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Computational treatment for the coupled system of viscous Burger's equations through non-central formula in the method of lines

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Abstract. Viscous Burger's equation is one of the most celebrated models with immense applications cutting across all aspects of mathematical physics. Thus, the present communication makes use of the non-central formula infused in the method of lines coupled with Runge-Kutta spatial discretization to computationally and efficiently treat the coupled system of viscous Burger's equations. Further, we numerically tested the derived schemes of the governing model amidst suitable initial and boundary data. In fact, we eventually analyzed the effectiveness of the method via L_2 and L_∞ norms on some test problems and found it to be robust; indeed, a comparison of the current scheme with some notable approaches in the literature has been established, which are realized to be in perfect agreement.

2020 Mathematics Subject Classifications: 35A25, 35A99, 35C08, 65G99, 65L06, 65M20

Key Words and Phrases: Non-central formula, method of lines, Nonlinear coupled system, viscous Burger's equations

1. Introduction

Viscous Burger's equation is among the most celebrated models with immense relevance in many science and engineering processes, including fluid dynamics, optics, wave scattering, shocks, and shallow water waves to list a few. In particular, a coupled variant of Burger's equation was devised as the coupled viscous Burger's equation, which was utilized in modeling the transmission of poly-dispersion in sedimentation processes; read Esipov [8]. More so, in line with the abundant applications of the coupled model, several researchers have in the past and present times deployed dissimilar computational approaches in search of an optimal numerical solution for the governing model. For instance,

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Mubaraki et al [1] examined the special class of a generalized reaction-advection-diffusion dynamical model that is called the system of coupled Burger's equations by employing two promising analytical integration schemes. Also, Rashid and Ismail [2] utilized the Fourier pseudo-spectral analysis method to effectively derive a universal computational scheme for treating the coupled one-dimensional Burger's equation. El-Shiekh and Al-Nowehy [20] applied the symmetry group method to obtain similarity reductions for generalized symmetric coupled Burgers-type equations. Abazari and Borhanifar [16] used the differential transformation technique on the class of Burger's and coupled Burger's equations to acquire some interesting semi-numerical results. Moreover, one can equally find the application of finite difference numerical method on the acquisition of optimal computational solution for the coupled $(2+1)$-dimensional Burger's equation in Srivaslava et. al [22–24]. In addition, other numerical methods that were used on the coupled Burger's equation include the higher-order trigonometric B-spline [6], semi-Lagrangian method [21], quartic B-spline approach [27], and the quintic B-spline coupled with adaptive time integrator method [26] to mention a few. To see more related work, we refer the reader to see [10, 13, 28].

However, the key aim of the present study is to come up with an effective numerical scheme for the complete treatment of the coupled system of Burger's equations in mathematical physics. In the proposed method, we improve the method of lines (MOL) by using a non-central formula instead of the classical schemes of the first and second derivatives to get a more accurate result. This technique focuses on converting a partial differential equation (PDE) into an ordinary differential equation (ODE), since several methods can used to solve the ODE. In fact, the non-central formula in the method of lines (MOL) with an infusion of time-discretization via the Runge-Kutta numerical method will be deployed for the derivation of the scheme. Besides, the classical MOL has been used by several authors to find efficient numerical solutions for different partial differential equations, including for instance, the Korteweg-de Vries (KdV) equations [25], the one-dimensional wave propagation problem [9], the KdV-Burger's equation [17] and the system of coupled elastic beam equations [15], to mention but just a few. Additionally, Sharaf and Bakodah [4] introduced a reliable spatial discretization in MOL for the computational treatment of a class of partial differential equations. Further, other related studies have made use of the same approach, given as in [4], alongside the non-central formula to successfully tackle the regularized-long wave [14] and the equal-width wave [12] equations, respectively; read also the good work of Bakodah [11], which made use of the 7-point formula infused in the spatial-discretization via MOL to solve various forms of Burgers' equations.

More precisely, the current study would devise an appropriate spatial-discretization in the MOL for the coupled system of viscous Burger's equations. In addition, this approach sets to convert the coupled model endowed with sufficient auxiliary conditions to a coupled system of ordinary differential equations (ODEs). What is more, the unfailing fourth-order Runge-Kutta numerical method will further be sought to numerically treat the resulting ODEs, as against the most widely used finite difference method, which was proven to be superior in the literature; read the computational efficiency of the fourth-order Runge-Kutta numerical method that was reported in [11, 12, 14] and some of the references therewith. Lastly, the manuscript is arranged as follows: Section 2 features the central coupled model, Section 3 gives the delineation of the MOL, and Section 4 gives the resulting numerical experiments and discussion; while Section 5 gives some finishing notes.

