $a \in K$. Since K is a σ-ideal of R, we have that $(a)^* \vee K = R$. Then there exist $s \in (a)^*, t \in K$ such that $s \vee t$ is a maximal element of R, say m. That implies $[m]_\phi = [s \vee t]_\phi = [s]_\phi \vee [t]_\phi$ and $[s]_\phi \in ([a]_\phi)^\circ$. Therefore, $[m]_\phi \in ([a]_\phi)^\circ \vee \widetilde{K}$ and hence $([a]_\phi)^\circ \vee \widetilde{K} = R/\phi$. Thus, $\widetilde{K}$ is a σ-ideal of R/ϕ.

We denote $\mathrm{Spec}_\alpha R$ and $\mathrm{Spec}_\alpha R/\phi$ as the sets of all prime α-ideals of R and R/ϕ, respectively.

Theorem 12. *For any congruence relation ϕ on R, the mapping is an order isomorphism of* $\mathrm{Spec}_{\alpha}R$ *onto* $\mathrm{Spec}_{\alpha}R/\phi$.

Proof. Let $M \in \mathrm{Spec}_{\alpha}R$. Then clearly, we have that $\widetilde{M}$ is an α-ideal of R/ϕ. Let $K \in \mathrm{Spec}_{\alpha}R/\phi$. Take $H = \{a \in R \mid [a]_{\phi} \in K\}$. Since K is an α-ideal of R/ϕ, we get that H is an ideal of R. Let $a, b \in R$ with $(a)^* = (b)^*$ and $a \in H$. Then $([a]_{\phi})^{\circ} = ([b]_{\phi})^{\circ}$ and $[a]_{\phi} \in K$. Since K is an α-ideal of R/ϕ, we get $[b]_{\phi} \in K$. That implies $b \in H$. Hence, H is an α-ideal of R. Therefore, $\widetilde{H} = K$. Clearly, we get that $H \in \mathrm{Spec}_{\alpha}R$. Let $M, N \in \mathrm{Spec}_{\alpha}R$ with $M \subseteq N$. Then $\widetilde{M} \subseteq \widetilde{N}$. Hence, the mapping is an order isomorphism of $\mathrm{Spec}_{\alpha}R$ onto $\mathrm{Spec}_{\alpha}R/\phi$.

Lemma 4. *In an ADL R, we have the following:*

(1) every proper σ-ideal contains no dense element

(2) every non-zero σ-ideal is non-dense

(3) every non-dense prime ideal is an α-ideal.

Proof. (1) Let K be any proper σ-ideal of R. Suppose a is a dense element of R with $a \in K$. Then $(a)^* \vee K = R$. That implies $(0] \vee K = R$. Hence, $K = R$, we get a contradiction. Therefore, every proper σ-ideal contains no dense element.

(2) Let K be a non-zero σ-ideal of R. Then there exists a non-zero element $a \in R$ such that $a \in K = K^{\sigma}$. That implies $(a)^* \vee K = R$. Therefore, $((a)^* \vee K)^* = R^*$ and hence $(a)^{**} \cap (K)^* = \{0\}$. Suppose $K^* = \{0\}$. Then $(a)^{**} = \{0\}$ and hence $a = 0$, we get a contradiction. Therefore, K is a non-dense ideal of R.

(3) Let Q be any non-dense prime ideal of R. Then $Q = (a)^*$ for some non-zero element a of R. Let $b \in Q$. Then $b \in (a)^*$. That implies $(b)^{**} \subseteq (a)^* = Q$. Hence, Q is an α-ideal of R.

We denote $\mathrm{Spec}^{\sigma}R$ and $\mathrm{Spec}^{\sigma}R/\phi$ as the sets of all prime σ-ideals of R and R/ϕ, respectively.

Theorem 13. *For any congruence relation ϕ on R and every member of* $\mathrm{Spec}R$ *is non-dense, there is an order isomorphism between* $\mathrm{Spec}^{\sigma}R$ *and* $\mathrm{Spec}^{\sigma}R/\phi$.

Proof. Let $P \in \mathrm{Spec}^{\sigma}R$. Then $\widetilde{P} \in \mathrm{Spec}^{\sigma}R/\phi$ and hence $\widetilde{P} \in \mathrm{Spec}_{\alpha}R/\phi$. Let $K \in \mathrm{Spec}^{\sigma}R/\phi$. Then we have that $K \in \mathrm{Spec}_{\alpha}R/\phi$. Take $H = \{a \in R \mid [a]_{\phi} \in K\}$. Since $K \in \mathrm{Spec}_{\alpha}R/\phi$, we get $H \in \mathrm{Spec}_{\alpha}R$. Hence, $\widetilde{H} = K$. Let $a \in H$. Then $[a]_{\phi} \in \widetilde{H} = K$. That implies $([a]_{\phi})^{\circ} \vee K = R/\phi$. We prove that $(a)^* \vee H = R$. Suppose $(a)^* \vee H \neq R$. Then $(a)^* \vee H \subseteq M$ for some $M \in \mathrm{Spec}R$. That implies $(a)^* \subseteq M$ and $H \subseteq M$. By our assumption, we get that M is non-dense. By the above result, we have that P is an α-ideal of R. Hence, $\widetilde{P} \in \mathrm{Spec}R/\phi$. Since $(a)^* \subseteq M$ and $H \subseteq M$, we get $([a]_{\phi})^{\circ} \subseteq \widetilde{M}$ and $\widetilde{H} \subseteq \widetilde{M}$. That implies $([a]_{\phi})^{\circ} \subseteq \widetilde{M}$ and $K \subseteq \widetilde{M}$. So that $R/\phi = ([a]_{\phi})^{\circ} \vee K \subseteq \widetilde{M}$, which gives $\widetilde{M} = R/\phi$, we get a contradiction. Therefore, $(a)^* \vee H = R$ and hence $a \in H^{\sigma}$. Thus, $H \subseteq H^{\sigma}$. Since $H^{\sigma} \subseteq H$, we get H is a σ-ideal of R.

