EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2299-2310
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

European Call Option under Stochastic Interest Rate in a Fractional Brownian Motion with Transaction Cost

Felipe R. Sumalpong, Jr.[1,*], Eric G. Lauron[2]

[1] *Department of Mathematics and Statistics, Faculty/Mindanao State University - Iligan Institute of Technology, Iligan City, Philippines*
[2] *J.H Cerilles State College - Dumingag Campus, Zamboanga del Sur, Philippines*

Abstract. This paper deals on the valuation of European call option price in a stochastic environment by employing three factors which are the stochastic model of the asset value, the stochastic interest rate and the transaction cost. We specify that our underlying asset and the stochastic interest rate, particularly Hull-White model, follows a fractional Brownian Motion governed by Hurst parameter H. We used the hedging and replicating technique to established the zero-coupon bond on the European option. Finally, we give a closed-form formula of the European call option price.

2020 Mathematics Subject Classifications: 62P05, 97M30

Key Words and Phrases: European Call Option, Fractional Brownian Motion, Fractional Hull-White Interest Rate Model, Transaction Cost, Hedging, Option Replication

1. Introduction

The study of option pricing can be traced back to the seminal papers of Black and Scholes[1] and Merton[2]. The Black-Scholes-Merton(BSM) model is a known mathematical model to evaluate the price of the option that utilizes five inputs namely as: the asset price, the strike price, the risk-free interest rates, time of expiration and the volatility. The initial equation of the BSM model was published in 1973 on the paper "The Pricing of Options and Corporate Liabilities" in *Journal of Political Economy.* By this model, Black, Scholes and Merton received a Nobel prize in Economics in 1997.

However, there are several drawbacks of the BSM model such as the assumptions that the interest rate and the volatility rate are constant over the period of the contract does not fit the actual scenario of the market, transaction cost may not be avoided, and the evolution of the asset price does not always obey a standard Brownian motion($H = 1/2$) for which Nualart[3] noted that for some stock process, the Hurst exponent H may not

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5107

Email addresses: `felipejr.sumalpong@g.msuiit.edu.ph` (F.Jr. Sumalpong), `eric.lauron@g.msuiit.edu.ph` (E. Lauron)

be exactly 1/2 but greater than or equal to 1/2. Hence, the standard Brownian Motion where $H = 1/2$ may not be a fitting stochastic process for some stock processes.

In this paper, we assumed that both the asset value and the interest rate follows the fractional Brownian Motion. We specify that the asset value $X(t)$ follows

$$dX(t) = r(t)X(t)dt + \sigma_X X(t)dB_1^H(t) \tag{1}$$

and the interest rate $r(t)$ follows the fractional Hull-White model

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma_r dB_2^H(t) \tag{2}$$

where σ_X is the volatility of the asset price, σ_r is the volatility of the interest rate, and B^H is a fractional Brownian motion with Hurst parameter H.

With the transaction cost

$$Cost = cX(t)|v(t)| \tag{3}$$

where c is a fixed proportion of the trading amount for the asset agreed between both parties, and $v(t)$ is the number of assets sold or bought, we formulate a European option price model on par with the BSM model.

For simplicity, we assumed that the volatilities for both the asset price and the interest rate are constant, no dividend and coupon payments, and we limit our analysis to European options only.

2. Model Formulation

This section presents our assumptions for the computations of the option price. The following assumptions are made:

1. Asset Price Model
   The asset price $X(t)$ follows a fractional Brownian motion given by

   $$dX(t) = r(t)X(t)dt + \sigma_X X(t)dB_1^H(t) \tag{4}$$

   where $r(t)$ is the total expected rate of return, σ_X is the volatility of the price and $B_1^H(t)$ is a fractional Brownian Motion with Hurst parameter H.

2. Interest Rate model
   The risk-free interest rate $r(t)$ follows a fractional Hull-White model given by

   $$dr(t) = [\theta(t) - ar(t)]dt + \sigma_r dB_2^H(t) \tag{5}$$

   where $\theta(t)$ is a deterministic function of time, a is constant, σ_r is the volatility of the interest rate which is assumed to be constant and $B_2^H(t)$ is a fractional Brownian Motion with Hurst parameter H.

