EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 2, 2024, 931-944
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Volterra-Composition Operators Acting on S^p Spaces and Weighted Zygmund Spaces

Waleed Al-Rawashdeh

Department of Mathematics, Zarqa University, 2000 Zarqa, 13110 Jordan

Abstract. Let φ be an analytic selfmap of the open unit disk $\mathbb{D}$ and g be an analytic function on $\mathbb{D}$. The Volterra-type composition operators induced by the maps g and φ are defined as

$$\left(I_g^{\varphi} f\right)(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta \text{ and } \left(T_g^{\varphi} f\right)(z) = \int_0^z f(\varphi(\zeta))g'(\zeta)d\zeta.$$

For $1 \leq p < \infty$, $S^p(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$ whose first derivative f' lies in the Hardy space $H^p(\mathbb{D})$, endowed with the norm $\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}$. Let $\mu : (0, 1] \to (0, \infty)$ be a positive continuous function on $\mathbb{D}$ such that for $z \in \mathbb{D}$ we define $\mu(z) = \mu(|z|)$. The weighted Zygmund space $\mathcal{Z}_\mu(\mathbb{D})$ is the space of all analytic functions f on $\mathbb{D}$ such that $\sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$ is finite. In this paper, we characterize the boundedness and compactness of the Volterra-type composition operators that act between S^p spaces and weighted Zygmund spaces.

2020 Mathematics Subject Classifications: 47B33, 47B38, 30H10, 30H20, 47B37, 30H05, 32C15

Key Words and Phrases: Weighted Zygmund Spaces, S^p spaces, Volterra operators, composition operators, bounded operators, compact operators

1. Introduction

Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{D} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$. For $1 \leq p < \infty$, the analytic Hardy space $H^p(\mathbb{D}$ on the unit disk $\mathbb{D}$ is the Banach space of all analytic functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where σ is the normalized Lebesgue measure on the boundary of the unit disk. For f belongs to $H^p(\mathbb{D})$, it is well known from Fatou's theorem that the radial limit

$$f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$$

DOI: https://doi.org/10.29020/nybg.ejpam.v17i2.5113

Email address: walrawashdeh@zu.edu.jo (W. Al-Rawashdeh)

exists for almost all ζ on $\partial\mathbb{D}$. Moreover,

$$\|f\|_{H^p}^p = \int_{\partial\mathbb{D}} |f^*(\zeta)|^p d\sigma(\zeta),$$

for all finite values of p. We often use a standard synonym of notation of writing f instead of f^*. For $1 \le p < \infty$, $S^p(\mathbb{D})$ is the space of all analytic functions on the unit disk $\mathbb{D}$ whose first derivative f' lies in the Hardy space $H^p(\mathbb{D})$ endowed with the norm

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

One can easily show that S^p is a Banach space with respect to this norm. It is well known that S^p is a Banach algebra when the norm of $f \in S^p$ is defined by $\|f\|_\infty + \|f'\|_{H^p}$. For more information about these spaces we refer the readers to the monograph [5], the papers ([1], [2]), and the references therein.

Let $\mu : (0, 1] \to (0, \infty)$ be a positive continuous function on $\mathbb{D}$ such that for $z \in \mathbb{D}$ we define $\mu(z) = \mu(|z|)$. The weighted Zygmund space $\mathcal{Z}_\mu$ is the space of all analytic functions f on the open unit disk $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

It is a Banach space with the norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|.$$

When $\mu(z) = 1 - |z|^2$, this space is known as the classical Zygmund space $\mathcal{Z}$. From Zygmund's theorem ([6], Theorem 5.3),we know that $f \in \mathcal{Z}$ if and only if f is continuous on $\overline{\mathbb{D}}$ and

$$\|f\| = \sup \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$.

Let g be an analytic function on $\mathbb{D}$, the Volterra type operator (see [18]) is defined as

$$(T_g f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta,$$

where $f \in \mathcal{H}(\mathbb{D})$ and $z \in \mathbb{D}$. Note that T_g can be viewed as a generalization of the cesâro operator whose first studied by Aleman and Siskkis [4].

It is natural to define another Volterra type operator I_g as follows

$$(I_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta.$$

Recently, many researchers considered these operators and characterized their boundedness and compactness between various spaces of analytic functions, for example see ([3], [8], [10], [11], [12], [13], [14], [16], [20]) and the references therein.