Corollary 2. *For any congruence relation ϕ on R, we have the following:*

(1) there is one-to-one correspondence between $\mathrm{Spec}_{\alpha}R$ *and* $\mathrm{Spec}_{\alpha}R/\phi$

(2) if every member of $\mathrm{Spec}R$ *is non-dense, then there is one-to-one correspondence between* $\mathrm{Spec}^{\sigma}R$ *and* $\mathrm{Spec}^{\sigma}R/\phi$.

Theorem 14. *For any congruence relation ϕ on R and every member of* $\mathrm{Spec}R$ *is non-dense, every α-ideal of R is a σ-ideal if and only if every α-ideal of R/ϕ is a σ-ideal.*

Proof. Assume that every α-ideal of R is a σ-ideal. Let K be an α-ideal of R/ϕ. Then $K = \widetilde{H}$ for some α-ideal H of R. By our assumption, we get that H is a σ-ideal of R. Hence, $\widetilde{H} = K$ is a σ-ideal of R/ϕ.

Conversely, assume that every α-ideal of R/ϕ is a σ-ideal. Let K be an α-ideal of R. Clearly, we have that $\widetilde{K}$ is an α-ideal of R/ϕ. By our assumption, we get $\widetilde{K}$ is a σ-ideal of R/ϕ. Let $a \in K$. Then $[a]_{\phi} \in \widetilde{K}$. We prove that $(a)^* \vee K = R$. If $(a)^* \vee K \neq R$, then $(a)^* \vee K \subseteq M$ for some $M \in \mathrm{Spec}R$. By the hypothesis, we have that M is non-dense. That implies M is an α-ideal of R. Clearly, we get $\widetilde{M} \in \mathrm{Spec}R/\phi$. Since $(a)^* \vee K \subseteq M$, we have that $(a)^* \subseteq M$ and $K \subseteq M$. That implies $([a]_{\phi})^{\circ} \subseteq \widetilde{M}$ and $\widetilde{K} \subseteq \widetilde{M}$. Therefore, $R/\phi = ([a]_{\phi})^{\circ} \vee \widetilde{K} \subseteq \widetilde{M}$ and hence $\widetilde{M} = R/\phi$, which is a contradiction. Thus, $(a)^* \vee K = R$, which gives $K \subseteq K^{\sigma}$. Since $K^{\sigma} \subseteq K$, we get $K^{\sigma} = K$. Hence, K is a σ-ideal of R.

4. σ-prime spectrum of normal ADLs

In this section, we derive the properties of prime σ-ideals of a normal ADL topologically.

Lemma 5. *A join of two σ-ideals of an ADL R is a σ-ideal of R. Also, the intersection of two σ-ideals is a σ-ideal of R.*

Proof. Let I, J be any two σ-ideals of R. Then $I^{\sigma} = I$ and $J^{\sigma} = J$. Clearly, we have that $(I \vee J)^{\sigma} \subseteq I \vee J$. And also, we have that $I^{\sigma} \vee J^{\sigma} \subseteq (I \vee J)^{\sigma}$ and hence $I \vee J \subseteq (I \vee J)^{\sigma}$. Therefore, $(I \vee J)^{\sigma} = I \vee J$.

Note that the set $\mathcal{I}^{\sigma}(R)$ of all σ-ideals of an ADL R is a sublattice of all ideals of R, which is closed under arbitrary joins.

Definition 12. *[6] An ADL R with maximal elements is said to be normal if for any $x, y \in L$ with $x \wedge y = 0$, there exist elements $a, b \in R$ such that $x \wedge a = 0, y \wedge b = 0$ and $a \vee b$ is maximal.*

Theorem 15. *Let R be an ADL with maximal elements. Then the following are equivalent:*

(1) R is normal

(2) every minimal prime ideal is a σ-ideal of R

(3) every prime ideal contains a unique minimal prime ideal

(4) for every ideal $Q, Q^{\mathcal{O}}$ is a prime ideal.