2. Coupled system for viscous Burger's equation

The coupled nonlinear system of viscous Burger's equation, which is extensively used in modeling turbulent fluids, theory of shock waves, and stochastic processes among others, is a nonlinear evolution equation, or precisely a nonlinear partial differential equation (PDE) that reads as follows [7, 8]

$$\begin{aligned} &u_t = u_{xx} - \eta u u_x - \gamma (vu)_x, \\ &v_t = v_{xx} - \eta v v_x - \xi (vu)_x, \\ &c < x < d, \quad t > 0, \end{aligned} \tag{1}$$

where γ, ξ and η are real constants, η is an arbitrary constant depending on the system parameters, more so, the first two constants $\gamma,\ \xi$ depend on the parameters of the system, like the Brownian diffusivity, Stokes velocity, and Peclet number, to mention a few. In addition, the celebrating coupled model in Eq.(1) admits broad relevance in various areas of mathematical physics, including, the nonlinear acoustic, fluid mechanics, traffic flow, and gas dynamics to mention but just a few. Furthermore, for a successful deployment and further implementation of the aiming computational approach, it is imperative to prescribe appropriate initial data. Thus, we make consideration of the initial data as follows

$$u(x,0) = f(x), \quad v(x,0) = g(x), \quad c \le x \le d.$$

Once more, we restate here that we would devise an appropriate spatial discretization in the MOL to solve the above initial-value problem for the coupled system of viscous Burgers' equations. Above and beyond, other recent methods applied to the model can be found in [16] that used the differential transformation method, and [19] that utilized finite-difference scheme among others.

3. Methodology

This section delineates the methodology of interest; the MOL coupled with an infusion of a reliable spatial discretization via finite difference method for the computational treatment of a class of PDE, the coupled system of viscous Burger's equations. Specifically, the present section starts off the methodology by converting the coupled model, which is endowed with sufficient auxiliary conditions to a coupled system of ODEs; in addition to the deployment of the consistent fourth-order Runge-Kutta technique for the numerical solution of the resulting ODEs. Accordingly, to begin with, we portray the spatial discretization by considering the spatial variable x, and discretize it into $N + 1$ uniformly spaced grid points as follows

$$x_i = x_{i-1} + \Delta x, \quad i = 1, 2, 3, ..., N,$$

where $\Delta x = x_i - x_{i-1} = 1/N$ is a constant spacing, x_0 and x_N are the two end-points (specifically, the boundary points), while $x_i's$ for $i = 2, 3, 4, ..., (N-1)$ are the interior points. Next, upon making use of the non-central difference 5-point scheme [4], the concerning first- and second-order derivatives are thus approximated as follows

$$\begin{aligned} \frac{\mathrm{d}u_i}{\mathrm{d}x} &= \frac{1}{4!(\Delta x)}\left[u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}\right], \\ \frac{\mathrm{d}^2u_i}{\mathrm{d}x^2} &= \frac{1}{4!(\Delta x)^2}\left[-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}\right]. \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{\mathrm{d}v_i}{\mathrm{d}x} &= \frac{1}{4!(\Delta x)}\left[v_{i-2} - 8v_{i-1} + 8v_{i+1} - v_{i+2}\right], \\ \frac{\mathrm{d}^2v_i}{\mathrm{d}x^2} &= \frac{1}{4!(\Delta x)^2}\left[-v_{i-2} + 16v_{i-1} - 30v_i + 16v_{i+1} - v_{i+2}\right]. \end{aligned} \tag{3}$$

Additionally, on using the approximations expressed above in Eq. (1), where $uu_x = \frac{1}{2}(u^2)_x$, and $vv_x = \frac{1}{2}(v^2)_x$, the following coupled system of first-order ODEs is thus revealed