   Under the fractional Brownian motion, the correlation coefficient between $B_1^H(t)$

and $B_2^H(t)$ for $t \geq 0$ is given by

$$Cov(B_1^H(t), B_2^H(t)) = \rho(dt)^{2H}. \tag{6}$$

Moreover, the following properties are applied to the fractional Brownian Motion:

$$\mathbb{E}[dB^H(t)] = 0, \tag{7}$$
$$\mathbb{E}[dtdB^H(t)] = 0, \tag{8}$$
$$\mathbb{E}[dB_1^H(t)dB_2^H(t)] = \rho dt^{2H}, \tag{9}$$
$$\mathbb{E}[(dB^H(t))^2] = (dt)^{2H}, \tag{10}$$
$$\mathbb{E}[(dt)^2] = 0. \tag{11}$$

Lemma 1. *The zero-coupon bond model with the terminal condition $P(r, t; T) = 1$ can derive the following formula*

$$P(r, t; T) = e^{-rB(t,T)-A(t,T)}, \tag{12}$$

with

$$B(t, T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$
$$A(t, T) = -\int_t^T \theta(u)B(u, T)du + \frac{1}{2}dt^{2H}\left[\int_t^T \sigma_r^2 B^2(u, T)du\right]$$

where σ_r is the volatility of the interest rate, $\theta(t)$ is a deterministic function of time, and a is a constant assumed to be nonzero.

3. Transaction cost
   Transaction cost is a fixed proportion c, depending on the individual investor, of the trading amount for the asset. We have

   $$Cost = cX(t)|v(t)|$$

   where $v(t)$ is the number of shares of the sold or bought at the price X_t. Specifically, $v(t) > 0$ indicates a bought share while $v(t) < 0$ indicates a sold shares.

4. Portfolio
   The portfolio is revised at time dt, where dt is a small time step from t to $t + dt$.

5. Expected return of the portfolio
   The expected return of the portfolio $\Pi(t)$ satisfies the equality

   $$\mathbb{E}[d\Pi(t)] = r(t)\Pi(t)dt \tag{13}$$

   where $r(t)$ is the interest rate.

From these assumptions, we can now begin to price the zero-coupon bond of the fractional Hull-White which we will use to come up an option price.

Theorem 1. *Under the fractional Hull-White Interest Rate Model, the zero-coupon bond price $P(r(t), t; T)$ obeys the following equation:*

$$\frac{\partial P}{\partial t} + [\theta(t) - ar(t) - \psi\sigma_r]\frac{\partial P}{\partial r} + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\sigma_r)^2(dt)^{2H-1} - rP = 0.$$

where $r(t)$ is the interest rate under Hull-White model, $\theta(t)$ is a deterministic function of time, a is constant, ψ is the market price of the risk with volatility σ and σ_r is the volatility of the interest rate which is assumed to be constant.

Proof: The equation can be attained by constructing a deterministic hedged portfolio that employs two coupon bonds $P_1(t), P_2(t)$, that is,

$$\Pi(t) = P_1(t) - \Delta P_2(t) \tag{14}$$

with different maturity T_1 and T_2 respectively. We long one share of bond $P_1(t)$ and short Δ shares of $P_2(t)$. Taking the derivative of Equation (14), we have

$$d\Pi(t) = dP_1(t) - \Delta dP_2(t). \tag{15}$$

Applying Ito's Lemma to the price function $P(r(t), t; T)$, the right-hand side of the equality in Equation 15 becomes,

$$\begin{aligned}
&\frac{\partial P_1}{\partial t}dt + [\theta(t) - ar(t)]\frac{\partial P_1}{\partial r}dt + \frac{1}{2}\frac{\partial^2 P_1}{\partial r^2}\big([\theta(t) - ar(t)]^2(dt)^2 \\
&\quad +2[\theta(t) - ar(t)]\sigma_r dt dB_2^H + (\sigma_r)^2(dB_2^H)^2\big) + \sigma_r\frac{\partial P_1}{\partial r}dB_2^H - \Delta\sigma_r\frac{\partial P_1}{\partial r}dB_2^H \\
&\quad - \Delta\bigg(\frac{\partial P_2}{\partial t}dt + [\theta(t) - ar(t)]\frac{\partial P_2}{\partial r}dt + \frac{1}{2}\frac{\partial^2 P_2}{\partial r^2}\bigg([\theta(t) - ar(t)]^2(dt)^2 \\
&\quad + 2[\theta(t) - ar(t)]\sigma_r dt dB_2^H + (\sigma_r)^2(dB_2^H)^2\bigg)\bigg)
\end{aligned}$$