Proof. (1) $\Rightarrow$ (2): Assume that R is normal. Let M be any minimal prime ideal of R. We prove that M is a σ-ideal of R. Clearly, we have that $M^{\sigma} \subseteq M$. Let $x \in M$. Since M is minimal, there exists an element $y \notin M$ such that $x \wedge y = 0$. Since R is normal, there exist elements $a, b \in R$ such that $x \wedge a = 0, y \wedge b = 0$ and $a \vee b$ is maximal. That implies $a \in (x)^{*}, b \in M$ and $a \vee b$ is maximal. Therefore, $(x)^{*} \vee M = R$ and hence M is a σ-ideal of R.

(2) $\Rightarrow$ (3): Assume (2). Let P be any prime ideal of R and M, N be any two minimal prime ideals of R such that $M \subseteq P$ and $N \subseteq P$. By our assumption, we have that M and N are σ-ideals of R. That implies $M^{\sigma} = M$ and $N^{\sigma} = N$. We prove that $M = N$. Suppose $M \neq N$. Then choose an element $x \in M$ such that $x \notin N$. Since $x \in M$, we get that $x \in M^{\sigma}$ and hence $(x)^{*} \vee M = R$. That implies $a \vee b$ is a maximal element of R for some $a \in (x)^{*}$ and $b \in M$. That implies $x \wedge a = 0, b \in M$ and $a \vee b$ is maximal. Since $x \notin N$, we get that $a \in N$. Since $M \subseteq P$ and $N \subseteq P$, we get that $a, b \in P$. That implies $a \vee b \in P$, which is a contradiction. Therefore, $M = N$. Hence, every prime ideal contains a unique minimal prime ideal of R.

(3) $\Rightarrow$ (4): Assume (3). Let Q be any prime ideal of R. We have that $Q^{\mathcal{O}}$ is the intersection of all minimal prime ideals contained in Q. By our assumption, we get that $Q^{\mathcal{O}}$ is a minimal prime ideal of R contained in Q. Therefore, $Q^{\mathcal{O}}$ is prime.

(4) $\Rightarrow$ (1): Assume (4). Let $a, b \in R$ with $a \wedge b = 0$. Suppose $(a)^{*} \vee (b)^{*} \neq R$. Then there exists $N \in \mathrm{Spec}R$ such that $(a)^{*} \vee (b)^{*} \subseteq N$. That implies $(a)^{*} \subseteq N$ and $(b)^{*} \subseteq N$, which give that $a \notin P^{\mathcal{O}}$ and $b \notin P^{\mathcal{O}}$, which is a contradiction to $P^{\mathcal{O}}$ is a prime ideal. Therefore, $(a)^{*} \vee (b)^{*} = R$. Hence, there exist elements $s \in (a)^{*}$ and $t \in (b)^{*}$ such that $s \vee t$ is maximal. Thus, R is normal.

Theorem 16. *Let R be an ADL with maximal elements. Then R is normal if and only if for every $a \in R, (a)^{*}$ is a σ-ideal of R.*

Proof. Assume that R is normal. Let a be any element of R. We prove that $(a)^{*}$ is a σ-ideal of R. Clearly, we have that $(a)^{*^{\sigma}} \subseteq (a)^{*}$. Let $b \in (a)^{*}$. Then $a \wedge b = 0$. Since R is normal, we have that $(a)^{*} \vee (b)^{*} = R$. That implies $b \in (a)^{*^{\sigma}}$ and hence $(a)^{*} \subseteq (a)^{*^{\sigma}}$. Therefore, $(a)^{*^{\sigma}} = (a)^{*}$. Thus, $(a)^{*}$ is a σ-ideal of R.

Conversely, assume that for every $a \in R, (a)^{*}$ is a σ-ideal of R. We prove that R is normal. It is enough to prove that every minimal prime ideal of R is a σ-ideal of R. Let M be any minimal prime ideal of R with $a \in M$. Then there exists an element $b \notin M$ such that $a \wedge b = 0$. That implies $b \in (a)^{*}$. By our assumption, we have that $(a)^{*}$ is a σ-ideal of R and hence $b \in (a)^{*^{\sigma}}$. That implies $(a)^{*} \vee (b)^{*} = R$. That implies $s \vee t$ is a maximal element for some $s \in (a)^{*}$ and $t \in (b)^{*} \subseteq M$. That implies $(a)^{*} \vee M = R$. Therefore, M is a σ-ideal of R.

Definition 13. *A σ-ideal I of an ADL R is said to be a prime σ-ideal if for any $I_1, I_2 \in \mathfrak{I}^{\sigma}(R), I_1 \cap I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$.*

Let $\mathrm{Spec}^{\sigma}(R)$ be the set of all prime σ-ideals of an ADL R. For any $G \subseteq R$, let $h(G) = \{M \in \mathrm{Spec}^{\sigma}(R) \mid G \nsubseteq M\}$ and for any $a \in R, h(a) = h(\{a\})$. For any two subsets G and H of R, it is obvious that $G \subseteq H$ implies $h(G) \subseteq h(H)$. The following observations can be verified directly:

Lemma 6. *For any $x, y \in R$, the following conditions hold:*

(1) $\bigcup_{a \in R} h(a) = \mathrm{Spec}^{\sigma}(R)$

(2) $h(a) \cup h(b) = h(a \vee b)$

(3) $h(a) \cap h(b) = h(a \wedge b)$

(4) $h(a) = \emptyset \Leftrightarrow a = 0$

(5) $h(a) = \mathrm{Spec}^{\sigma}(R) \Leftrightarrow a$ *is a maximal element of* R.