$$
\begin{aligned}
\dot{U}_1 &= \frac{1}{4(\Delta x)^2}\left[35U_0 - 104U_1 + 114U_2 - 56U_3 + 11U_4\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[-25U_0^2 + 48U_1^2 - 36U_2^2 + 16U_3^2 - 3U_4^2\right] \\
&- \frac{\gamma}{4!(\Delta x)}\left(U_1\left[-25V_0 + 48V_1 - 36V_2 + 16V_3 - 3V_4\right] - V_1\left[-25U_0 + 48U_1 - 36U_2 + 16U_3 - 3U_4\right]\right), \\
\dot{U}_2 &= \frac{1}{4(\Delta x)^2}\left[11U_1 - 20U_2 + 6U_3 + 4U_4 - 1U_5\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[-3U_1^2 - 10U_2^2 + 18U_3^2 - 6U_4^2 + 1U_5^2\right] \\
&- \frac{\gamma}{4!(\Delta x)}\left(U_2\left[-3V_1 - 10V_2 + 18V_3 - 6V_4 + 1V_5\right] - V_2\left[-3U_1 - 10U_2 + 18U_3 - 6U_4 + 1U_5\right]\right), \\
\dot{U}_i &= \frac{1}{4!(\Delta x)^2}\left[-U_{i-2} + 16U_{i-1} - 30U_i + 16U_{i+1} - U_{i+1}\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[U_{i-2}^2 - 8U_{i-1}^2 + 8U_{i+1}^2 - U_{i+2}^2\right] \\
&- \frac{\gamma}{4!(\Delta x)}\left(U_i\left[V_{i-2} - 8V_{i-1} + 8V_{i+1} - V_{i+2}\right] - V_i\left[U_{i-2} - 8U_{i-1} + 8U_{i+1} - U_{i+2}\right]\right), \\
&\qquad U_0 = U(x_0), U_i = U(x_i), \qquad for \quad i = 3(1)N-3, \\
\dot{U}_{N-2} &= \frac{1}{4!(\Delta x)^2}\left[-U_{N-5} + 4U_{N-4} + 6U_{N-3} - 20U_{N-2} + 11U_{N-1}\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[-U_{N-5}^2 + 6U_{N-4}^2 - 18U_{N-3}^2 + 10U_{N-2}^2 + 3U_{N-1}^2\right] \\
&- \frac{\gamma}{4!(\Delta x)}(U_{N-2}\left[-V_{N-5} + 6V_{N-4} - 18V_{N-3} + 10V_{N-2} + 3V_{N-1}\right] \\
&\qquad - V_{N-2}\left[-U_{N-5} + 6U_{N-4} - 18U_{N-3} + 10U_{N-2} + 3U_{N-1}\right]), \\
\dot{U}_{N-1} &= \frac{1}{4!(\Delta x)^2}\left[11U_{N-4} - 56U_{N-3} + 114U_{N-2} - 104U_{N-1} + 35U_N\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[3U_{N-4}^2 - 16U_{N-3}^2 + 36U_{N-2}^2 - 48U_{N-1}^2 + 25U_N^2\right] \\
&- \frac{\gamma}{4!(\Delta x)}(U_{N-1}\left[3V_{N-4} - 16V_{N-3} + 36V_{N-2} - 48V_{N-1} + 25V_N\right] \\
&\qquad - V_{N-1}\left[3U_{N-4} - 16U_{N-3} + 36U_{N-2} - 48U_{N-1} + 25U_N\right]),
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\dot{V}_1 &= \frac{1}{4(\Delta x)^2}\left[35V_0 - 104V_1 + 114V_2 - 56V_3 + 11V_4\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[-25V_0^2 + 48V_1^2 - 36V_2^2 + 16V_3^2 - 3V_4^2\right] \\
&- \frac{\xi}{4!(\Delta x)}\left(U_1\left[-25V_0 + 48V_1 - 36V_2 + 16V_3 - 3V_4\right] - V_1\left[-25U_0 + 48U_1 - 36U_2 + 16U_3 - 3U_4\right]\right), \\
\dot{V}_2 &= \frac{1}{4(\Delta x)^2}\left[11V_1 - 20V_2 + 6V_3 + 4V_4 - 1V_5\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[-3V_1^2 - 10V_2^2 + 18V_3^2 - 6V_4^2 + 1V_5^2\right] \\
&- \frac{\xi}{4!(\Delta x)}\left(U_2\left[-3V_1 - 10V_2 + 18V_3 - 6V_4 + 1V_5\right] - V_2\left[-3U_1 - 10U_2 + 18U_3 - 6U_4 + 1U_5\right]\right), \\
\dot{V}_i &= \frac{1}{4!(\Delta x)^2}\left[-V_{i-2} + 16V_{i-1} - 30V_i + 16V_{i+1} - V_{i+1}\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[V_{i-2}^2 - 8V_{i-1}^2 + 8V_{i+1}^2 - V_{i+2}^2\right] \\
&- \frac{\xi}{4!(\Delta x)}\left(U_i\left[V_{i-2} - 8V_{i-1} + 8V_{i+1} - V_{i+2}\right] - V_i\left[U_{i-2} - 8U_{i-1} + 8U_{i+1} - U_{i+2}\right]\right), \\
&\qquad V_0 = V(x_0), V_i = V(x_i), \qquad for \quad i = 3(1)N-3, \\
\dot{V}_{N-2} &= \frac{1}{4!(\Delta x)^2}\left[-V_{N-5} + 4V_{N-4} + 6V_{N-3} - 20V_{N-2} + 11V_{N-1}\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[-V_{N-5}^2 + 6V_{N-4}^2 - 18V_{N-3}^2 + 10V_{N-2}^2 + 3V_{N-1}^2\right] \\
&- \frac{\xi}{4!(\Delta x)}(U_{N-2}\left[-V_{N-5} + 6V_{N-4} - 18V_{N-3} + 10V_{N-2} + 3V_{N-1}\right] \\
&\qquad - V_{N-2}\left[-U_{N-5} + 6U_{N-4} - 18U_{N-3} + 10U_{N-2} + 3U_{N-1}\right]), \\
\dot{V}_{N-1} &= \frac{1}{4!(\Delta x)^2}\left[11V_{N-4} - 56V_{N-3} + 114V_{N-2} - 104V_{N-1} + 35V_N\right] \\
&- \frac{\eta}{2(4!)(\Delta x)}\left[3V_{N-4}^2 - 16V_{N-3}^2 + 36V_{N-2}^2 - 48V_{N-1}^2 + 25V_N^2\right] \\
&- \frac{\xi}{4!(\Delta x)}(U_{N-1}\left[3V_{N-4} - 16V_{N-3} + 36V_{N-2} - 48V_{N-1} + 25V_N\right] \\
&\qquad - V_{N-1}\left[3U_{N-4} - 16U_{N-3} + 36U_{N-2} - 48U_{N-1} + 25U_N\right]).
\end{aligned} \tag{5}
$$

