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Strongly Geodesic Log-Preinvex Functions

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Abstract. In this article, we delve into the intriguing concept of strongly geodesic log-preinvex functions in Riemannian manifolds. We present essential preliminaries and fundamental results that shed light on this specialized area of study. By examining the properties and implications of these functions, we aim to contribute to the growing body of knowledge in convexity theory within the context of Riemannian manifolds.

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1. Introduction and Preliminaries

Let $\vartheta \subseteq \mathbb{R}$ be an interval. A function $\xi : \vartheta \longrightarrow \mathbb{R}$ is said to be strongly convex with modulus $\varepsilon > 0$ if

$$\xi(\varsigma u_1+(1-\varsigma)u_2) \leq \varsigma\xi(u_1)+(1-\varsigma)\xi(u_2)-\varepsilon\varsigma(1-\varsigma)(u_1-u_2)^2, \;\; \forall\mu_1,\mu_2 \in \vartheta, \;\; \varsigma \in [0,1]. \quad (1)$$

The concept of strongly convex functions, initially introduced by Polyak (1966) [13], holds substantial relevance in the fields of optimization theory and mathematical economics. An extensive exploration of their properties and applications is well-documented across various studies, including those by Angulo et al. [1], Awan et al. [2], and Merentes et al. [6], among others.

The notion of convexity has been expanded to encompass strong convexity of order n on $\mathbb{R}^n$, as defined by Lin et al. [5]:

A function ξ, defined on a subset ϑ of the real numbers ($\mathbb{R}$), is termed strongly convex of order n if it satisfies the following condition for all μ_1 and μ_2 within ϑ and for ς in the range $[0,1]$: There exists a positive constant $\varepsilon > 0$ such that:

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$$\xi(\varsigma\mu_1 + (1-\varsigma)\mu_2) \leq \varsigma\xi(\mu_1) + (1-\varsigma)\xi(\mu_2) - \varepsilon\varsigma(1-\varsigma)\|\mu_1 - \mu_2\|^n.$$

This condition characterizes the strong convexity property of the function ξ in the context of mathematical analysis.

Additional applications, numerical techniques, and variational-like inequalities for convex functions are discussed in [10, 11]. It is noteworthy that log-convex functions, as opposed to convex functions, have been shown to yield more precise results and inequalities. Numerous aspects related to exponentially preinvex functions and their variants are introduced in [8, 9].

Let (N, ξ) be a complete m-dimensional Riemannian manifold equipped with a Riemannian connection ∇. Consider a piecewise C^1 path $\gamma : [\mu_1, \mu_2] \longrightarrow \vartheta$ connecting a_1 to a_2, where $\gamma(\mu_1) = a_2$ and $\gamma(\mu_2) = a_1$. The length of γ is defined as:

$$L(\gamma) = \int_{u_1}^{u_2} \|\acute{\gamma}(\lambda)\|_{\gamma(\lambda)} d\lambda.$$

For any two points a_1 and a_2 in N, we introduce the following metric:

$$d(a_1, a_2) = \inf \left\{ L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ path connecting } a_1 \text{ to } a_2 \right\}.$$

This metric d induces the original topology on N.

In every Riemannian manifold, there exists a uniquely determined Riemannian connection known as the Levi-Civita connection, denoted by $\nabla_X Y$, for any vector fields X and Y in ϑ. Furthermore, a smooth path γ is considered a geodesic if and only if its tangent vector is a parallel vector field along the path γ, i.e., γ satisfies the equation $\nabla_{\gamma'}\gamma' = 0$.

Any path γ that connects μ_1 and μ_2 in N such that $L(\gamma) = d(\mu_1, \mu_2)$ is a geodesic and is referred to as a minimal geodesic.

Let N be a C^∞ complete n-dimensional Riemannian manifold with metric g and Levi-Civita connection ∇. Additionally, consider the points μ_1 and μ_2 in N, and let $\gamma : [0, 1] \longrightarrow N$ be a geodesic connecting μ_1 and μ_2, i.e., $\gamma_{\mu_1,\mu_2}(0) = \mu_2$ and $\gamma_{\mu_1,\mu_2}(1) = \mu_1$.