From the above Lemma, it can be easily observed that the collection $\{h(a) \mid a \in R\}$ forms a base for a topology on $\mathrm{Spec}^{\sigma}(R)$ which is called a hull-kernel topology.

Definition 14. *For any ideal I of an ADL R, define $I^{\triangle} = \bigcup\{J \in \mathfrak{I}^{\sigma}(R) \mid J \subseteq I\}$.*

Lemma 7. *Let I be any ideal of R. Then $I^{\triangle}$ is the largest σ-ideal of R contained in I.*

Proof. Clearly, we have that $\{0\}$ is a σ-ideal of R and $\{0\} \subseteq I$. That implies $\{0\} \in I^{\triangle}$ and hence $I^{\triangle} \neq \emptyset$. Let $x \in I^{\triangle}$. Then $x \in \bigcup\{J \in \mathfrak{I}^{\sigma}(R) \mid J \subseteq I\}$. Then there exists $J \in \mathfrak{I}^{\sigma}(R)$ such that $J \subseteq I$ and $x \in J$. That implies $x \in I$. Therefore, $I^{\triangle} \subseteq I$. Let $x, y \in I^{\triangle}$. Then there exist $J, K \in \mathfrak{I}^{\sigma}(R)$ such that $x \in J, J \subseteq I$ and $y \in K, K \subseteq I$. That implies $x \vee y \in J \vee K \subseteq I$. Since the join of two σ-ideals J and K is a σ-ideal of R, we get that $x \vee y \in I^{\triangle}$. Let $x \in I^{\triangle}$. Then there exists $J \in \mathfrak{I}^{\sigma}(R)$ such that $x \in J \subseteq I$. Let r be any element of R. Then $x \wedge r \in J \subseteq I$. That implies $x \wedge r \in I^{\triangle}$. Hence, $I^{\triangle}$ is a σ-ideal of R contained in I. Clearly, $I^{\triangle}$ is the largest σ-ideal of R contained in I.

Lemma 8. *Let R be an ADL with maximal elements. Then for any ideal K of R, we have $K^{\triangle} \subseteq K^{\sigma}$.*

Proof. Let K be any ideal of R with $a \in K^{\triangle}$. Then there exists $H \in \mathfrak{I}^{\sigma}(R)$ such that $a \in H \subseteq K$. Since H is a σ-ideal of R and $H \subseteq K$, we get that $(a)^* \vee K = R$. That implies $a \in K^{\sigma}$. Therefore, $K^{\triangle} \subseteq K^{\sigma}$.

Lemma 9. *Let K be an ideal of a normal ADL R with maximal elements. Then $K^{\triangle} = K^{\sigma}$. Moreover, K^{σ} is a σ-ideal of R.*

Proof. Clearly, we have that $K^{\triangle} \subseteq K^{\sigma}$. Let $a \in K^{\sigma}$. Then $(a)^* \vee K = R$. Then there exist elements $s \in (a)^*$ and $t \in I$ such that $a \vee b$ is maximal. Since $s \in (a)^*$, we have that $a \wedge s = 0$. Since R is normal, there exist elements $s_1, t_1 \in R$ such that $s \wedge s_1 = 0, a \wedge t_1 = 0$ and $s_1 \vee t_1$ is maximal. Since $s \vee t$ is maximal, $s \in (s_1)^*$ and $t \in K$, we get that $(s_1)^* \vee K = R$. That implies $s_1 \in K^\sigma$. Since $t_1 \in (a)^*$, $s_1 \vee t_1$ is maximal and $s_1 \in K^\sigma$, we get that $(a)^* \vee K^\sigma = R$. That implies $a \in (K^\sigma)^\sigma$, so $K^\sigma \subseteq (K^\sigma)^\sigma$. Therefore, K^σ is a σ-ideal of R contained in K. Since $K^\triangle$ is the largest σ-ideal of R contained K, we get that $K^\sigma \subseteq K^\triangle$. Hence, $K^\triangle = \sigma(K)$. Thus, K^σ is a σ-ideal of R.