Therefore, to numerically solve the coupled system of ODEs outlined in Eqs. (4)-(5), we then deploy the reliable fourth-order Runge-Kutta method. In fact, we re-express these equations in a more compact form as follows

$$\dot{U}_i = F(U_i, V_i), \quad i = 1, 2, ..., N-1, \tag{6}$$

$$\dot{V}_i = F(U_i, V_i), \quad i = 1, 2, ..., N-1. \tag{7}$$

Moreover, the implementation of MOL begins with the replacement of the involving spatial derivatives, which arise from the governing coupled Burger's model with discretized algebraic approximations. In light of this, explicit representation of the spatial derivatives is not realistic. Hence, a coupled system of ODEs is revealed and further solved numerically, which then gives the approximation of the novel PDE.

3.1. MOL via the fourth-order Runge-Kutta method

Here, the approach starts by deploying the fourth-order Runge-Kutta method to solve the resulting ODEs [19], in Eqs. (4)-(5), which were revealed independently with the help of spatial discretization of the finite difference approach. Certainly, in trying to incorporate the application of MOL, it is pertinent to recall that highly accurate computational results are achieved when deploying multistep approaches, as against the use of one-step methods, where the intermediate values of the solutions and the related derivatives are produced and utilized at each computational stage.

At first, the following definition

$$U^n = \left[U_1^n, U_2^n, ..., U_{N-1}^n\right]^t, \quad and \quad V^n = \left[V_1^n, V_2^n, ..., V_{N-1}^n\right]^t,$$

is considered; then followed by the definition of the Runge-Kutta method in the following pattern

$$U^{n+1} = U^n + \frac{1}{6}\left[K_{11} + 2K_{12} + 2K_{13} + K_{14}\right], \tag{8}$$

$$V^{n+1} = V^n + \frac{1}{6}\left[K_{21} + 2K_{22} + 2K_{23} + K_{24}\right], \tag{9}$$

where,

$$K_{11} = \Delta t F\left(U^n, V^n\right), \tag{10}$$

$$K_{21} = \Delta t G\left(U^n, V^n\right), \tag{11}$$

$$K_{12} = \Delta t F\left(U^n + \frac{K_{11}}{2}, V^n + \frac{K_{21}}{2}\right), \tag{12}$$

$$K_{22} = \Delta t G\left(U^n + \frac{K_{11}}{2}, V^n + \frac{K_{21}}{2}\right), \tag{13}$$

$$K_{13} = \Delta t F\left(U^n + \frac{K_{12}}{2}, V^n + \frac{K_{22}}{2}\right), \tag{14}$$

$$K_{23} = \Delta t G\left(U^n + \frac{K_{12}}{2}, V^n + \frac{K_{22}}{2}\right), \tag{15}$$

$$K_{14} = \Delta t F\left(U^n + K_{13}, V^n + K_{23}\right), \tag{16}$$

$$K_{24} = \Delta t G\left(U^n + K_{13}, V^n + K_{23}\right), \tag{17}$$

with Δt representing the temporal step size. In addition, and without further delay, the application of the Runge-Kutta method requires the evaluation of eight vector functions at every computational time step; thus, we only report the resulting numerical results in the subsequent section to suppress the length of the paper.

4. Numerical experiments and discussion

The present section considers certain test models to portray the compliance and competence of the devised scheme on the initial-value problem of the coupled system of Burger's equations. However, to assess the accuracy of the proposed numerical results, we resort to the engagement of the norm two L_2 and norm infinity L_∞ error estimators, which take the following explicit definitions

$$L_2(u) = \sqrt{\left[h\sum_{i=1}^{N-1} |u_i^n - U_i^n|^2\right]}, \tag{18}$$

$$L_\infty(u) = \max_{1\leq i\leq N-1} |u_i^n - U_i^n|. \tag{19}$$

In addition, we are set to establish a comparative assessment between the obtained/proposed computational results, via the devised approach with the available exact solutions in the literature. Moreover, we will be assessing the proposed scheme with other numerical schemes that were once proposed for the governing model. Lastly, all the computational simulations of the study will be performed with the aid of *Maple* 17.