Let φ be an analytic function maps $\mathbb{D}$ into itself, the composition operator induced by φ is defined on the space $\mathcal{H}(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ by

$$C_\varphi f(z) = f(\varphi(z)),$$

for all $f \in \mathcal{H}(\mathbb{D})$ and $z \in \mathbb{D}$. It is well known that the composition operator $C_\varphi f = f \circ \varphi$ defines a linear operator C_φ which acts boundedly on various spaces of analytic or harmonic functions on $\mathbb{D}$. These operators have been studied on many spaces of analytic functions. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of C_φ in terms of the function-theoretic properties of the induced map φ. We refer the reader to the monographs ([5], [7], [9], [15], [17], [22], [23]) and the references therein.

Let g be a fixed analytic function on $\mathbb{D}$, f an analytic function of $\mathbb{D}$ and $z \in \mathbb{D}$. The Volterra type composition operators are defined as

$$\left(T_g^\varphi f\right)(z) = \int_0^z f(\varphi(\zeta))g'(\zeta)d\zeta,$$

$$\left(I_g^\varphi f\right)(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta.$$

The classical Volterra operators are obtained in the case when $\varphi(z) = z$. These operators have been studied by many researchers, for example see ([3], [11], [12], [19], [21], [24], [25]) and the references therein.

In this paper, we are investigating the boundedness and compactness of the Volterra type composition operators T_g^φ and I_g^φ acting between $S^p(\mathbb{D})$ spaces and weighted Zygmund spaces $\mathcal{Z}_\mu$.

2. Preliminaries

In this section we present some well known, but useful, information that are curial for the main results of this paper. The following lemma is a well known fact that can be proven by using Cauchy estimates, so we omit the proof.

Lemma 1. *If $\{f_n\}$ is a sequence converges to zero on compact subsets of $\mathbb{D}$, then $\{f_n'\}$ also converges to zero on compact subsets of $\mathbb{D}$ as $n \to \infty$. In particular if K is a compact subset of $\mathbb{D}$, then $\lim_{n\to\infty} \sup_{w\in K} |f'(w)| = 0$.*

The following lemma is a know fact, for the readers who are interested in its proof we refer them to (Theorem 1, [16]).

Lemma 2. *If $1 \leq p < \infty$, the following are true:*

(i) $S^p(\mathbb{D}) \subset S^1(\mathbb{D}) \subset H^\infty$;

(ii) $\|f\|_\infty \leq \pi \|f\|_{S^1(\mathbb{D})} \leq \pi \|f\|_{S^p(\mathbb{D})}$;

(iii) $S^p(\mathbb{D})$ *is a Banach algebra;*

(iv) polynomials are dense in $S^p(\mathbb{D})$.

3. Boundedness and Compactness of I_g^φ

In this section, we characterize the boundedness and compactness of the operator I_g^φ acting between S^p spaces and weighted Zygmund spaces $\mathcal{Z}_\mu$. The results will be written in terms of

$$K_1(z) = \frac{\mu(z)|g'(z)|}{(1 - |\varphi(z)|^2)^{1/p}},$$

and

$$K_2(z) = \frac{\mu(z)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1+p)/p}},$$

where $z \in \mathbb{D}$, $g \in \mathcal{H}(\mathbb{D})$, and φ is the analytic selfmap of $\mathbb{D}$.

In the following Theorem 1, we characterize the boundedness of I_g^φ that acts between S^p spaces and weighted Zygmund spaces.

Theorem 1. *Let g be an analytic function on $\mathbb{D}$ and φ be an analytic selfmap of $\mathbb{D}$. Then $I_g^\varphi : S^p \to \mathcal{Z}_\mu$ is bounded if and only if*

$$M_1 = \sup_{z \in \mathbb{D}} K_1(z) < \infty \quad \textit{and} \quad M_2 = \sup_{z \in \mathbb{D}} K_2(z) < \infty.$$

Proof. Suppose that $I_g^\varphi : S^p \to \mathcal{Z}_\mu$ is bounded.