To eliminate the risk, take

$$\Delta = \frac{\partial P_1}{\partial r}\Big/\frac{\partial P_2}{\partial r}, \tag{16}$$

such that $\frac{\partial P_2}{\partial r} \neq 0$. Thus,

$$\begin{aligned}
&\frac{\partial P_1}{\partial t}dt + [\theta(t) - ar(t)]\frac{\partial P_1}{\partial r}dt + [\theta(t) - ar(t)]^2(dt)^2 \\
&\quad + [\theta(t) - ar(t)]\sigma_r\frac{\partial^2 P_1}{\partial r^2}dt dB_2^H + (\sigma_r)^2\frac{1}{2}\frac{\partial^2 P_1}{\partial r^2}(dB_2^H)^2 \\
&\quad - \left(\frac{\partial P_1}{\partial r}\Big/\frac{\partial P_2}{\partial r}\right)\frac{\partial P_2}{\partial t}dt - [\theta(t) - ar(t)]\frac{\partial P_2}{\partial r}\left(\frac{\partial P_1}{\partial r}\Big/\frac{\partial P_2}{\partial r}\right)dt
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\frac{\partial^2 P_2}{\partial r^2}\left(\frac{\partial P_1}{\partial r}/\frac{\partial P_2}{\partial r}\right)[\theta(t) - ar(t)]^2(dt)^2 \\
&- [\theta(t) - ar(t)]\sigma_r \frac{\partial^2 P_2}{\partial r^2}\left(\frac{\partial P_1}{\partial r}/\frac{\partial P_2}{\partial r}\right) dt dB_2^H \\
&- (\sigma_r)^2 \frac{1}{2}\frac{\partial^2 P_2}{\partial r^2}\left(\frac{\partial P_1}{\partial r}/\frac{\partial P_2}{\partial r}\right)(dB_2^H)^2.
\end{aligned}
$$

By non-arbitrage principle,

$$
\begin{aligned}
&\frac{\partial P_1}{\partial t} + [\theta(t) - ar(t)]\frac{\partial P_1}{\partial r} + (\sigma_r)^2\frac{1}{2}\frac{\partial^2 P_1}{\partial r^2}(dt)^{2H-1} - rP_1(t) \\
&= \left(\frac{\partial P_1}{\partial r}/\frac{\partial P_2}{\partial r}\right)\left[\frac{\partial P_2}{\partial t} + [\theta(t) - ar(t)]\,\frac{\partial P_2}{\partial r} + (\sigma_r)^2\frac{1}{2}\frac{\partial^2 P_2}{\partial r^2}(dt)^{2H-1} - rP_2(t)\right].
\end{aligned}
$$

This is one equation in two unknowns. However, the left-hand side is a function of T_1 and the right-hand side is a function of T_2. The only way for this equality to be possible is for both side to be independent of the maturity date. Thus dropping the subscripts of P and introducing the market price of the risk ψ, we have

$$
\frac{\frac{\partial P}{\partial t} + [\theta(t) - ar(t)]\frac{\partial P}{\partial r} + (\sigma_r)^2\frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dt)^{2H-1} - rP(t)}{\frac{\partial P}{\partial r}} = \psi\sigma_r.
$$

Simplifying this, the equation in the theorem can be arrived. □

Theorem 2. *If the number of assets traded during the time interval $[t, dt]$ is v, then*

$$
\begin{aligned}
\mathbb{E}[|v|] = \sqrt{\frac{2}{\pi}}(dt)^H \Bigg(&\left(\frac{\partial^2 V}{\partial X^2}\right)^2 \sigma_X^2 X^2(t) \\
&+ \left(\frac{\partial^2 V}{\partial r \partial X}\right)^2 \sigma_r^2 + 2\rho\sigma_X\sigma_r X(t)\frac{\partial^2 V}{\partial X^2}\frac{\partial^2 V}{\partial r \partial X}\Bigg)^{\frac{1}{2}}
\end{aligned}
$$

where $V(t)$ is the option price, $X(t)$ is the market price of the asset at time t, $r(t)$ is the interest rate that follows the fractional Hull-White model with Hurst parameter H, σ_X is the volatility of the asset price, σ_r is the volatility of the interest rate and ρ is the correlation coefficient between the interest rate and the asset price.