Definition 1. *[3]. Let a set $\vartheta \subset N$ be geodesic invex w.r.t. $\eta : N \times N \longrightarrow TN$. A function $\xi : \vartheta \longrightarrow \mathbb{R}$is said to be geodesic preinvex w.r.t.η iff*

$$\xi(\gamma_{\mu_1,\mu_2}) \leq (1-\varsigma)\xi(\mu_1) + \varsigma\xi(\mu_2), \forall \mu_1, \quad \mu_2 \in \vartheta, \quad \varsigma \in [0, 1].$$

The strongly geodesic convexity of order n on a Riemannian manifold is elaborated in [4].

Definition 2. *Suppose $\vartheta \subseteq N$ is a geodesically convex subset of N. A function $\xi : \vartheta \longrightarrow \mathbb{R}$ is termed strongly geodesically convex of order $n > 0$ on ϑ if there exists a positive constant $\varepsilon > 0$ such that for all μ_1 and μ_2 in ϑ and for t in the interval $[0, 1]$, the following inequality holds:*

$$\xi(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq \varsigma\xi(\mu_1) + (1-\varsigma)\xi(\mu_2) - \varepsilon\varsigma(1-\varsigma)\|\gamma'_{\mu_1,\mu_2}(\varsigma)\|^n.$$

Pini [12] conducted an investigation into various properties of invex functions on Riemannian manifolds, while Mititelu [7] explored its generalization. Noor and Noor [9] introduced a novel concept known as exponentially preinvex functions. Subsequently, a multitude of papers have emerged in the literature, delving into the realm of (generalized) convexity on Riemannian manifolds, we refer readers to [14–16].

In this article, we present some introductory concepts and fundamental results pertaining to strongly geodesic log-preinvex functions in Riemannian manifolds.

2. Main results

In this article, we introduce an innovative concept of generalized convexity in the context of Riemannian manifolds. Specifically, we define the concept of strongly geodesic log-preinvex functions, which serves as a comprehensive generalization and extension of various previously introduced notions of generalized convexity in the existing literature.

Definition 3. *A function $\xi : \vartheta \longrightarrow \mathbb{R}_+^*$ is considered strongly geodesic log-preinvex w.r.t a bifunction η if there exists a constant $\varepsilon \geq 0$ such that:*

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq (1-\varsigma)\log \xi(\mu_1) + \varsigma \log \xi(\mu_2) - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2 \tag{2}$$

for all $\mu_1, \mu_2 \in \vartheta$ and $\varsigma \in [0,1]$.

Definition 4. *A function $\xi : \vartheta \longrightarrow \mathbb{R}_+^*$ is considered to be strongly geodesic log-quasi preinvex w.r.t the bifunction η if:*

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq \max\{\log \xi(\mu_1), \log \xi(\mu_2)\} - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2, \forall \mu_1,\mu_2 \in \vartheta, \varsigma \in [0,1].$$

Definition 5. *A function $\xi : \vartheta \longrightarrow \mathbb{R}_+^*$ is said to be first kind of strongly geodesic log-preinvex w.r.t. any arbitrary bifunction η , if*

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq (\log \xi(\mu_1))^{1-\varsigma}(\log \xi(\mu_2))^{t} - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2, \forall \mu_1,\mu_2 \in \vartheta, \varsigma \in [0,1].$$

From the above defintions, we have

(i)

$$\begin{aligned}\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) &\leq (\log \xi(\mu_1))^{1-\varsigma}(\log \xi(\mu_2))^{t} - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2 \\ &\leq (1-\varsigma)\log \xi(\mu_1) + \varsigma \log \xi(\mu_2) - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2, \forall \\ &\leq \max\{\log \xi(\mu_1), \log \xi(\mu_2)\} - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2, \forall \mu_1,\mu_2 \in \vartheta, \varsigma \in [0,1].\end{aligned}$$

This demonstrates that every first kind of strongly geodesic log-preinvex function is indeed a strongly geodesic log-preinvex function, and a strongly geodesic log-preinvex function can be considered a strongly geodesic log-quasipreinvex function. However, it's important to note that the converse is not necessarily true.