First, for a fixed $w \in \mathbb{D}$ we consider the test function

$$f_{1,w}(z) = \frac{\left(1 - |\varphi(w)|^2\right)^{(2p-1)/p}}{\overline{\varphi(w)}(1 - \overline{\varphi(w)}z)}.$$

By direct calculations, we get

$$\|f_{1,w}\|_{S^p} = \|f'_{1,w}\|_{H^p} \leq 2^{(2p-2)/p}, \tag{1}$$

$$f'_{1,w}(\varphi(w)) = \frac{1}{(1 - |\varphi(w)|^2)^{1/p}}, \tag{2}$$

and

$$f''_{1,w}(\varphi(w)) = \frac{2\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{(p+1)/p}}. \tag{3}$$

Therefore, we obtain the following

$$\begin{aligned}\left(I_g^{\varphi} f_{1,w}\right)''(w) &= \left(f'_{1,w}(\varphi(w))g(w)\right)' \\ &= f''_{1,w}(\varphi(w))g(w)\varphi'(w) + f'_{1,w}(\varphi(w))g'(w) \\ &= \frac{2\overline{\varphi(w)}\varphi'(w)g(w)}{(1-|\varphi(w)|^2)^{(p+1)/p}} + \frac{g'(w)}{(1-|\varphi(w)|^2)^{1/p}} \end{aligned} \tag{4}$$

Moreover, we consider another test function

$$f_{2,w}(z) = \frac{\left(1-|\varphi(w)|^2\right)^{(3p-1)/p}}{2\overline{\varphi(w)}(1-\overline{\varphi(w)}z)^2}.$$

By direct calculations, we get

$$\|f_{2,w}\|_{S^p} = \|f'_{2,w}\|_{H^p} \le 2^{(3p-2)/p},$$

$$f'_{2,w}(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{1/p}},$$

and

$$f''_{2,w}(\varphi(w)) = \frac{3\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{(p+1)/p}}.$$

Similarly, we obtain the following

$$\left(I_g^{\varphi} f_{2,w}\right)''(w) = \frac{3\overline{\varphi(w)}\varphi'(w)g(w)}{(1-|\varphi(w)|^2)^{(p+1)/p}} + \frac{g'(w)}{(1-|\varphi(w)|^2)^{1/p}} \tag{5}$$

Now, using equations (4) and (5), we get

$$\left(I_g^{\varphi} f_{2,w}\right)''(w) - \left(I_g^{\varphi} f_{1,w}\right)''(w) = \frac{\overline{\varphi(w)}\varphi'(w)g(w)}{(1-|\varphi(w)|^2)^{(p+1)/p}}.$$

Hence, by the boundedness of $I_g^{\varphi} : S^p \to \mathcal{Z}_\mu$ we have

$$\begin{aligned}\frac{\mu(w)\left|\overline{\varphi(w)}\varphi'(w)g(w)\right|}{(1-|\varphi(w)|^2)^{(p+1)/p}} &\le \|I_g^{\varphi}\|\|f_{2,w}\|_{S^p} + \|I_g^{\varphi}\|\|f_{1,w}\|_{S^p} \\ &\le \|I_g^{\varphi}\|\left(2^{(3p-2)/p} + 2^{(2p-2)/p}\right) \\ &\le C_1 \end{aligned} \tag{6}$$

On the other hand, Lemma 2 tells us that polynomials are dense in S^p spaces. Thus $P_n(z) = z^n$ in S^p and we get the following

$$\begin{aligned}\left(I_g^{\varphi} P_1\right)''(z) &= \left(\int_0^z P_1'(\varphi(w))g(w)dw\right)'' \\ &= \left(\int_0^z g(w)dw\right)'' \\ &= g'(z).\end{aligned}$$

Similarly, we obtain the following

$$\begin{aligned}\left(I_g^{\varphi} P_2\right)''(z) &= \left(\int_0^z P_2'(\varphi(w))g(w)dw\right)'' \\ &= \left(\int_0^z 2\varphi(w)g(w)dw\right)'' \\ &= 2\varphi(z)g'(z) + 2\varphi'(z)g(z).\end{aligned}$$

Hence, using the previous equations, we get

$$2\left(\varphi' g\right)(z) = \left(I_g^{\varphi} P_2\right)''(z) - 2\varphi(z)\left(I_g^{\varphi} P_1\right)''(z). \tag{7}$$

Therefore, using equation (7) we get

$$\begin{aligned}\sup_{w\in\mathbb{D}} \mu(w)|\varphi'(w)g(w)| &\leq \frac{1}{2}\|I_g^{\varphi} P_2\|_{\mathcal{Z}_\mu} + \sup_{w\in\mathbb{D}}\left(\|I_g^{\varphi} P_1\|_{\mathcal{Z}_\mu}\sup_{w\in\mathbb{D}}|\varphi(w)|\right) \\ &\leq \|I_g^{\varphi}\|\|P_2\|_{S^P} + \|I_g^{\varphi}\|\|P_1\|_{S^P} \\ &\leq C_2\end{aligned}$$