Proof: Suppose v is the number of asset traded during the time interval $[t, dt]$. At the short time t, the number of assets hold is given by

$$
\Delta_1 = \frac{\partial V}{\partial X}(X, r, t). \tag{17}
$$

After the time step dt, the number of assets held is

$$\Delta_{t+dt} = \frac{\partial V}{\partial X}(X + dt, r + dt, t + dt). \tag{18}$$

Since the time step dt is assumed to be so small, we have

$$dX \simeq \sigma_X X(t) dB_1^H(t) \tag{19}$$

and

$$dr \simeq \sigma_r dB_2^H. \tag{20}$$

The number of assets traded v during the time interval $[t, t + dt]$ given by

$$v = \frac{\partial^2 V}{\partial X^2}\sigma_X X(t) dB_1^H(t) + \frac{\partial^2 V}{\partial r \partial X}\sigma_r dB_2^H(t) \tag{21}$$

with mean

$$\mathbb{E}[v] = 0, \tag{22}$$

and variance

$$\begin{aligned}\mathbb{E}[v^2] = &\left(\frac{\partial^2 V}{\partial X^2}\right)^2 \sigma_X^2 X^2(t)(dt)^{2H} + \left(\frac{\partial^2 V}{\partial r \partial X}\right)^2 \sigma_r^2 (dt)^{2H} \\ &+ 2\rho\sigma_X\sigma_r X(t)\frac{\partial^2 V}{\partial X^2}\frac{\partial^2 V}{\partial r \partial X}(dt)^{2H}.\end{aligned}$$

If we let the variance of v to be β^2, then we have $\mathbb{E}[v^2] = \beta^2$. Since v is normally distributed, the probability density function of v is given as

$$f(v) = \frac{1}{\beta\sqrt{2\pi}} e^{\frac{-(v)^2}{2\beta^2}}. \tag{23}$$

Hence, we have

$$\begin{aligned}\mathbb{E}[|v|] &= \int_{-\infty}^{+\infty} |v| f(v) dv \\ &= \int_{-\infty}^{+\infty} |v| \frac{1}{\beta\sqrt{2\pi}} e^{\frac{-(v)^2}{2\beta^2}} dv\end{aligned}$$

By letting $u = \frac{-(v)^2}{2\beta^2}$, we have $dv = \frac{-\beta^2 du}{v}$. Substituting we have

$$\mathbb{E}[|v|] = \sqrt{\frac{2}{\pi}}(dt)^H \times \left[\left(\frac{\partial^2 V}{\partial X^2}\right)^2 \sigma_X^2 X^2(t)\right.$$

$$+\left(\frac{\partial^2 V}{\partial r\partial X}\right)^2\sigma_r^2+2\rho\sigma_X\sigma_r X(t)\frac{\partial^2 V}{\partial X^2}\frac{\partial^2 V}{\partial r\partial X}\Bigg].$$

□

Theorem 3. *Using the Hull-White Interest rate model and with transaction cost in a fractional Brownian Motion, the value of the option is modeled as:*

$$\begin{aligned}\frac{\partial V}{\partial t}&+\frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma_X^2(X(t))^2(dt)^{2H-1}+\frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2(dt)^{2H-1}+\frac{\partial V}{\partial X}rX(t)\\&+\frac{\partial^2 V}{\partial X\partial r}\sigma_r\sigma_X X(t)\rho(t)(dt)^{2H-1}+\frac{\partial V}{\partial r}[\theta(t)-ar(t)-\psi\sigma_r]\\&-rV(t)+cX(t)\sqrt{\frac{2}{\pi}}(dt)^H\times\left[\left(\frac{\partial^2 V}{\partial X^2}\right)^2\sigma_X^2X^2(t)+\left(\frac{\partial^2 V}{\partial r\partial X}\right)^2\sigma_r^2\right.\\&\left.+2\rho\sigma_X\sigma_r X(t)\frac{\partial^2 V}{\partial X^2}\frac{\partial^2 V}{\partial r\partial X}\right]^{\frac{1}{2}}=0.\end{aligned}$$

where $V(t)$ is the option price, $X(t)$ is the market value of the asset at time t, $r(t)$ is the interest rate, σ_X is the volatility of the asset, σ_r is the volatility of the interest rate, c is fixed proportion of the trading amount for the asset agreed by both parties, and H is the Hurst parameter.