(ii) If $\varepsilon = 0$, then

- **(a)** Strongly geodesic log-preinvex function is called geodesic log-preinvex function.
- **(b)** Strongly geodeic log-quasi preinvex is called geodesic log-quasi preinvex.
- **(c)** The first kind of strongly geodesic log-preinvex is called the first kind of geodesic log-preinvex.

(iii) If $\varsigma = 1$, then Definitions 3 and 5 will be become

$$\xi(\mu_1) \leq \xi(\mu_2), \forall \mu_1, \mu_2 \in \vartheta.$$

Definition 6. *A function $\xi : \vartheta \longrightarrow \mathbb{R}_+^*$ is said to be a strongly affine geodesic log-preinvex w.r.t. the bifunction η , if there exists a constant $\varepsilon \geq 0$, such that*

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) = (1-\varsigma) \log \xi(\mu_1) + \varsigma \log \xi(\mu_2) - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2, \tag{3}$$

$\forall \mu_1, \quad \mu_2 \in \vartheta$ *and* $\varsigma \in [0,1]$.

Example 1. *Assume that map $\eta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as*

$$\eta(\mu_1,\mu_2) = \begin{cases} 0 & if \mu_1 = \mu_2, \\ 1 - mu_1 & if \mu_1 \neq \mu_2, \end{cases}$$

also,

$$\gamma_{\mu_1,\mu_2}(\varsigma) = \begin{cases} \mu_2 & if \mu_1 = \mu_2, \\ \mu_2 + \varsigma(1-\mu_1) & if \mu_1 \neq \mu_2, \end{cases}$$

Assume that $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}$, where $\xi(\mu) = \exp^{\mu}$, then ξ is strongly geodesic log-preinvex w.r.t. the bifunction η.

Example 2. *In this example, we give some new parallegram low of uniformly Banach spaces involving the notion of strongly affine geodesic log-preinvex w.r.t. the bifunction η Assume that*

$$\eta(\mu_1,\mu_2) = \begin{cases} 0 & if \mu_1 = \mu_2, \\ mu_1 - mu_2 & if \mu_1 \neq \mu_2, \end{cases}$$

also,

$$\gamma_{\mu_1,\mu_2}(\varsigma) = \begin{cases} \mu_2 & if \mu_1 = \mu_2, \\ \mu_2 + \varsigma(\mu_1 - \mu_2) & if \mu_1 \neq \mu_2, \end{cases}$$

Frome equality (3), we get

$$\|\log \xi(\mu_2 + \varsigma(\mu_1-\mu_2))\|^2 = (1-\varsigma)\|\log \xi(\mu_1)\|^2 + \varsigma\|\log \xi(\mu_2)\|^2 - \varepsilon\varsigma(1-\varsigma)\|\mu_2 - \mu_1\|^2, \tag{4}$$

$\forall \mu_1, \quad \mu_2 \in \vartheta$ *and* $\varsigma \in [0,1]$. *Taking $\varsigma = \frac{1}{2}$in (4), we get*

$$\|\log \xi(\frac{\mu_1+\mu_2}{2})\|^2 + \frac{\varepsilon}{4}\|\mu_2-\mu_1\|^2 = \frac{1}{2}\left(\|\log \xi(\mu_1)\|^2 + \|\log \xi(\mu_2)\|^2\right), \tag{5}$$

$\forall \mu_1, \quad \mu_2 \in \vartheta$. *Which ξ is called the log-paralleogram for the inner product spaces. By putting $\log \xi(\mu_1) = \|\mu_1\|^2$ in (5), we have the paralleogram for the inner product spaces.*

3. Some Aspects of Geodesic Log-Preinvexity Properties

In this section, we examine fundamental properties of geodesic log-preinvex functions.