Now, for a fixed $0 < r < 1$, consider $w \in \mathbb{D}$ such that $0 \leq |\varphi(w)| \leq r < 1$. Then we get

$$\begin{aligned}\frac{\mu(w)|\overline{\varphi(w)}\varphi'(w)g(w)|}{(1-|\varphi(w)|^2)^{(p+1)/p}} &\leq \frac{\mu(w)|\varphi'(w)g(w)|}{(1-|\varphi(w)|^2)^{(p+1)/p}} \\ &\leq \frac{C_2}{(1-r^2)^{1+1/p}}\end{aligned} \tag{8}$$

Moreover, consider $w \in \mathbb{D}$ such that $r < |\varphi(w)| < 1$. Then we get

$$\begin{aligned}\frac{\mu(w)|r\varphi'(w)g(w)|}{(1-|\varphi(w)|^2)^{(p+1)/p}} &\leq \frac{\mu(w)|\overline{\varphi(w)}\varphi'(w)g(w)|}{(1-|\varphi(w)|^2)^{(p+1)/p}} \\ &\leq \|I_g^{\varphi}\|\left(2^{(3p-2)/p} + 2^{(2p-2)/p}\right).\end{aligned}$$

Hence, using equation (6), we get

$$\frac{\mu(w)|\varphi'(w)g(w)|}{(1-|\varphi(w)|^2)^{(p+1)/p}} \leq \frac{C_1}{r}. \tag{9}$$

Therefore, using inequalities (8) and (9), we get

$$\begin{aligned} M_2 &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} \\ &\leq \max\left\{\frac{C_1}{r} + \frac{C_2}{(1-r^2)^{1+1/p}}\right\} \\ &< \infty. \end{aligned}$$

Second, for a fixed $w \in \mathbb{D}$, using equations (2), (3) and (4) we get

$$\begin{aligned} \frac{\mu(w)|g'(w)|}{(1-|\varphi(w)|^2)^{1/p}} &= \mu(w)\left|g'(w)f'_{1,w}(\varphi(w)) + f''_{1,w}(\varphi(w))g(w)\varphi'(w) - f''_{1,w}(\varphi(w))g(w)\varphi'(w)\right| \\ &= \mu(w)\left|\left(I_g^{\varphi} f_{1,w}\right)''(w) - \frac{2\overline{\varphi(w)}g(w)\varphi'(w)}{(1-|\varphi(w)|^2)^{(p+1)/p}}\right| \\ &\leq \|I_g^{\varphi} f_{1,w}\|_{\mathcal{Z}_\mu} + \frac{2\mu(w)|\overline{\varphi(w)}g(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{(p+1)/p}} \\ &\leq \|I_g^{\varphi}\|\|f_{1,w}\|_{S^p} + 2M_2. \end{aligned}$$

Taking the supremum over all $w \in \mathbb{D}$, we get that $M_1 < \infty$.

Conversely, Suppose that conditions M_1 and M_2 are finite. Let $f \in S^p$, then it is well known, see [6] or [23], that for all $z \in \mathbb{D}$ we have

$$|f'(z)| \leq \frac{\|f'\|_{H^p}}{(1-|z|^2)^{1/p}},$$

and

$$|f''(z)| \leq \frac{\|f'\|_{H^p}}{(1-|z|^2)^{1+1/p}}.$$

Therefore, for $z \in \mathbb{D}$, we have

$$\begin{aligned} \mu(z)\left|\left(I_g^{\varphi} f\right)''(z)\right| &= \mu(z)\left|\left(\int_0^z f'(\varphi(w))g(w)dw\right)''\right| \\ &= \left(f'(\varphi(z))g(z)\right)' \\ &= \mu(z)\left|f''(\varphi(z))\varphi'(z)g(z) + f'(\varphi(z))g'(z)\right| \\ &\leq \frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}}\|f'\|_{H^p} + \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{1/p}}\|f'\|_{H^p} \end{aligned}$$

$$\leq (M_1 + M_2)\left(\|f\|_{S^p} - |f(0)|\right)$$
$$\leq (M_1 + M_2)\|f\|_{S^p}.$$

Taking the supremum over all $z \in \mathbb{D}$, we get

$$\|\left(I_g^{\varphi} f\right)(z)\|_{\mathcal{Z}_\mu} \leq (M_1 + M_2)\|f\|_{S^p}.$$

Hence, I_g^{φ} is bounded, as desired.