Example 1. *Let us make consideration of the governing coupled system of Burger's equation* (1) *with* $\eta = 2$, *while generalized values for* γ *and* ξ *are assumed. In fact, the exact hyperbolic solution of the coupled equation is reported by Soliman [3] as follows*

$$\begin{aligned} u(x,t) &= D\left(1 - \tanh(B(x-2Bt))\right), \\ v(x,t) &= D\left(\frac{2\xi-1}{2\gamma-1} - \tanh(B(x-2Bt))\right), \end{aligned} \tag{20}$$

with $D = 0.05$ *and* $B = D(4\gamma\xi - 1)/(4\gamma - 2)$. *Moreover, sufficient initial (at* $t = 0$*) and boundary (at* $x = c$ *and* $x = d$*) data follow accordingly from the above exact solution.*

Hence, we have established in Tables 1 and 2, a comparative analysis between the proposed solution via the devised approach, and that obtained by the Chebyshev spectral collocation method [5]. Moreover, the earlier stated error norm estimators are deployed for the present assessment, which are, L_2 and L_∞; while fixing $D = 0.05$ at two different time levels, that is, $t = 0.5$ and $t = 1.0$, respectively. Indeed, it can be noted from these tables that the obtained results through the proposed approach outperformed the challenging methodology.

Additionally, the numerical solution obtained by 5-point non-central MOL and the exact solution of $u(x,t)$ and $v(x,t)$ for a fixed time $t = 0.5$ and for various values of h are

γ	ξ	t	L_2	L_∞	
				Present	Ref [5]
0.1	0.3	0.5	6.35232×10^{-6}	4.49177×10^{-6}	4.16×10^{-5}
		1	1.019395×10^{-5}	7.20821×10^{-6}	8.23×10^{-5}
0.3	0.03	0.5	6.38257×10^{-6}	4.51316×10^{-6}	4.59×10^{-5}
		1	4.105386×10^{-5}	2.902946×10^{-5}	9.16×10^{-5}

Table 1: Comparison of error norms for the solution pair $u(x,t)$ when $N = 10$, $c = -10$, $d = 10$, $\Delta t = 0.1$, and $D = 0.05$ of Example 1.

γ	ξ	t	L_2	L_∞	
				Present	Ref [5]
0.1	0.3	0.5	1.49091×10^{-6}	1.05423×10^{-6}	4.99×10^{-5}
		1	1.019395×10^{-5}	7.20821×10^{-6}	9.92×10^{-5}
0.3	0.03	0.5	3.513746×10^{-5}	2.484593×10^{-5}	1.81×10^{-4}
		1	4.105384×10^{-5}	2.902945×10^{-5}	3.62×10^{-4}

Table 2: Comparison of error norms for the solution pair $v(x,t)$ when $N = 10$, $c = -10$, $d = 10$, $\Delta t = 0.1$, and $D = 0.05$ of Example 1.

given in Tables 3 and 4. Moreover, Tables 5 and 6 further made use of L_∞ as an error norm estimator to establish a comparative study between the present computational results and those obtained using various approaches, including the Fourier pseudo-spectral method (FPM) [2], Chebyshev spectral collocation (CSC) [5], and the cubic B-spline collocation (CBC) [18]. Indeed, among all the comparing methods, the currently devised approach happens to proffer better results with high accuracy in contrast with the aforementioned approaches in [2, 5, 18]. In addition, we have shown in figures 1 and 2 the graphical display of the obtained computational and exact solution fields $u(x,t)$ and $v(x,t)$, while fixing the following values: $N = 10$, $\Delta t = 0.1$, $t = 0.5$, $\gamma = 1$, $\xi = 2$, and $D = 0.1$.

Δt	t	h	$u(x,t)$		L_∞
			present	Exact	
0.1	0.5	2	0.0527592618	0.0527547700	4.4918×10^{-6}
		1	0.0581890811	0.0581832978	5.7833×10^{-6}
		0.6666666667	0.0599622104	0.0599560894	6.1209×10^{-6}
		0.5714285714	0.0604642722	0.0604580646	6.2076×10^{-6}
		0.5	0.0608393804	0.0608331108	6.2696×10^{-6}
		0.3921568627	0.0614032873	0.0613969292	6.3581×10^{-6}

Table 3: Comparison between MOL and exact solutions for the solution pair $u(x,t)$ when $c = -10$, $d = 10$, $\gamma = 0.1$, $\xi = 0.3$, and $D = 0.05$ using various mesh sizes of Example 1.

Δt	t	h	$v(x,t)$		L_∞
			present	Exact	
0.1	0.5	2	0.02775582427	0.0277547700	1.0542×10^{-6}
		1	0.0331857077	0.0331832978	2.4100×10^{-6}
		0.6666666667	0.034958886	0.0349560894	2.7965×10^{-6}
		0.5714285714	0.0354609639	0.0354580646	2.8993×10^{-6}
		0.5	0.0358360849	0.0358331108	2.9742×10^{-6}
		0.3921568627	0.0364000124	0.0363969291	3.0833×10^{-6}

Table 4: Comparison between MOL and exact solutions for the solution pair $v(x,t)$ when $c = -10$, $d = 10$, $\gamma = 0.1$, $\xi = 0.3$, and $D = 0.05$ using various mesh sizes of Example 1.