Theorem 17. *Let P be a proper ideal of an ADL R with maximal elements. Then σ-ideal P of $R, P = \bigcap\limits_{P\subseteq N} N^{\mathcal{O}}$, where N runs over all maximal ideals of R containing P.*

Proof. Let P be any proper σ-ideal of R and N be a maximal ideal of R with $P \subseteq N$. Let $a \in P$. Then $(a)^* \vee P = R$. That implies there exist elements $s \in (a)^*$ and $t \in P$ such that $s \vee t$ is maximal. Since $s \in (a)^*$ and $t \in P$, we have that $a \wedge s = 0$ and $t \in N$. Since $s \vee t$ is maximal and $t \in N$, we get that $s \notin N$. Since $a \wedge s = 0$ and $s \notin N$, we get that $a \in N^{\mathcal{O}}$. Therefore, $P \subseteq \bigcap\limits_{P\subseteq N} N^{\mathcal{O}}$. Let $a \in \bigcap\limits_{P\subseteq N} N^{\mathcal{O}}$. Then $a \in N^{\mathcal{O}}$ for all maximal ideal N of R containing P. Now, we prove that $a \in P$. Suppose $a \notin P$. Then there exists a prime ideal Q of R such that $a \notin Q$ and $P \subseteq Q$. We know that a proper ideal is contained in a maximal ideal. So that $Q \subseteq M$, where M is a maximal ideal. That implies $M^{\mathcal{O}} \subseteq Q^{\mathcal{O}} \subseteq Q$. That implies $M^{\mathcal{O}} \subseteq Q$. Since $a \notin Q$, we get that $a \notin M^{\mathcal{O}}$ and $P \subseteq M$, which is a contradiction. Therefore, $a \in P$ and hence $\bigcap\limits_{P\subseteq N} N^{\mathcal{O}} \subseteq P$. Thus, $P = \bigcap\limits_{P\subseteq N} N^{\mathcal{O}}$.

Theorem 18. *Let K be a σ-ideal of a normal ADL R with maximal elements. Then $\frac{R}{\theta(K)}$ is a normal ADL.*

Definition 15. *A σ-ideal M of R is said to be maximal if it is maximal among σ-ideals of R.*

Lemma 10. *Let R be an ADL with maximal elements. Then*

(1) every maximal σ-ideal is a prime σ-ideal

(2) every prime σ-ideal is contained in a maximal σ-ideal.

Proof. (1) Let M be any maximal σ-ideal of R. Let I, J be two σ-ideals of R such that $I \cap J \subseteq M$. Now, we prove that $I \subseteq M$ or $J \subseteq M$. Suppose $I \nsubseteq M$ and $J \nsubseteq M$. Then there exist elements $a \in I$ and $b \in J$ such that $a, b \notin M$. Since M is σ-maximal, we have $M \vee (a] = R$ and $M \vee (b] = R$. That implies $M \vee (a \wedge b] = R$. Therefore, $a \wedge b \notin M$. Since I, J are σ-ideals of R and $a \in I, b \in J$, we get that $(a)^* \vee I = R$ and $(b)^* \vee J = R$. That implies $(a \wedge b)^* \vee I \cap J = R$. Since $I \cap J \subseteq M$, we get that $(a \wedge b)^* \vee M = R$. That implies $a \wedge b \in M^\sigma = M$ and hence $a \wedge b \in M$, which is a contradiction to $a \wedge b \notin M$. Therefore, $I \subseteq M$ or $J \subseteq M$. Thus, M is a prime σ-ideal of R.

(2) Let P be a prime σ-ideal of R. Consider $\mathfrak{F} = \{I \mid I \text{ is a } \sigma\text{-ideal of } R, P \subseteq I\}$. Clearly, $P \in \mathfrak{F}$. Let $\{I_\alpha\}_{\alpha\in\Delta}$ be a chain in $\mathfrak{F}$. Clearly, we have that $\bigcup\limits_{\alpha\in\Delta} I_\alpha$ is a σ-ideal of R and it is an upper bound for $\{I_\alpha \mid \alpha \in \Delta\}$. By Zorn's lemma, $\mathfrak{F}$ has a maximal element; let it be M. Therefore, prime σ-ideal P is contained in a maximal σ-ideal M of R.

Theorem 19. *Let I be a σ-ideal of R and F a closed under $\wedge$ such that $I \cap F = \emptyset$. Then there exists a prime σ-ideal P of R such that $I \subseteq P$ and $F \cap P = \emptyset$.*

Proof. Consider $\mathfrak{F} = \{J \mid J \text{ is a } \sigma\text{-ideal of } R, I \subseteq P \text{ and } I \cap F = \emptyset\}$. Clearly, $I \in \mathfrak{F}$. Let $\{J_\alpha\}_{\alpha \in \Delta}$ be a chain in $\mathfrak{F}$. Clearly, we have that $\bigcup_{\alpha \in \Delta} J_\alpha$ is a σ-ideal of R and it is an upper bound for $\{J_\alpha \mid \alpha \in \Delta\}$. By Zorn's lemma, $\mathfrak{F}$ has a maximal element; let it be M. Let I_1, I_2 be any two σ-ideals of R such that $I_1 \cap I_2 \subseteq M$. Now, we prove that $I_1 \subseteq M$ or $I_2 \subseteq M$. Suppose $I_1 \nsubseteq M$ and $I_2 \nsubseteq M$. Then there exist elements $a \in I_1$ and $b \in I_2$ such that $a, b \notin M$. Since $a \in I_1$ and $b \in I_2$, we have that $a \wedge b \in I_1 \cap I_2 \subseteq M$. Since $a \notin M$ and $b \notin M$, we get that $M \vee (a] \subseteq M$ and $M \vee (b] \subseteq M$. That implies $M \vee (a \wedge b] \subseteq M$. By maximality of M in $\mathfrak{F}$, we have that $(M \vee (a]) \cap F \neq \emptyset$ and $(M \vee (b]) \cap F \neq \emptyset$. Then choose elements $x \in (M \vee (a]) \cap F$ and $y \in (M \vee (b]) \cap F$. Since F is closed under $\wedge$, we have $x \wedge y \in ((M \vee (a]) \cap F) \cap ((M \vee (b]) \cap F) = (M \vee (a \wedge b]) \cap F$. By maximality of M in $\mathfrak{F}$, we have that $a \wedge b \notin M$, which is a contradiction to $a \wedge b \in M$. Therefore, $I_1 \subseteq M$ or $I_2 \subseteq M$. Hence, M is a prime σ-ideal of R containing I and $M \cap F = \emptyset$.