Theorem 1. *If ξ is a strongly geodesic log-preinvex fun ction, then any point that serves as a local minimum is also considered a global minimum.*

Proof.
Considering that the function ξ is geodesic log-preinvex and possesses a local minimum at $\mu_1 \in \vartheta$. Let's assume the contrary, which is, $\xi(\mu_2) < \xi(\mu_1)$ for some $\mu_2 \in \vartheta$. Since ξ is a geodesic log-preinvex function, we have:

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) < (1-\varsigma)\log \xi(\mu_1) + \varsigma \log \xi(\mu_2),$$

for $\varsigma \in (0,1)$.

Thus, we can derive:

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) - \log \xi(\mu_1) < t\left(\log \xi(\mu_2) - \log \xi(\mu_1)\right) < 0, \tag{6}$$

From which it follows that:

$$\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) < \log \xi(\mu_1),$$

for arbitrary small $\varsigma > 0$. This contradicts the fact that the function has a minimum at the point μ_1.

Theorem 2. *A positive function $\mathcal{F}$ is considered to be geodesic log-preinvex if and only if the set $epi(\mathcal{F}) = \{(u,v) : u \in U, \log \mathcal{F}(u) \leq v, v \in \mathbb{R}\}$ is a geodesic invex set.*

Proof. Let $\mathcal{F}$ be a geodesic log-preinvex function. Let $(u_1,v_1),(u_2,v_2) \in epi(\mathcal{F})$. Then $log\mathcal{F}(u_1) \leq v_1$ and $log\mathcal{F}(u_2) \leq v_2$. Thus

$$\begin{aligned} Log\mathcal{F}(\gamma_{u_1,u_2}(\varsigma)) &\leq (1-\varsigma)log\mathcal{F}(u_1) + \varsigma log\mathcal{F}(u_2) \\ &\leq (1-\varsigma)v_1 + \varsigma v_2 . \forall \varsigma \in [0,1]. \end{aligned}$$

That means

$$(\gamma_{u_1,u_2}(\varsigma), (1-\varsigma)v_1 + \varsigma v_2) \in epi(\mathcal{F}).$$

This $epi(\mathcal{F})$ is geodesic invex set.

Conversely, suppose $epi(\mathcal{F})$ is a geodesic invex set. Let $u,v \in U$. Then, we have $(u, \log \mathcal{F}(u)) \in epi(\mathcal{F})$ and $(v, \log \mathcal{F}(v)) \in epi(\mathcal{F})$. Since $epi(\mathcal{F})$ is a geodesic invex set, it follows that:

$$(\gamma_{u,v}(\varsigma), (1-\varsigma)\log \mathcal{F}(u) + \varsigma \log \mathcal{F}(v)) \in epi(\mathcal{F}),$$

which implies:

$$\log \mathcal{F}(\gamma_{u,v}(\varsigma)) \leq (1-\varsigma)\log \mathcal{F}(u) + \varsigma \log \mathcal{F}(v).$$

This demonstrates that $\mathcal{F}$ is a geodesic log-preinvex function.

4. Proporties of Strongly Geodesic Log-Preinvex Functions

In this section, we delve into fundamental properties of functions exhibiting strongly geodesic log-preinvexity.