In the following Theorem 2, we characterize the compactness of I_g^{φ} that acts between S^p spaces and weighted Zygmund spaces.

Theorem 2. *Let g be an analytic function on $\mathbb{D}$, φ be an analytic selfmap of $\mathbb{D}$ and $I_g^{\varphi} : S^p \to \mathcal{Z}_\mu$ be bounded. Then I_g^{φ} is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} K_1(z) = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1} K_2(z) = 0. \tag{10}$$

Proof. Suppose I_g^{φ} is compact. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in the open unit disk $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, consider the test functions $f_{1,w}$ and $f_{2,w}$ we used in the proof of Theorem 1 with $w = z_n$.

Then, we get

$$f_{1,z_n}(\varphi(z_n)) = \varphi(z_n)\left(1 - |\varphi(z_n)|^2\right)^{1-1/p},$$

and

$$f_{2,z_n}(\varphi(z_n)) = \frac{\varphi(z_n)}{2}\left(1 - |\varphi(z_n)|^2\right)^{1-1/p}\left(2 - |\varphi(z_n)|^2\right).$$

Hence, the sequences $\{f_{1,z_n}\}$ and $\{f_{2,z_n}\}$ converge to zero uniformly on $\overline{\mathbb{D}}$. Then, by the compactness of I_g^{φ}, we get

$$\lim_{n\to\infty} \|I_g^{\varphi} f_{1,z_n}\|_{\mathcal{Z}_\mu} = 0 \text{ and } \lim_{n\to\infty} \|I_g^{\varphi} f_{2,z_n}\|_{\mathcal{Z}_\mu} = 0. \tag{11}$$

Now, following similar argument as in the proof of Theorem 1, we get

$$\frac{\mu(z_n)\left|\overline{\varphi(z_n)}\varphi'(z_n)g(z_n)\right|}{(1 - |\varphi(z_n)|^2)^{(p+1)/p}} \leq \|I_g^{\varphi} f_{1,w}\|_{\mathcal{Z}_\mu} + \|I_g^{\varphi} f_{2,w}\|_{\mathcal{Z}_\mu}. \tag{12}$$

Hence, using equations (11) and (12), we get

$$\lim_{n\to\infty} \frac{\mu(z_n)\,|\varphi'(z_n)g(z_n)|}{(1 - |\varphi(z_n)|^2)^{(p+1)/p}} = 0. \tag{13}$$

Moreover, following similar argument as in the proof of Theorem 1, we get

$$\frac{\mu(z_n)\,|g'(z_n)|}{(1 - |\varphi(z_n)|^2)^{1/p}} \leq \|I_g^{\varphi} f_{1,w}\|_{\mathcal{Z}_\mu}\| + \frac{2\mu(z_n)\,|\varphi'(z_n)g(z_n)|}{(1 - |\varphi(z_n)|^2)^{(p+1)/p}}. \tag{14}$$

Hence, using equations (11) and (14), we get

$$\lim_{n\to\infty} \frac{\mu(z_n)\,|g'(z_n)|}{(1-|\varphi(z_n)|^2)^{1/p}} = 0. \tag{15}$$

Therefore, equations (13) and (15) give us the desired conditions

$$\lim_{|\varphi(z)|\to 1} K_1(z) = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1} K_2(z) = 0.$$

Conversely, suppose conditions (10) hold. Then for $\epsilon > 0$, there is $\delta \in (0,1)$ such that $K_1(z) < \epsilon$ and $K_2(z) < \epsilon$ whenever $\delta < |\varphi(z)| < 1$.

Let $\{f_n\}$ be a bounded sequence in S^p such that $\sup_{n\in\mathbb{N}} \|f_n\|_{S^p} < L$ and $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $U = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$.

Now, it is clear that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu(z) \left| \left(I_g^{\varphi} f_n\right)''(z) \right| \\
&\le \sup_{z\in\mathbb{D}\setminus U} \mu(z) \left| \left(I_g^{\varphi} f_n\right)''(z) \right| + \sup_{z\in U} \mu(z) \left| \left(I_g^{\varphi} f_n\right)''(z) \right|
\end{aligned}$$

First, we consider the case $|\varphi(z)| > \delta$ then we have

$$\begin{aligned}
\mu(z) \left| \left(I_g^{\varphi} f_n\right)''(z) \right| &= \mu(z) \left| f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z) \right| \\
&\le \frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^{(p+1)/p}} \|f_n'\|_{H^p} + \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{1/p}} \|f_n'\|_{H^p} \\
&\le (K_2(z) + K_1(z)) \|f_n\|_{S^p} \\
&< 2L\epsilon. \qquad (16)
\end{aligned}$$