Theorem 20. *Let I be a proper σ-ideal of R. Then $I = \bigcap\{P \mid P \in \operatorname{Spec}^\sigma R$ and $I \subseteq P\}$.*

Proof. Consider $I_\circ = \bigcap\{P \mid P \in \operatorname{Spec}^\sigma R \text{ and } I \subseteq P\}$. Let $a \notin I$. Then there exists a prime σ-ideal of R such that $a \notin P$ and $I \subseteq P$. That implies $a \notin I_\circ$. Therefore, $I_\circ \subseteq I$. Let $x \in I$. Then $(x)^* \vee I = R$. Since I is a proper σ-ideal of R, choose a maximal σ-ideal M such that $I \subseteq M$. That implies $(x)^* \vee M = R$ and M is a prime σ-ideal of R. That implies $x \in M^\sigma = M$ and $I \subseteq M$. That implies $a \in I_\circ$. Therefore, $I \subseteq I_\circ$ and hence $I = I_\circ$.

Theorem 21. *In an ADL R, the mapping $I \to D(I)$ from the set $\mathfrak{I}^\sigma(R)$ of all σ-ideals to the set $\{k(I) \mid I \in \mathfrak{I}^\sigma(R)\}$ is an isomorphism.*

Proof. Let $I, J \in \mathfrak{I}^\sigma(R)$ such that $I = J$. Clearly, we have that $h(I) = h(J)$. Assume that $h(I) = h(J)$. We prove that $I = J$. Let $x \in I$. Suppose that $x \notin J$. Then there exists a prime σ-ideal P of R such that $x \notin P$ and $J \subseteq P$. Since $J \subseteq P$, we have that $P \notin h(J) = h(I)$. That implies $I \subseteq P$. Since $x \notin P$, we get that $x \notin I$, which is a contradiction. Therefore, $x \in J$ and hence $I \subseteq J$. Similarly, we get that $J \subseteq I$. Therefore, $I = J$ and hence this map is one-one. Clearly, it is onto and homomorphism. Thus, it is an isomorphism.

Theorem 22. *Let M be a prime σ-ideal of a normal ADL R with maximal elements. Then M is a maximal σ-ideal of R.*

Proof. Suppose there is a proper σ-ideal Q such that $M \subseteq Q$. Now, we prove that $Q \subseteq M$. Suppose that $Q \nsubseteq M$. Then, there exists an element $x \in Q$ such that $x \notin M$. Since $x \in Q$, we have that $(x)^* \vee Q = R$. That implies there exist $a \in (x)^*$ and $b \in Q$ such that $a \vee b$ is a maximal element of R. Since $a \in (x)^*$, we have that $a \wedge x = 0$. Since R is normal, there exist elements $c, d \in R$ such that $a \wedge c = 0, x \wedge d = 0$ and $c \vee d$ is a maximal element. Since $c \vee d$ is maximal, we have that $(c)^* \cap (d)^* = (0] \subseteq M$. Since $a \in (c)^*$ and $a \notin Q$, we have that $(c)^* \nsubseteq Q$ and hence $(c)^* \nsubseteq M$. Since M is a prime σ-ideal of $R, (c)^* \cap (d)^* \subseteq M$ and $(c)^* \nsubseteq M$, we get that $(d)^* \subseteq M$. Since $d \wedge x = 0$, we have that $x \in (d)^* \subseteq M$. That implies $x \in M$, which contradicts $x \notin M$. Therefore, $Q \subseteq M$ and hence $M = Q$. Thus, M is a maximal σ-ideal of R.