Theorem 3. *Assume that the function ξ is differentiable on ϑ° . If ξ is geodesic log-preinvex, then*

$$\log\xi(\mu_2) - \log\xi(\mu_1) \geq \frac{d\xi_{\mu_1}\eta(\mu_2,\mu_1)}{\xi(\mu_1)} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2 \tag{7}$$

Proof. Assume that the function ξ is strongly geodesic log-preinvex, one has that

$$\log\xi(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq (1-\varsigma)\log\xi(\mu_1) + \varsigma\log\xi(\mu_2) - \varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2,$$

hence

$$\log\xi(\mu_2) - \log\xi(\mu_1) \geq \frac{\log\xi(\gamma_{\mu_1,\mu_2}(\varsigma)) - \log\xi(\mu_1)}{t} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2.$$

Upon approaching the limit as ς tends towards zero in the preceding inequality, we obtain:

$$\log\xi(\mu_2) - \log\xi(\mu_1) \geq \frac{d_{\mu_1}\xi\gamma_{\mu_1,\mu_2}}{\xi(u)} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2.$$

Which (7), the required result.

Remark 1. *Form (7), we have*

$$\xi(\mu_2) \geq \xi(\mu_1)\exp\left\{\frac{d_{\mu_1}\xi\gamma_{\mu_1,\mu_2}}{\xi(\mu_1)} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2\right\}, \forall\mu_1, \quad \mu_2 \in \vartheta.$$

By interchanging the roles of μ_1 and μ_2 in the inequality above, we also obtain:

$$\xi(\mu_1) \geq \xi(\mu_2)\exp\left\{\frac{d_{\mu_2}\xi\gamma_{\mu_1,\mu_2}}{\xi(\mu_2)} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2\right\}, \forall\mu_1, \quad \mu_2 \in \vartheta.$$

Therefore, we can deduce the subsequent inequality:

$$\begin{aligned}\xi(\mu_1) + \xi(\mu_2) \geq\ & \xi(\mu_2)\exp\left\{\frac{d_{\mu_2}\xi\gamma_{\mu_1,\mu_2}}{\xi(\mu_2)} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2\right\} \\ & + \xi(\mu_1)\exp\left\{\frac{d_{\mu_1}\xi\gamma_{\mu_1,\mu_2}}{\xi(\mu_1)} + \varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2\right\}, \forall\mu_1, \quad \mu_2 \in \vartheta.\end{aligned}$$

The last Theorem allows us to introduce the concept of geodesic log-monotone operators, which appears to be a novel addition to the field.

Definition 7. *(i) The differential f' is considered to be strongly geodesic log-monotone , if*

$$\frac{df_{\mu_1}\eta(\mu_2,\mu_1)}{f(\mu_1)} + \frac{df_{\mu_2}\eta(\mu_1,\mu_2)}{f(v)} \leq -\varepsilon\left\{\|\eta(\mu_2,\mu_1)(\varsigma)\|^2 + \|\eta(\mu_1,\mu_2)(\varsigma)\|^2\right\}, \forall\mu_1,\mu_2 \in \vartheta.$$

(ii) The differential f' is considered to be geodesic log-monotone , if

$$\frac{df_{\mu_1}\eta(\mu_2,\mu_1)}{f(\mu_1)}+\frac{df_{\mu_2}\eta(\mu_1,\mu_2)}{f(\mu_2)}\leq 0,\forall\mu_1,\mu_2\in\vartheta.$$

(iii) The differential f' is considered to be geodesic log- pseudo-monotone , if

$$\frac{df_{\mu_1}\eta(v,\mu_1)}{f(\mu_1)}\geq 0\Longrightarrow -\frac{df_{v}\eta(\mu_1,v)}{f(v)}\geq 0\,\forall\mu_1,v\in\vartheta.$$

From these definitions, we can deduce that strongly geodesic log-monotonicity entails geodesic log-monotonicity, which, in turn, implies geodesic log-pseudo-monotonicity. It's crucial to emphasize that the reverse may not always hold true.