Second, we consider the case $|\varphi(z)| \le \delta$. Since I_g^{φ} is bounded and polynomials are dense in $S^p(\mathbb{D})$, by taking $f(z) = z$ we get

$$\sup_{z\in\mathbb{D}} \mu(z) \left| \left(I_g^{\varphi} f\right)''(z) \right| = \sup_{z\in\mathbb{D}} \mu(z)|g'(z)| < \infty, \tag{17}$$

and by taking $f(z) = z^2$ we get

$$\sup_{z\in\mathbb{D}} \mu(z) \left| \left(I_g^{\varphi} f\right)''(z) \right| = 2\sup_{z\in\mathbb{D}} \mu(z)|\varphi(z)g'(z) + \varphi'(z)g(z)| < \infty. \tag{18}$$

Using equations (17) and (18), and the boundedness of $\varphi(z)$ we get

$$C_1 = \sup_{z\in\mathbb{D}} \mu(z)|g'(z)| < \infty,$$

and

$$C_2 = \sup_{z\in\mathbb{D}} \mu(z)|\varphi'(z)g(z)| < \infty$$

Hence, for $|\varphi(z)| \leq \delta$, using these facts we get

$$\mu(z)\left|\left(I_g^{\varphi} f_n\right)''(z)\right| \leq C_2\left|f_n''(\varphi(z))\right| + C_1\left|f_n'(\varphi(z))\right|. \tag{19}$$

Since $\{f_n\}$ is bounded in S^p and converges to zero on $\{w \in \mathbb{D} : |w| \leq \delta\}$, so do the sequences $\{f_n'\}$ and $\{f_n''\}$ by Cauchy's estimate.

Thus, there exists $N \in \mathbb{D}$ such that for all $n \geq N$ and $w \in \mathbb{D}$ with $|w| \leq \delta$ we have

$$|f_n'(w)| < \epsilon \qquad \text{and} \qquad |f_n''(w)| < \epsilon$$

Hence, using inequality (19), we get

$$\begin{aligned}\sup_{z\in\mathbb{D}} \mu(z)\left|\left(I_g^{\varphi} f_n\right)''(z)\right| &\leq C_2 \sup_{|w|<\delta} |f_n''(w)| + C_1 \sup_{|w|<\delta} |f_n'(w)|. \\ &< (C_1 + C_2)\epsilon. \end{aligned} \tag{20}$$

Now, using inequalities (16) and (20), we get

$$\begin{aligned}\|I_g^{\varphi} f_n\|_{\mathcal{Z}_\mu} &= |f_n'(\varphi(0))g(0)| + \sup_{z\in\mathbb{D}} \mu\left|\left(I_g^{\varphi} f_n\right)''(z)\right| \\ &\leq |f_n'(\varphi(0))g(0)| + 2L\epsilon + (C_1 + C_2)\epsilon\end{aligned}$$

Since $\{f_n'\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, it converges pointwise. Thus, $|f_n'(\varphi(0))g(0)| \to 0$ as $n \to 0$.

Hence, for arbitrary $\epsilon > 0$, we get $\|I_g^{\varphi} f_n\|_{\mathcal{Z}_\mu} \to 0$ as $n \to 0$. Therefore, I_g^{φ} is compact, which completes the proof.

4. Boundedness and Compactness of T_g^{φ}

In this section, we characterize the boundedness and compactness of the operator T_g^{φ} acting between S^p spaces and weighted Zygmund spaces $\mathcal{Z}_\mu$. In the following Theorem 3, we characterize the boundedness of T_g^{φ} that acts between S^p spaces and weighted Zygmund spaces.

Theorem 3. *Let g be an analytic function on $\mathbb{D}$ and φ be an analytic selfmap of $\mathbb{D}$. Then $T_g^{\varphi} : S^p \to \mathcal{Z}_\mu$ is bounded if and only if $g \in \mathcal{Z}_\mu$ and*

$$M_3 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1/p}} < \infty.$$

Proof. Suppose $T_g^\varphi : S^p \to \mathcal{Z}_\mu$ is bounded. Since polynomials are dense in S^p, $P_1(z) = 1 \in S^p$. By the boundedness of T_g^φ, we get $\|T_g^\varphi P_1\|_{\mathcal{Z}_\mu} < \infty$.