Proof: Let $V(t)=V(t,X(t),r(t))$ be the option price. Define the portfolio

$$\Pi(t)=V(t)-\Delta_1X(t)-\Delta_2P(t)$$

where Δ_1,Δ_2 are the respective shares of the asset price $X(t)$ and the zero-coupon bond $P(t)$.

The value of the change of portfolio at time $[t,t+dt]$ with transaction cost is now

$$d\Pi(t)=V(t)-\Delta_1dX(t)-\Delta_2dP(t)+c|v(t)|X(t).$$

Taking the expectation we have

$$\mathbb{E}[d\Pi(t)]=\mathbb{E}\left[V(t)\right]-\mathbb{E}\left[\Delta_1dX(t)\right]-\mathbb{E}\left[\Delta_2dP(t)\right]+cX(t)\mathbb{E}\left[|v(t)|\right].$$

By non-arbitrage principle,

$$\begin{gathered}\mathbb{E}\left[V(t)\right]-\mathbb{E}\left[\Delta_1dX(t)\right]-\mathbb{E}\left[\Delta_2dP(t)\right]-rV(t)dt+\Delta_1rX(t)dt\\+\Delta_2rP(t)dt+cX(t)\mathbb{E}\left[|v(t)|\right]=0\end{gathered}$$

To eliminate the risk, take

$$\Delta_1=\frac{\partial V}{\partial X}\tag{24}$$

and

$$\Delta_2 = \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r}. \tag{25}$$

Hence,

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma_X^2 (X(t))^2 (dt)^{2H-1} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2 (dt)^{2H-1} + \frac{\partial V}{\partial X} r X(t) \\
&\quad + \frac{\partial^2 V}{\partial X \partial r}\sigma_r \sigma_X X(t)\rho(t)(dt)^{2H-1} + \frac{\partial V}{\partial r}[\theta(t) - ar(t) - \psi\sigma_r] \\
&\quad - rV(t) + cX(t)\mathbb{E}\left[|v(t)|\right] = 0.
\end{aligned}$$

Employing the transaction cost, we have can derived the equation in the theorem. □

3. The Model

Theorem 4. *Based on the price model in Theorem 3 under a fractional Brownian motion, the closed form formula for the European call option price is given by*

$$V_C(X, r, t) = XN(d_1) - KP(r, t, T)N(d_2) \tag{26}$$

where

$$\begin{aligned}
d_1 &= d_2 + M \\
d_2 &= \frac{\ln \frac{X}{KP(r,t;T)} - M}{N} \\
\hat{\sigma^2} &= \sigma_X^2 + \sigma_2^2 B^2 + 2\rho\sigma_X\sigma_2 B \\
B &= \frac{1}{P}\frac{\partial P}{\partial r} \\
M &= \sqrt{2\int_t^T \left[\frac{1}{2}(ds)^{2H-1}\hat{\sigma^2} + c(ds)^{H-1}\sqrt{\frac{2}{\pi}}\tilde{\sigma}\right] ds} \\
N &= \sqrt{2\int_t^T \left[\frac{1}{2}(ds)^{2H-1}\tilde{\sigma}^2 + c(ds)^{H-1}\sqrt{\frac{2}{\pi}}\tilde{\sigma}\right] ds}
\end{aligned}$$