Theorem 4. *Let f be differentiable strongly geodesic log-preinvex function on the geodesic set. If (7) holds, then f' satisfies*

$$\frac{df_{\mu_1}\eta(\mu_2,\mu_1)}{f(\mu_1)}+\frac{df_{\mu_2}\eta(\mu_1,v)}{f(\mu_2)}\leq -\varepsilon\left\{\|\eta(\mu_2,\mu_1)(\varsigma)\|^2+\|\eta(\mu_1,\mu_2)(\varsigma)\|^2\right\},\forall\mu_1,\mu_2\in\vartheta. \quad (8)$$

Proof.

$$\log f(\mu_2)-\log f(\mu_1)\geq\frac{df_{\mu_1}\eta(\mu_2,\mu_1)}{f(\mu_1)}+\varepsilon\|\eta(\mu_2,\mu_1)(\varsigma)\|^2. \quad (9)$$

By swapping the roles of μ_1 and μ_2 in inequality (9), we obtain:

$$\log f(\mu_1)-\log f(\mu_2)\geq\frac{df_{\mu_2}\eta(\mu_1,\mu_2)}{f(\mu_2)}+\varepsilon\|\eta(\mu_1,\mu_2)(\varsigma)\|^2. \quad (10)$$

Adding (9) and (10), we have

$$\frac{df_{\mu_1}\eta(\mu_2,\mu_1)}{f(\mu_1)}+\frac{df_{\mu_2}\eta(\mu_1,\mu_2)}{f(\mu_2)}\leq -\varepsilon\left\{\|\eta(\mu_2,\mu_1)(\varsigma)\|^2+\|\eta(\mu_1,\mu_2)(\varsigma)\|^2\right\},\forall\mu_1,\mu_2\in\vartheta, \quad (11)$$

demonstrating that the derivative f' is strongly geodesic log-monotone.

Definition 8. *The function f is considered to be sharply geodesic log-pseudo preinvex, if*

$$\frac{df_{\mu_1}\eta(\mu_2,\mu_1)}{f(\mu_1)}\geq 0\Longrightarrow f(\mu_2)\geq\log f(\gamma_{\mu_1,\mu_2}(\varsigma)),\forall\mu_1,\mu_2\in\vartheta,\varsigma\in[0,1].$$

Theorem 5. *Assume that f is a sharply geodesic log-pseudo preinvex function on A. Then*

$$\frac{df_{\mu_2}\eta(\mu_1,\mu_2)}{f(\mu_2)}\geq 0,\forall\mu_1,\mu_2\in\vartheta.$$

Proof. Let f be a sharply geodesic log-pseudo preinvex function on ϑ, then

$$f(\mu_2)\geq\log f(\gamma_{\mu_1,\mu_2}(\varsigma)),\forall\mu_1,\mu_2\in\vartheta,\varsigma\in[0,1].$$

Hence, by taking $\varsigma\longrightarrow 0$, we have the result.

Definition 9. *A function ξ is considered geodesic log-pseudo preinvex w.r.t. a strictly positive bifunction β , such that*

$$\log \xi(\mu_2) < \log \xi(\mu_1) \Longrightarrow \log \xi(\gamma_{\mu_1,\mu_2}) < \log \xi(\mu_1) + \varsigma(\varsigma - 1)\beta(\mu_2, \mu_1),$$

$\forall \mu_1, \quad \mu_2 \in \vartheta$ *and* $\varsigma \in [0, 1]$.

Theorem 6. *If ξ is a strongly geodesic log-preinvex function such that*

$$\log \xi(\mu_2) < \log \xi(\mu_1),$$

then ξ is strongly geodesic log-pseudo preinex.

Proof. Since $\log \xi(\mu_2) < \log \xi(\mu_1)$ and ξ is a strongly geodesic log-preinvex function, then $\forall \mu_1, \mu_2 \in \vartheta, \varsigma \in [0, 1]$, we have