Method	t	γ	ξ	L_∞
Present (RK4)	0.5	0.1	0.3	$4.49177\ \times10^{-6}$
		0.3	0.03	4.51316×10^{-6}
	1	0.1	0.3	7.20821×10^{-6}
		0.3	0.03	2.902946×10^{-5}
FPM [2]	0.5	0.1	0.3	9.619×10^{-4}
		0.3	0.03	4.310×10^{-4}
	1	0.1	0.3	1.153×10^{-3}
		0.3	0.03	1.268×10^{-3}
CSC [5]	0.5	0.1	0.3	4.38×10^{-5}
		0.3	0.03	4.58×10^{-5}
	1	0.1	0.3	8.66×10^{-5}
		0.3	0.03	9.16×10^{-5}
CBC [18]	0.5	0.1	0.3	4.16×10^{-5}
		0.3	0.03	4.59×10^{-5}
	1	0.1	0.3	8.25×10^{-5}
		0.3	0.03	9.18×10^{-5}

Table 5: Comparison of norm L_∞ errors for various approaches at different times for the solution pair $u(x,t)$ of Example 1.

Method	t	γ	ξ	L_∞
Present (RK4)	0.5	0.1	0.3	1.05423×10^{-6}
		0.3	0.03	2.48459×10^{-5}
	1	0.1	0.3	7.2082×10^{-6}
		0.3	0.03	2.90294×10^{-5}
FPM [2]	0.5	0.1	0.3	3.33×10^{-4}
		0.3	0.03	1.14×10^{-3}
	1	0.1	0.3	1.16×10^{-3}
		0.3	0.03	1.63×10^{-3}
CSC [5]	0.5	0.1	0.3	4.99×10^{-5}
		0.3	0.03	1.81×10^{-4}
	1	0.1	0.3	9.92×10^{-5}
		0.3	0.03	3.62×10^{-4}
CBC [18]	0.5	0.1	0.3	1.48×10^{-4}
		0.3	0.03	5.72×10^{-4}
	1	0.1	0.3	4.77×10^{-5}
		0.3	0.03	3.61×10^{-4}

Table 6: Comparison of norm L_∞ errors for various approaches at different times for the solution pair $v(x,t)$ of Example 1.

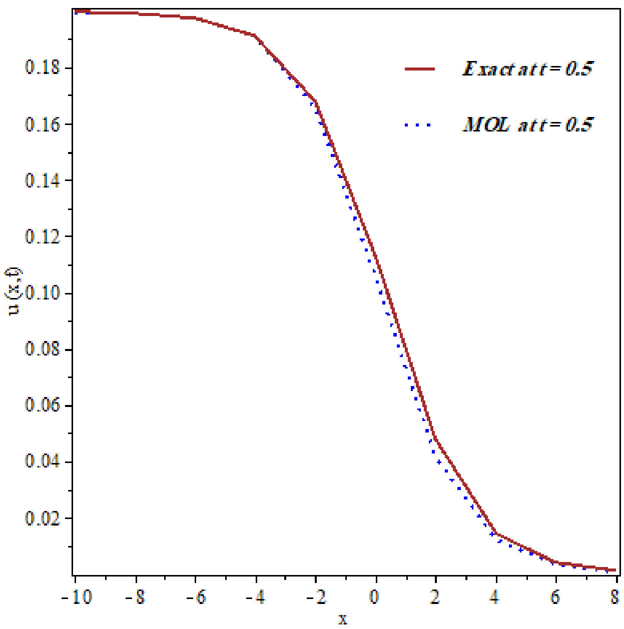

Figure 1: Numerical and exact solutions for $u(x,t)$ of Example 1 when $N=10$, $\Delta t=0.1$, $\gamma=1$, $\xi=2$, and $D=0.1$.

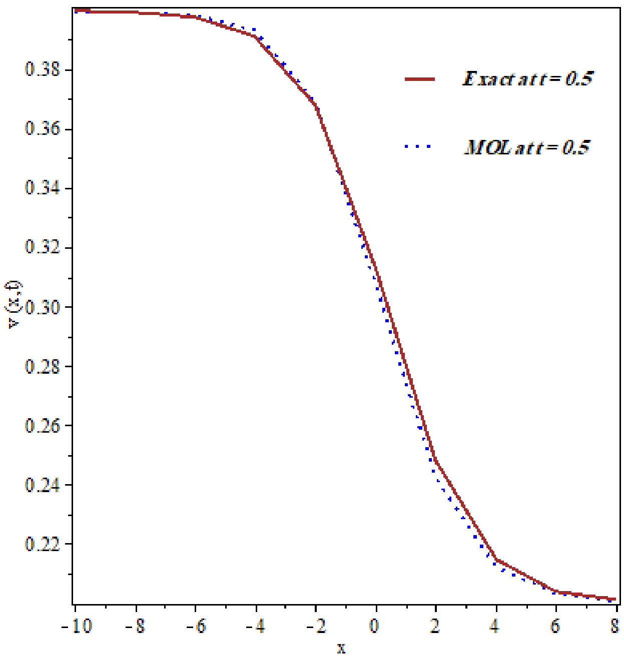

Figure 2: Numerical and exact solutions for $v(x,t)$ of Example 1 when $N = 10$, $\Delta t = 0.1$, $\gamma = 1$, $\xi = 2$, and $D = 0.1$.