Proof: Theorem 3 can be solved by the transformation of independent variables

$$y = \frac{X}{P(r, t; T)} \tag{27}$$

and a new unknown function denoted as

$$\hat{V}(y, t) = \frac{V(X, r, t)}{P(r, t; T)}. \tag{28}$$

We have the following computations,

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \hat{V}\frac{\partial P}{\partial t} + P\frac{\partial \hat{V}}{\partial t} - y\frac{\partial P}{\partial t}\frac{\partial \hat{V}}{\partial y},\\
\frac{\partial V}{\partial r} &= \hat{V}\frac{\partial P}{\partial r} - y\frac{\partial P}{\partial r}\frac{\partial \hat{V}}{\partial r},\\
\frac{\partial V}{\partial X} &= \frac{\partial \hat{V}}{\partial y},\\
\frac{\partial^2 V}{\partial r^2} &= \hat{V}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial \hat{V}}{\partial y}\frac{\partial^2 P}{\partial r^2} - y^2\frac{\partial^2 \hat{V}}{\partial y^2}\frac{1}{P}\left(\frac{\partial P}{\partial r}\right)^2,\\
\frac{\partial^2 V}{\partial r \partial X} &= -y\frac{\partial^2 \hat{V}}{\partial y^2}\frac{1}{P}\frac{\partial P}{\partial r},\\
\frac{\partial^2 V}{\partial X^2} &= \frac{1}{P}\frac{\partial^2 \hat{V}}{\partial y^2}.
\end{aligned}
$$

Substituting these equations gives,

$$
\begin{aligned}
\frac{\partial \hat{V}}{\partial t} &+ \left[\frac{1}{2}y^2(dt)^{2H-1}\frac{\partial^2 \hat{V}}{\partial y^2}\right]\left[\sigma_X^2 + \frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2(\sigma_r)^2 - \frac{2}{P}\frac{\partial P}{\partial r}\sigma_r\sigma_X\rho(t)\right]\\
&+ cy^2\sqrt{\frac{2}{\pi}}(dt)^{H-1}\left|\frac{\partial^2 \hat{V}}{\partial y^2}\right| \times \left[\sigma_X^2 + \frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2\sigma_r^2\right.\\
&\left. -2\rho\sigma_X\sigma_r\left(\frac{1}{P}\right)\left(\frac{\partial P}{\partial r}\right)\right]^{1/2} = 0
\end{aligned}
$$

Taking another transformation by letting

$$z = \ln y.$$

Hence we have,

$$\frac{\partial \hat{V}}{\partial y} = \frac{\partial \hat{V}}{\partial z}\frac{1}{y} \tag{29}$$

and

$$\frac{\partial^2 \hat{V}}{\partial y^2} = \left(\frac{1}{y}\right)^2\left(\frac{\partial^2 \hat{V}}{\partial z^2} - \frac{\partial \hat{V}}{\partial z}\right).$$

Furthermore, let $B = \frac{1}{P}\frac{\partial P}{\partial r}$ and $\tilde{\sigma}^2 = \sigma_X^2 + \sigma_r^2 B^2 + 2\rho\sigma_X\sigma_r B$.

Therefore, we have

$$\frac{\partial \hat{V}}{\partial t} + \left[\frac{1}{2}(dt)^{2H-1}\tilde{\sigma}^2 + c(dt)^{H-1}\sqrt{\frac{2}{\pi}}\right] \times \left(\frac{\partial^2 \hat{V}}{\partial z^2} - \frac{\partial \hat{V}}{\partial z}\right) = 0$$

Finally we take the following transformation. Let $\hat{V}(z,t) = \mu(\eta,\tau)$, where $\eta = z + \alpha(t)$ and $\tau = \gamma(t)$. At the expiry date T of the contract, $\alpha(T) = \gamma(t) = 0$. So we have the following calculations,

$$\frac{\partial \hat{V}}{\partial t} = \frac{\partial \mu}{\partial \eta}\alpha'(t) + \frac{\partial \mu}{\partial \tau}\gamma'(t) \tag{30}$$

and

$$\frac{\partial \hat{V}}{\partial z} = \frac{\partial \mu}{\partial \eta}, \tag{31}$$

$$\frac{\partial^2 \hat{V}}{\partial z^2} = \frac{\partial^2 \mu}{\partial \eta^2}. \tag{32}$$

We let

$$\alpha'(t) = \frac{1}{2}(dt)^{2H-1}\tilde{\sigma}^2 + c(dt)^{H-1}\sqrt{(2/\pi)}\tilde{\sigma}$$

and

$$\gamma'(t) = -\left[\frac{1}{2}(dt)^{2H-1}\tilde{\sigma}^2 + c(dt)^{H-1}\sqrt{(2/\pi)}\tilde{\sigma}\right].$$

Substituting these, we have

$$\frac{\partial \mu}{\partial \eta}\alpha'(t) + \frac{\partial \mu}{\partial \tau}(-\alpha'(t)) + \alpha'(t)\left[\frac{\partial^2 \mu}{\partial \eta^2} - \frac{\partial \mu}{\partial \eta}\right] = 0.$$