$$\begin{aligned}
\log \xi(\gamma_{\mu_1,\mu_2}(\varsigma)) &\leq \log \xi(\mu_1) + \varsigma \left(\log \xi(\mu_2) - \log \xi(\mu_1)\right) - \varepsilon\varsigma(1 - \varsigma)\|\eta(\mu_2, \mu_1)(\varsigma)\|^2 \\
&< \log \xi(\mu_1) + \varsigma(1 - \varsigma) \left(\log \xi(\mu_2) - \log \xi(\mu_1)\right) - \varepsilon\varsigma(1 - \varsigma)\|\eta(\mu_2, \mu_1)(\varsigma)\|^2 \\
&= \log \xi(\mu_1) + \varsigma(\varsigma - 1) \left(\log \xi(\mu_1) - \log \xi(\mu_2)\right) - \varepsilon\varsigma(1 - \varsigma)\|\eta(\mu_2, \mu_1)(\varsigma)\|^2 \\
&< \log \xi(\mu_1) + \varsigma(1 - \varsigma)\beta(\mu_2, \mu_1) - \varepsilon\varsigma(1 - \varsigma)\|\eta(\mu_2, \mu_1)(\varsigma)\|^2,
\end{aligned}$$

where $\beta(\mu_2, \mu_1) = \log \xi(\mu_1) - \log \xi(v) > 0$. This demonstrates that ξ is a strongly geodesic log-preinvex function.

Now, we demonstrate that the subtraction of a strongly geodesic log-preinvex function and an affine geodesic strongly log-preinvex function results in another geodesic log-preinvex function.

Theorem 7. *Let f be affine strongly geodesic log-preinvex function. If $\mathcal{F}$ is a strongly geodesic log-preinvex , then $\mathcal{F} - f$ is a geodesic log-preinvex function.*

Proof. Assume that f is an affine strongly geodesic log-preinvex function.Then

$$\log f(\gamma_{\mu_1,\mu_2}(\varsigma)) = (1-\varsigma) \log f(\mu_1)+\varsigma \log f(\mu_2)-\varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2, \mu_1)(\varsigma)\|^2, \forall \mu_1, \mu_2 \in \vartheta, \varsigma \in [0, 1]. \tag{12}$$

From the strongly geodesic log-preivexity of $\mathcal{F}$, we have

$$\log \mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma)) = (1-\varsigma) \log \mathcal{F}(\mu_1)+\varsigma \log \mathcal{F}(\mu_2)-\varepsilon\varsigma(1-\varsigma)\|\eta(\mu_2, \mu_1)(\varsigma)\|^2, \forall \mu_1, \mu_2 \in \vartheta, \varsigma \in [0, 1]. \tag{13}$$

From (1) and (12), we have

$$\log \mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma)) - \log f(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq (1 - \varsigma)(\log \mathcal{F}(\mu_1) - \log(\mu_1)) + \varsigma(\log \mathcal{F}(\mu_2) - \log f(\mu_2)). \tag{14}$$

$$\log \mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma))-\log f(\gamma_{\mu_1,\mu_2}(\varsigma)) \leq (1-\varsigma)(\log \mathcal{F}(\mu_1)-\log f(\mu_1))+\varsigma(\log \mathcal{F}(\mu_2)-\log f(\mu_2)).$$

This demonstrates that $\mathcal{F} - f$ is a geodesic log-preinvex function.

Remark 2. *We observe that if a strictly positive function* $\mathcal{F}$ *is strongly geodesic log-preinvex, then the following inequality holds for all* $\mu_1, \mu_2 \in \vartheta$ *and* $\varsigma \in [0,1]$*, known as the Wright strongly geodesic Log-preinvex function:*

$$\log \mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma)) + \log \mathcal{F}(\gamma_{\mu_2,\mu_1}(\varsigma)) \leq \log \mathcal{F}(\mu_1) + \log \mathcal{F}(\mu_2) - 2\varsigma(1-\varsigma)\|\eta(\mu_2,\mu_1)(\varsigma)\|^2. \quad (15)$$

From (15)*, we can deduce the following:*

$$\begin{aligned} \mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma))\mathcal{F}(\gamma_{\mu_2,\mu_1}(\varsigma)) &= \log \mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma)) + \log \mathcal{F}(\gamma_{\mu_2,\mu_1}(\varsigma)) \\ &\leq \log \mathcal{F}(\mu_1) + \log \mathcal{F}(\mu_2) \\ &= \log \mathcal{F}(\mu_1)\mathcal{F}(\mu_2), \forall \mu_1, \mu_2 \in \vartheta, \varsigma \in [0,1]. \end{aligned}$$