Theorem 23. *Let R be a normal ADL with maximal elements and P a minimal prime ideal of R. For every maximal ideal M containing $P, P = M^{\mathcal{O}}$ and P is a prime σ-ideal of R.*

Proof. Let P be a minimal prime ideal of R and M be a maximal ideal with $P \subseteq M$. We prove that $M^{\mathcal{O}} = P$. Let $x \in P$. Then there exists an element $y \notin P$ such that $x \wedge y = 0$. Since R is normal, there exist elements $a, b \in R$ such that $x \wedge a = 0, y \wedge b = 0$ and $a \vee b$ is maximal. Since $y \notin P$, we get that $b \in P$. Since $P \subseteq M$, we get that $a \notin M$. Since $a \notin M$ and $x \wedge a = 0$, we get that $x \in M^{\mathcal{O}}$. Therefore, $P \subseteq M^{\mathcal{O}}$. Let $x \in M^{\mathcal{O}}$. Then there exists an element $y \notin M$ such that $x \wedge y = 0$. Since $P \subseteq M$, we get that $y \notin P$. Since $x \wedge y = 0$, we get that $x \in P$. Therefore, $M^{\mathcal{O}} \subseteq P$. Hence, $P = M^{\mathcal{O}}$. We prove that P is a prime σ-ideal of R. It is enough to prove that P is a maximal σ-ideal of R. Suppose Q is a proper σ-ideal of R such that $P \subseteq Q$. Let $x \in Q$. Then $(x)^* \vee Q = R$. That implies there exist elements $a \in (x)^*$ and $b \in Q$ such that $a \vee b$ is a maximal element. That implies $a \notin Q$ and hence $a \notin P$. Since $a \wedge x = 0$, we get that $x \in P$. That implies $Q \subseteq P$. Therefore, $Q = P$ and hence P is a maximal σ-ideal of R. Thus, P is a prime σ-ideal of R.

Theorem 24. *Let R be a normal ADL and* Min$R$ *be the set of all minimal prime ideals of R. Then there is a continuous bijection $\psi : \mathrm{Min}R \to \mathrm{Spec}^\sigma R$ defined by $\psi(P) = P$.*

Proof. Clearly, ψ is a bijection. Now, we prove that ψ is continuous. Let $h(a)$ be open in $\mathrm{Spec}^\sigma(R)$. We prove that $\psi^{-1}(h(a))$ is open in Min$R$. Now, $\psi^{-1}(h(a)) = \{M \in \mathrm{Min}R \mid \psi(M) \in h(a)\} = \{M \in \mathrm{Min}R \mid M \in h(a)\} = \{M \in \mathrm{Min}R \mid a \notin M\} = h(a) \cap \mathrm{Min}R$. Therefore, $\psi^{-1}(h(a))$ is open in Min$R$. Hence, ψ is continuous.

Theorem 25. *An ADL R is Stone if and only if R is normal and* Min$R$ *is compact.*

Theorem 26. *Let R be a normal ADL. Then $\psi : \mathrm{Min}R \to \mathrm{Spec}^\sigma R$ defined above is a homeomorphism if and only if R is a Stone ADL.*

Definition 16. *[8] Let R be an ADL with maximal elements and B a Birkhoff center of R. An ideal I of R is said to be a B-ideal of R if for any $x \in I$ there exists an element $e \in I \cap B$ such that $e \wedge x = x$.*

Lemma 11. *Let R be an ADL with maximal elements. Then every B-ideal of R is a σ-ideal of R.*

Proof. Let I be any B-ideal of R. We prove that I is a σ-ideal of R. Clearly, we have that $I^\sigma \subseteq I$. Let $x \in I$. Since I is a B-ideal of R, there exists an element $b \in I \cap B$ such that $b \wedge x = x$. Since B is a Birkhoff center and $b \in B$, there exists an element $c \in R$ such that $b \wedge c = 0$ and $b \vee c$ is maximal. Now, $x \wedge c = b \wedge x \wedge c = 0$. That implies $c \in (x)^*$. Since $b \in I$ and $c \vee b$ is maximal, we get that $(x)^* \vee I = R$. That implies $x \in I^\sigma$. Therefore, $I \subseteq I^\sigma$ and hence $I^\sigma = I$. Thus, I is a σ-ideal of R.

Definition 17. *An ADL R is said to be strongly normal if for any $a \in R$, $(a)^*$ is a B-ideal of R.*

Theorem 27. *An ADL R with maximal elements is strongly normal if and only if every σ-ideal of R is a B-ideal.*

Proof. Assume that R is strongly normal. Let I be any σ-ideal of R. Let $x \in I$. Then $I \vee (x)^* = R$. That implies there exist elements $i \in I$ and $y \in (x)^*$ such that $i \vee y$ is maximal. That implies $y \wedge x = 0$ and $i \vee y$ is maximal. By our assumption, we have that $(x)^*$ is a B-ideal of R. Since $y \in (x)^*$, there exists an element $b \in (x)^* \cap B$ such that $b \wedge y = y$. Since $b \in B$, there exists an element $c \in R$ such that $b \wedge c = 0$ and $b \vee c$ is maximal. Since $b \wedge c = 0$, we have that $y \wedge c = 0$. Now, $i \wedge c = (i \wedge c) \vee 0 = (i \wedge c) \vee (y \wedge c) = (i \vee y) \wedge c = c$, since $i \vee y$ is maximal. Since $i \in I$, we get that $c \in I$. Now, $c \wedge x = 0 \vee (c \wedge x) = (b \wedge x) \vee (c \wedge x) = (b \vee c) \wedge x = x$, since $b \vee c$ is maximal. Therefore, I is a B-ideal of R.