Therefore,

$$\begin{aligned}\sup_{z\in\mathbb{D}} \mu|g''(z)| &= \sup_{z\in\mathbb{D}} \mu \left| \left(T_g^\varphi P_1\right)'' \right| \\ &\le \|T_g^\varphi P_1\|_{\mathcal{Z}_\mu} \\ &< \infty,\end{aligned}$$

which gives us that $g \in \mathcal{Z}_\mu$.

Second, consider the test function $f_{1,w}$ we defined in the proof of Theorem 1. Then,

$$\begin{aligned}\left(T_g^\varphi f_{1,w}\right)''(w) &= \left(f_{1,w}(\varphi(w))g'(w)\right)' \\ &= f_{1,w}(\varphi(w))g''(w) + f'_{1,w}(\varphi(w))\varphi'(w)g'(w) \\ &= f_{1,w}(\varphi(w))g''(w) + \frac{\varphi'(w)g'(w)}{(1-|\varphi(w)|^2)^{1/p}}.\end{aligned}$$

Therefore, by the boundedness of T_g^φ and equation (1), we get

$$\begin{aligned}\frac{\mu(w)|\varphi'(w)g'(w)|}{(1-|\varphi(w)|^2)^{1/p}} &\le \mu(w)\left|\left(T_g^\varphi f_{1,w}\right)''(w)\right| + \mu(w)|g''(w)||f_{1,w}(\varphi(w))| \\ &\le \|T_g^\varphi f_{1,w}\|_{\mathcal{Z}_\mu} + \|g\|_{\mathcal{Z}_\mu}\|f_{1,w}\|_{S^p} \\ &\le \|T_g^\varphi\|\|f_{1,w}\|_{S^p} + \|g\|_{\mathcal{Z}_\mu}\|f_{1,w}\|_{S^p} \\ &\le \left(\|T_g^\varphi\| + \|g\|_{\mathcal{Z}_\mu}\right) 2^{(2p-2)/p}.\end{aligned}$$

Taking the supremum over all $w \in \mathbb{D}$, we get

$$\sup_{w\in\mathbb{D}} \frac{\mu(w)|\varphi'(w)g'(w)|}{(1-|\varphi(w)|^2)^{1/p}} < \infty.$$

Conversely, suppose $g \in \mathcal{Z}_\mu$ and condition M_3 is finite.

Let $f \in S^p$ and $z \in \mathbb{D}$. Then, by using Lemma 2, we get

$$\begin{aligned}\mu(z)\left|\left(T_g^\varphi f\right)''(z)\right| &= \mu(z)\left|\left(f(\varphi(z))g'(z)\right)'\right| \\ &= \mu(z)\left|f(\varphi(z))g''(z) + f'(\varphi(z))\varphi'(z)g'(z)\right| \\ &\le \|g\|_{\mathcal{Z}_\mu}\|f\|_\infty + \frac{\mu(z)\,|\varphi'(z)g'(z)|}{(1-|\varphi(z)|^2)^{1/p}}\|f'\|_{H^p} \\ &\le \pi\|g\|_{\mathcal{Z}_\mu}\|f\|_{S^p} + M_3\|f\|_{S^p}.\end{aligned}$$

Since $g \in \mathcal{Z}_\mu$ and $f \in S^p$, by taking supremum over all $z \in \mathbb{D}$, we get

$$\sup_{z\in\mathbb{D}} \mu(z) \left| \left(T_g^\varphi f\right)'' (z) \right| < C\|f\|_{S^p}, \tag{21}$$

for some constant C.

Finally, using equation (21), we get

$$\begin{aligned} \|T_g^\varphi f\|_{\mathcal{Z}_\mu} &= \left| \left(T_g^\varphi f\right)(0) \right| + \left| \left(T_g^\varphi f\right)'(0) \right| + \sup_{z\in\mathbb{D}} \mu(z) \left| \left(T_g^\varphi f\right)''(z) \right| \\ &\le \left| \left(T_g^\varphi f\right)'(0) \right| + C\|f\|_{S^p} \\ &< (C^* + C)\|f\|_{S^p}, \end{aligned}$$

for some constant C^*. Hence, T_g^φ is bounded, as desired.

The following Theorem 4 characterizes the compactness of $T_g^\varphi : S^p \to \mathcal{Z}_\mu$ whose proof is similar to that of Theorem 2 and Theorem 3. So the details are omitted.