This implies that

$$\mathcal{F}(\gamma_{\mu_1,\mu_2}(\varsigma))\mathcal{F}(\gamma_{\mu_2,\mu_1}(\varsigma)) \leq \mathcal{F}(\mu_1)\mathcal{F}(\mu_2), \forall \mu_1, \mu_2 \in \vartheta, \varsigma \in [0,1],$$

which demonstrates that every strictly positive function $\mathcal{F}$ *is multiplicative Wright Strongly Geodesic Log-Preinvex.*

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References

[1] Hiliana Angulo, José Giménez, Ana Milena Moros, and Kazimierz Nikodem. On strongly h-convex functions. *Annals of functional analysis*, 2(2):85–91, 2011.

[2] Muhammad Uzair Awan, Muhammad Aslam Noor, Khalida Inayat Noor, and Farhat Safdar. On strongly generalized convex functions. *Filomat*, 31(18):5783–5790, 2017.

[3] A Barani and MR Pouryayevali. Invex sets and preinvex functions on riemannian manifolds. *Journal of Mathematical Analysis and Applications*, 328(2):767–779, 2007.

[4] Akhlad Iqbal and Izhar Ahmad. Strong geodesic convex functions of order m. *Numerical Functional Analysis and Optimization*, 40(15):1840–1846, 2019.

[5] Gui-Hua Lin and Masao Fukushima. Some exact penalty results for nonlinear programs and mathematical programs with equilibrium constraints. *Journal of Optimization Theory and Applications*, 118:67–80, 2003.

[6] Nelson Merentes, Kazimierz Nikodem, and Sergio Rivas. Remarks on strongly wright-convex functions. *Ann. Polon. Math*, 102(3):271–278, 2011.

[7] Stephan Mititelu. Generalized invexity and vector optimization on differentiable manifolds. *Differ. Geom. Dyn. Syst*, 3(1):21–31, 2001.

[8] Muhammad Aslam Noor and Khalida Inayat Noor. New classes of strongly exponentially preinvex functions. *AIMS Math*, 4(6):1554–1568, 2019.

[9] Muhammad Aslam Noor and Khalida Inayat Noor. Some properties of exponentially preinvex functions. *FACTA Universitat (NIS). Ser.Math. Inform*, 34(5), 2019.

[10] Muhammad Aslam Noor, Khalida Inayat Noor, and Muhammad Uzair Awan. Hermite hadamard inequalities for modified h-convex functions. *Transylvanian Journal Of Mathematics And Mechanics*, 6:1–10, 2014.

[11] Ammara Nosheen, Sana Ijaz, Khuram Ali Khan, Khalid Mahmood Awan, Marwan Ali Albahar, and Mohammed Thanoon. Some q-symmetric integral inequalities involving s-convex functions. *Symmetry*, 15(6):1169, 2023.

[12] R Pini. Convexity along curves and indunvexity. *Optimization*, 29(4):301–309, 1994.

[13] Boris Teodorovich Polyak. Existence theorems and convergence of minimizing sequences for extremal problems with constraints. In *Doklady Akademii Nauk*, volume 166, pages 287–290. Russian Academy of Sciences, 1966.

[14] Wedad Saleh. Some properties of geodesic strongly eb-vex functions. *International Journal of Analysis and Applications*, 17(3):388–395, 2019.

[15] Constantin Udriste. *Convex functions and optimization methods on Riemannian manifolds*, volume 297. Springer Science & Business Media, 2013.

[16] Taiyin Zhao, Muhammad Shoaib Saleem, Waqas Nazeer, Imran Bashir, and Ijaz Hussain. On generalized strongly modified h-convex functions. *Journal of Inequalities and Applications*, 2020:1–12, 2020.