Conversely, assume that every σ-ideal of R is a B-ideal. We prove that R is strongly normal. Let $x \in R$. We prove that $(x)^*$ is a B-ideal of R. Let $y \in (x)^*$. Then $x \wedge y = 0$. Since L is normal, there exist elements $a, b \in R$ such that $x \wedge a = 0, y \wedge b = 0$ and $a \vee b$ is maximal. That implies $(y)^* \vee (x)^* = R$. That implies $(x)^* \subseteq (x)^{*^\sigma}$ and hence $(x)^{*^\sigma} = (x)^*$. Therefore, $(x)^*$ is a σ-ideal of R. By our assumption, we get that $(x)^*$ is a B-ideal of R.

Theorem 28. *For any σ-ideal I of an ADL R with maximal elements, $h(I)$ is clopen in $\mathrm{Spec}^\sigma(R)$ if and only if there exists an element $e \in B$ such that $I = (e]$.*

Proof. Let I be any σ-ideal of R. Assume that $h(I)$ is clopen. Then $\mathrm{Spec}^\sigma R \setminus h(I) = h(J)$. That implies $h(I) \cup h(J) = \mathrm{Spec}^\sigma R$ and $h(I) \cap h(J) = \emptyset$. That implies $h(I \vee J) = \mathrm{Spec}^\sigma R$ and $h(I \cap J) = \emptyset$. That implies $I \vee J = R$ and $I \cap J = \{0\}$. Let $x \in I$. Then $x \in I^\sigma$. That implies $(x)^* \vee I = R$. That implies there exist elements $y \in (x)^*$ and $i \in I$ such that $y \vee i$ is maximal. Since $I \vee J = R$ and $I \cap J = \{0\}$, we have that $i \vee j$ is maximal and $i \wedge j = 0$ for all $j \in J$. That implies $i \in B$. Now, we prove that $I \subseteq (i]$. Let $x \in I$. Now, $x = (i \vee j) \wedge x = (i \wedge x) \vee (j \wedge x) = (i \wedge x) \vee 0 = i \wedge x$. That implies $x \in (i]$ and hence $I \subseteq (i]$. Therefore, $I = (i]$.

Conversely, assume that there is an element $e \in B$ such that $I = (e]$. Since $e \in B$, there exists an element $f \in R$ such that $e \wedge f = 0$ and $e \vee f$ is maximal. Now, $h(e) \cap h(f) = h(e \wedge f) = h(0) = \emptyset$ and $h(e) \cup h(f) = h(e \vee f) = \mathrm{Spec}^\sigma(R)$, since $e \vee f$ is maximal. Therefore, $h(I)$ is clopen.

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References

[1] G. Birkhoff. *Lattice theory.* American Mathematical Society Colloquium Publications XXV, Providence, U.S.A., 1967.

[2] W.H. Cornish. Annulets and α-ideals in distributive lattices. *Journal of the Australian Mathematical Society*, 15:70–77, 1973.

[3] W.H. Cornish. O-ideals, congruences, sheaf representation of distributive lattices. *Revue Roumaine de Mathématiques Pures et Appliquées*, 22:1059–1067, 1977.

[4] G. Gratzer. *General lattice theory.* Academic Press, New York, San Francisco, 1978.

[5] G.C. Rao. *Almost distributive lattices.* Doctoral Thesis, Department of Mathematics, Andhra University, Visakhapatnam, 1980.

[6] G.C. Rao and S. Ravi Kumar. Normal almost distributive lattices. *Southeast Asian Bulletin of Mathematics*, 32:831–841, 2008.

[7] G.C. Rao and S. Ravi Kumar. Minimal prime ideals in almost distributive lattices. *International Journal of Contemporary Mathematical Sciences*, 4(9-12):475–484, 2009.

[8] G.C. Rao, N. Rafi, and R. Bandaru. S-ideals in almost distributive lattices. *Southeast Asian Bulletin of Mathematics*, 35:825–836, 2011.

[9] G.C. Rao and M. Sambasiva Rao. α-ideals and prime ideals in almost distributive lattices. *International Journal of Algebra*, 3(5):221–229, 2009.

[10] G.C. Rao and M. Sambasiva Rao. Annulets in almost distributive lattices. *European Journal of Pure and Applied Mathematics*, 2(1):58–72, 2009.

[11] G.C. Rao and M. Sambasiva Rao. σ-ideals of almost distributive lattices. *Asian-European Journal of Mathematics*, 5(4):1250057, 2012.

[12] U.M. Swamy and S. Ramesh. Birkhoff centre of an almost distributive lattice. *International Journal of Algebra*, 3(11):539–546, 2009.

[13] U.M. Swamy and G.C. Rao. Almost distributive lattices. *Journal of the Australian Mathematical Society (Series A)*, 31:77–91, 1981.

[14] U.M. Swamy, G.C. Rao, and G. Nanaji Rao. Pseudo-complementation on almost distributive lattices. *Southeast Asian Bulletin of Mathematics*, 24:95–104, 2000.

[15] U.M. Swamy, G.C. Rao, and G. Nanaji Rao. Stone almost distributive lattices. *Southeast Asian Bulletin of Mathematics*, 24:513–526, 2000.