- We compare use 3-point formula, 5-point formula, and 7-point formula results.

  (i) The effect of increasing points formulas .

$\Delta t = 0.01,\ t = 0.1,\ N = 50$

Z-point formula	$u(x,t)$	
	L_2	L_∞
3-point formula	1.25330×10^{-6}	1.98163×10^{-6}
5-point formula	3.7714×10^{-7}	5.9630×10^{-7}
7-point formula	1.21805×10^{-6}	1.92591×10^{-6}

Table 7: Influence of increasing points formulas for the solution pair $u(x,t)$ of Example 1 when $\gamma = 0.1,\ \xi = 0.3$.

$\Delta t = 0.01,\ t = 0.1,\ N = 50$

Z-point formula	$v(x,t)$	
	L_2	L_∞
3-point formula	8.2861×10^{-7}	1.31015×10^{-6}
5-point formula	1.6555×10^{-7}	2.6176×10^{-7}
7-point formula	7.9487×10^{-7}	1.25680×10^{-6}

Table 8: Influence of increasing points formulas for the solution pair $v(x,t)$ of Example 1 when $\gamma = 0.1,\ \xi = 0.3$.

Finally, we conclude that the results obtained by using the proposed method are better than other methods introduced in Tables 1-6 at different values of time. Furthermore, from Table 7 and Table 8, we observe that the accuracy of the numerical solutions at each field $u(x,t)$, $v(x,t)$ with the 5-point formula is more efficient.

Example 2. *Let us again make consideration of the governing coupled system of Burgers' equation* (1) *when* $\eta = -2$, *and* $\gamma = \xi = \frac{5}{2}$. *In addition, the exact solution of the coupled model for* $0 \leq x \leq 1$ *was analytically reported as follows [16]*

$$u(x,t) = v(x,t) = \mu\left[1 - \tanh\left(1.5\mu(40(x-0.5) - 3\mu t)\right)\right] \tag{21}$$

where μ *is a constant, which is indeed arbitrary. More so, the related initial and boundary data follow accordingly from the above analytical solution.*

Accordingly, we have reported in Table 9 the acquired computational results for different fixed values of μ. In addition, we have shown in figures (3) and (4) the comparison between the acquired computational solution through MOL and the analytical solution, on the other hand, using the following data $\mu = 0.05, 0.1$ and $\Delta t = 0.005$ at $t = 0.05$. Notably, it can be observed from Table 9 that a decrease in the value of μ increases the accuracy. Besides, it is also noted that the error gets smaller as time grows.

	t	$u(x,t) = v(x,t)$		Absolute Error
		Numerical results	Exact	
$\mu = 0.05$	0.005	0.0693850254	0.0693855034	5.7161×10^{-5}
	0.01	0.0129083018	0.0129085548	5.6108×10^{-5}
$\mu = 0.1$	0.005	0.1674079059	0.1674103611	8.1566×10^{-4}
	0.01	0.0042990998	0.0042990998	1.3146×10^{-4}

Table 9: Numerical solution with the exact solution of Example 2 when $\gamma = \xi = \frac{5}{2}$, $\mu = 0.05, 0.10$, $N = 11$, $c = 0$, $d = 1$, and $\Delta t = 0.001$.

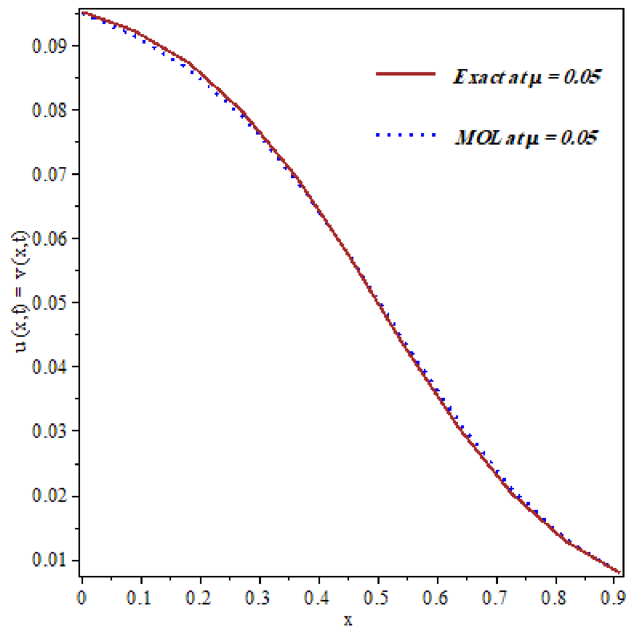

Figure 3: The behavior of the numerical and analytical solution for the 5-point formula of Example 2 for $N = 11$, $\Delta t = 0.005$, $t = 0.005$, $\gamma = \xi = \frac{5}{2}$.