Hence, this can be reduced further to

$$\frac{\partial \mu}{\partial \tau} - \frac{\partial^2 \mu}{\partial \eta^2} = 0 \tag{33}$$

with the initial condition

$$\mu(\eta, T) = (e^{\eta} - K)^{+}. \tag{34}$$

The solution for this equation is

$$\mu(\eta,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{+\infty} \mu_0(\xi) e^{-(\eta-\xi)^2/4\tau} d\xi \tag{35}$$

Now for $e^{\xi} - K \geq 0$,

$$\xi \geq \ln K. \tag{36}$$

Hence, the integration domain can be equivalently $[-\infty, \ln K]$. Hence, we can write the solution as

$$\begin{aligned}\mu(\eta,\tau) &= \frac{1}{2\sqrt{\pi\tau}}\int_{\ln K}^{+\infty} (e^{\xi} - K) e^{-(\eta-\xi)^2/4\tau} d\xi \\ &= \frac{1}{2\sqrt{\pi\tau}}\int_{\ln K}^{+\infty} e^{\xi} e^{-(\eta-\xi)^2/4\tau} d\xi - \frac{1}{2\sqrt{\pi\tau}}\int_{\ln K}^{+\infty} K e^{-(\eta-\xi)^2/4\tau} d\xi\end{aligned}$$

Let $\xi = \eta + \sqrt{2\tau}\omega$, then we can have

$$\omega = \frac{\xi - \eta}{\sqrt{2\tau}} \tag{37}$$

and

$$d\xi = \sqrt{2\tau} d\omega. \tag{38}$$

Therefore,

$$\mu(\eta, \tau) = \frac{1}{\sqrt{2\pi}} e^{\eta+\tau} \int_{-\infty}^{\frac{\eta - \ln K + 2\tau}{\sqrt{2\tau}}} e^{-\frac{\zeta^2}{2}} d\zeta - \frac{1}{\sqrt{2\pi}} K \int_{-\infty}^{\frac{\eta - \ln K}{\sqrt{2\tau}}} e^{-\omega^2/2} d\omega.$$

This can be written as

$$\mu(\eta, \tau) = e^{\eta+\tau} N(d_1) - KN(d_2) \tag{39}$$

where

$$d_1 = \frac{\eta - \ln K + 2\tau}{\sqrt{2\tau}} \tag{40}$$

$$d_2 = \frac{\eta - \ln K}{\sqrt{2\tau}} \tag{41}$$

and

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{\frac{-1}{2}\zeta^2} d\zeta \tag{42}$$

$$N(d_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{\frac{-1}{2}\omega^2} d\omega. \tag{43}$$

By inverse change of variables, the formula in the theorem can be derived. □

Corollary 5. *Without the transaction cost and under a fractional Brownian motion, the closed form formula for the European call option price $V_C(X, r, t)$ is given by*

$$V_C(X, r, t) = X(t)N(d_1) - KP(r, t, T)N(d_2) \tag{44}$$

where

$$\begin{aligned}
d_1 &= d_2 + \sqrt{\tilde{\sigma}^2 \int_t^T (ds)^{2H}}, \\
d_2 &= \frac{\ln \frac{X}{KP(r,t;T)} - \frac{1}{2}\tilde{\sigma}^2 \int_t^T (ds)^{2H}}{\sqrt{\tilde{\sigma}^2 \int_t^T (ds)^{2H}}}, \\
\tilde{\sigma^2} &= \sigma_X^2 + \sigma_2^2 B^2 + 2\rho\sigma_X\sigma_2 B, \\
B &= \frac{1}{P}\frac{\partial P}{\partial r},
\end{aligned}$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{\frac{-1}{2}y^2} dy.$$

4. Conclusion and Recommendations

This paper presents an extension of the BSM model for European call option introduced by Black, Scholes and Merton by considering a stochastic rate under a fractional Brownian Motion instead of a constant rate and adding a transaction cost. A closed-form formula for European call option is derived under risk-neutral measure using replication techniques. Also, the fractional Brownian motion employs the flexible dependence of the increments of the evolution of the prices making it more closer to the real world scenario. For further studies, a similar formula can be derived also that employs stochastic volatilities for both asset value and interest rate. Moreover, this pricing can be extended further by considering geometric fractional Brownian Motion.

References

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