Theorem 4. *Let g be an analytic function on $\mathbb{D}$ and φ be an analytic selfmap of $\mathbb{D}$. Then $T_g^\varphi : S^p \to \mathcal{Z}_\mu$ is compact if and only if $g \in \mathcal{Z}_\mu$ and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1/p}} = 0.$$

Acknowledgements

This research is partially funded by Zarqa University. The author would like to express his sincerest thanks to Zarqa University for the financial support. The author would like to express his sincerest thanks to the referees for their valuable comments and various useful suggestions.

References

[1] W. Al-Rawashdeh. Weighted composition operators between weighted Bergman and S^p spaces. *Bulletin of Math. Anal. Appl.*, 5:54–64, 2013.

[2] W. Al-Rawashdeh. Composition operators between weighted Bergman and S^p spaces. *New Zealand J. Mathematics*, 47:141–150, 2017.

[3] W. Al-Rawashdeh. Generalized composition operators on weighted Hilbert spaces of analytic functions. *Int. J. Adv. Res. in Math.*, 10:1–13, 2017.

[4] A. Aleman and A.G. Siskakkis. An integral operator on H^p. *Complex Variables Theory and Applications*, 28:149–158, 1995.

[5] C. Cowen and B. MacCluer. *Composition Operators on Spaces of Analytic Functions.* CRC press, Boca Raton, 1995.

[6] P. Duren. *Theory of H^p Spaces.* Academic press, New York, 1970.

[7] P. Duren and A. Schuster. *Bergman Spaces.* American Mathematical Society, New York, 2004.

[8] Z. Guo. Riemann-Stieltjes operators between Zygmund-type spaces. *J. Appl. Math. and Comp.*, 6(3):332–342, 2022.

[9] B. Korenblum H. Hedenmalm and K. Zhu. *Theory of Bergman Spaces.* Springer-Verlag, New York, 2000.

[10] S. Li and S. Stević. Volterra-type operators on Zygmund spaces. *Journal of Inequalities and Applications*, 10:Article ID 32124, 10 pages, 2007.

[11] S. Li and S. Stević. Generalized composition operators on Zygmund spaces and Bloch type spaces. *J. Math. Anal. Appl.*, 338:1282–1295, 2008.

[12] S. Li and S. Stević. Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to the Zygmund space. *J. Math. Anal. Appl.*, 345:40–52, 2008.

[13] Y. Liang. Volterra-type operators from weighted Bergman-Orlicz space to β-Zygmund-Orlicz and γ-Bloch-Orlicz spaces. *Monatsheftefür Mathematik*, 182:877–897, 2017.

[14] Y. Liu and Y. Yu. Riemann-Stieltjes operator from mixed norm spaces to Zygmund-type spaces on the unit ball. *Taiwanese Journal of Mathematics*, 17:1751–1764, 2013.

[15] S. Demiriz M. Başarır and E.E. Kara. Composition Operators on Second-Order Cesàro Function Spaces. *Approximation Theory, Sequence Spaces and Applications*, 1:21–33, 2022.

[16] J. Liu Q. Lin and Y. Wu. Volterra type operators on $S^p(\mathbb{D})$ spaces. *J. Math. Anal. and Appl.*, 461:1100–1114, 2018.

[17] J.H. Shapiro. *Composition Operators and Classical Function Theory.* Springer-Verlag, New York, 1993.

[18] A. Siskakis and R. Zhao. A Volterra type operator on spaces of analytic functions. *Contemporary Mathematics*, 232:299–311, 1999.

[19] E. Wolf. Volterra composition operators between weighted Bergman spaces and weighted Bloch type spaces. *Collect. Math.*, 61:57–63, 2010.

[20] Y. Yu Y. Liu and X. Liu. Riemann-Stieltjes operator from the general space to Zygmund-type spaces on the unit ball. *Complex Analysis and Operator Theory*, 9:985–997, 2015.

[21] W. Yang and X. Meng. Generalized composition operators from F(p,q,s) spaces to Bloch-type spaces. *Appl. Math. Comput.*, 217:2513–2519, 2010.

[22] K. Zhu. *Spaces of Holomorphic Functions in the Unit Ball.* Springer-Verlag, New York, 2005.

[23] K. Zhu. *Operator Theory in Function Spaces.* American Mathematical Society, New York, 2007.

[24] X. Zhu. Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces. *Indian J. Math.*, 49:139–149, 2007.

[25] X. Zhu. Generalized composition operators and Volterra composition operators on Bloch spaces in the unit ball. *Complex Var. Elliptic Equ.*, 54:95–102, 2009.