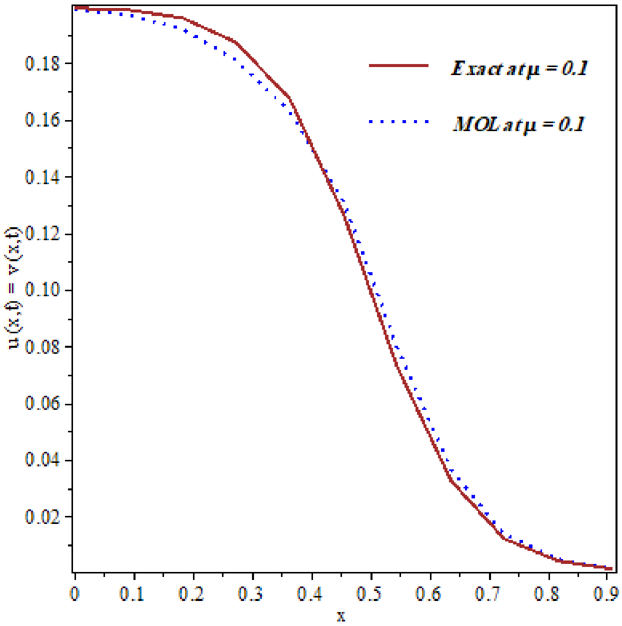

Figure 4: The behavior of the numerical and analytical solution for the 5-point formula of Example 2 for $N = 11$, $\Delta t = 0.005$, $t = 0.005$, $\gamma = \xi = \frac{5}{2}$.

- 3-point formula

  (i) The effect of increasing node points N.

$\Delta t = 0.001$, $t = 0.005$

N	$u(x,t)$=$v(x,t)$		Absolute Error
	Numerical results	Analytical results	
11	0.0693318373	0.0693855034	5.36661×10^{-5}
21	0.0863297722	0.0864966205	1.668483×10^{-4}
31	0.0901130213	0.0902545100	1.414887×10^{-4}
51	0.0925043358	0.0926179683	1.136325×10^{-4}
61	0.0930218434	0.0931283850	1.065417×10^{-4}
71	0.0933731026	0.0934746418	1.015391×10^{-4}

Table 10: The effect of increasing node points N of the numerical and analytical solution with $\mu = 0.05$ of Example (2).

(ii) The effect of increasing time.

$\Delta t = 0.01$, $N = 51$

t	$u(x,t)$=$v(x,t)$	
	L_2	L_∞
0.010	6.306876×10^{-5}	4.5040103×10^{-4}
0.050	1.5838520×10^{-4}	1.13109658×10^{-3}
0.1	4.7700242×10^{-4}	3.40647865×10^{-3}
0.2	1.91607952×10^{-3}	0.01368354479

Table 11: The effect of increasing time when $\mu = 0.1$ of the numerical and analytical solution of Example (2).

- We compare by using 3-point formula, 5-point formula, and 7-point formula results.

(i) The effect of increasing points formulas .

$\Delta t = 0.001$, $t = 0.050$

Z-point formula	$u(x,t) = v(x,t)$	
	L_2	L_∞
3-point formula	6.26752×10^{-6}	4.475902×10^{-5}
5-point formula	5.02258×10^{-6}	3.586836×10^{-5}
7-point formula	2.89533×10^{-5}	2.067682×10^{-4}

Table 12: Influence of increasing points formulas of Example (2) when $\mu = 0.1$, $N = 51$.

It can be seen that the accuracy of the numerical solutions with the 5-point formula is more efficient. But when $N = 11$ and $t = 0.009$, the 5-point formula will be the best also:

$\Delta t = 0.001$, $t = 0.009$

Z-point formula	$u(x,t) = v(x,t)$	
	L_2	L_∞
3-point formula	9.365749×10^{-5}	3.1062676×10^{-4}
5-point formula	9.192620×10^{-5}	3.0488472×10^{-4}
7-point formula	1.3584093×10^{-3}	4.5053339×10^{-3}

Table 13: Influence of increasing points formulas of Example (2) when $\mu = 0.1$, $N = 11$.

5. Conclusions

In conclusion, a coupled system of one-dimensional viscous Burger's equation has been computationally tackled through the application of the 5-point non-central formulas incorporated in MOL. The devised approach was further assessed by considering two test models of interest. Further, the obtained computational results when compared with their exact counterparts were found to be in perfect agreement. Indeed, the values of L_2 and L_∞ norms that were utilized as error estimators were discovered to be negligible; meaning, the devised scheme was reliable and effective. Based on the results, it is noted that in some cases the 5-point formula effect in significant improvement over the 3- and 7- points formulas. Notably, the acquired computational scheme concerning the proposed scheme was found to be better with the non-central point, as against the central point that gave results with relatively higher errors. As a result, we suggest that the proposed methods should be applied to other types of viscous Burger's equations. Furthermore, we will address a higher order of viscous Burger's equations in our future work.

Disclosure statement

No potential conflict of interest was reported by the authors.

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