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On the isomorphism problem for central extensions II

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Abstract. In this paper, we study the isomorphism problem for central extensions. More precisely, in some new situations, we provide necessary and sufficient conditions for two central extensions to be isomorphic. We investigate the case when the quotient group is simple or purely non-abelian. Furthermore, we characterize isomorphisms leaving the quotient group invariant. Finally, we deal with isomorphisms of central extensions where the kernel group and the quotient group are isomorphic.

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1. Introduction

The classification of groups in a certain class is one of the most classical problems in group theory. For groups with composition series, the Jordan–Hölder Theorem states that if we can list all simple groups, and solve the extension problem then we can construct and classify all groups. The classification of simple groups has been achieved in the finite case, hence we need to solve the extension problem. The extension problem for two groups G_1 and G_2 is the problem of finding all groups G with G_1 as a normal subgroup of G, and the quotient group G/G_1 isomorphic to G_2. Such a group G is called an extension of G_1 by G_2 [6]. The classification of extensions with non-abelian kernel group may be found in many texts, but the famous references for these extensions are Schreier's paper [7] and Eilenberg-Mac Lane's paper [4]. In this work, we will focus on extensions with abelian kernel group. In particular, if G_1 is a central subgroup of G, then we say that G is a central extension of G_1 by G_2. For central extensions, an answer to the extension problem has been given by Hölder and Schreier by using the group cohomology [6, Theorem 7.59]. However, this answer will not enable us to construct all possible non-isomorphic central extensions of G_1 by G_2 (the isomorphism problem). In fact, it is very hard to solve the isomorphism problem, but it has been discussed for some special cases in [8–10]. In fact,

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those results do not concern general isomorphisms, but only those of certain type, namely leaving the kernel group or both the two factors invariant, inducing the identity or a commuting automorphism on the quotient group. In this work, necessary and sufficient conditions for two central extensions of G_1 by G_2 to be isomorphic are given in some new situations. More precisely, we study the case when the quotient group is simple or purely non-abelian. Furthermore, we characterize isomorphisms leaving the quotient group invariant, and deal with isomorphisms of central extensions where the kernel group G_1 and the quotient group G_2 are isomorphic.

Throughout this paper, we denote by $Z(G)$, G', $Aut(G)$ and $End(G)$, respectively, the center, the derived subgroup, the automorphism group, and the monoid of endomorphisms of G. For any two groups H and K, let $Hom(H, K)$ denote the set of all homomorphisms from H to K.

2. Central extension

In this paper, aspects of group cohomology will be used frequently. Therefore, we recall in this section some basic facts of this theory and fix additional notations and terminology.

Let $1 \to G_1 \to G \to G_2 \to 1$ be a group extension, where for convenience we regard the kernel group G_1 as a subgroup of G and G_2 is identified with the quotient group G/G_1. If G_1 is a central subgroup of G, then we say that G is a central extension of G_1 by G_2. Two central extensions G and G' of G_1 by G_2 are said to be equivalent if there exists a homomorphism $\varphi : G \to G'$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \to & G_1 & \to & G & \to & G_2 & \to & 1 \\ & & \| & & \downarrow \varphi & & \| & & \\ 1 & \to & G_1 & \to & G' & \to & G_2 & \to & 1 \end{array}$$

commutes.

Let G_2 be a group which acts trivially on a group G_1. A 2-cocycle of G_2 with coefficients in G_1 is a map $\varepsilon : G_2 \times G_2 \to G_1$ satisfying the 2-cocycle condition, that is

$$\varepsilon(h, g)\varepsilon(hg, k) = \varepsilon(g, k)\varepsilon(h, gk) \quad \text{for all} \quad g, h, k \in G_2.$$

We always assume that ε is normalized, i.e. $\varepsilon(g, 1) = \varepsilon(1, g) = 1$ for all $g \in G_2$. Note that 2-cocycles are known by factor sets in many books (see for example [1–3, 5, 6, 12]).

The set of normalized 2-cocycles of G_2 with coefficients in G_1 is denoted by $Z^2(G_2, G_1)$. The trivial 2-cocycle is the 2-cocycle c with $c(g, h) = 1$ for all $g, h \in G_2$. Let ε_1, $\varepsilon_2 \in Z^2(G_2, G_1)$. We write $\varepsilon_1 \sim \varepsilon_2$ and say that ε_1 and ε_2 are cohomologous, if there is a map $t : G_2 \to G_1$ such that $\varepsilon_2(g, h) = t(g)t(h)\varepsilon_1(g, h)t(gh)^{-1}$ for all g, $h \in G_2$. Then $(\sim)$ is an equivalence relation on $Z^2(G_2, G_1)$. The cohomology class of $\varepsilon \in Z^2(G_2, G_1)$ is denoted by $[\varepsilon]$. The set of all cohomology classes of G_2 with coefficients in G_1 is denoted by $H^2(G_2, G_1)$ and called the second cohomology of G_2 with coefficients in G_1.

From now, G_1 will always considered an abelian group. Then $Z^2(G_2, G_1)$ is an abelian group and we have $H^2(G_2, G_1) = Z^2(G_2, G_1)/B^2(G_2, G_1)$ where $B^2(G_2, G_1)$ is the subgroup of $Z^2(G_2, G_1)$ which consists of all functions $\psi : G_2 \times G_2 \to G_1$ satisfying that for all

$g, h \in G_2$: $\psi(h, g) = \delta(g)\delta(hg)^{-1}\delta(h)$ for some $\delta : G_2 \to G_1$. The elements of $B^2(G_2, G_1)$ are called 2-coboundaries. The set of all normalized 2-cocycles which are symmetric forms a subgroup of $Z^2(G_2, G_1)$ and denoted by $SZ^2(G_2, G_1)$. The famous Schreier theorem says that the central extensions of G_1 by G_2 are classified by the non-trivial elements of the second cohomology group $H^2(G_2, G_1)$ with coefficients in G_1. In particular, a central extension of G_1 by G_2 splits if and only if the corresponding 2-cocycle is trivial in $H^2(G_2, G_1)$.

A 2-cocycle $\varepsilon \in Z^2(G_2, G_1)$ gives rise to a central extension $G = G_1 \underset{\varepsilon}{\times} G_2$ of G_1 by G_2 induced by ε, with group operation given by

$$(x, y) \underset{\varepsilon}{\bullet} (x', y') = (xx'\varepsilon(y, y'), yy')$$

for all $x,\ x' \in G_1$ and $y,\ y' \in G_2$. Conversely, given a central extension $1 \to G_1 \to G \xrightarrow{j} G_2 \to 1$ and choose a based section $\lambda : G_2 \to G$, i.e. a set map with $\lambda(1)$ is the identity element of G and $j \circ \lambda = id_{G_2}$. The based section λ induces a 2-cocycle $\varepsilon_\lambda : G_2 \times G_2 \longrightarrow G_1$ given by $\varepsilon_\lambda(h, g) = \lambda(h)\lambda(g)\lambda(hg)^{-1}$ and therefore, the group G is isomorphic to the group $G_1 \underset{\varepsilon_\lambda}{\times} G_2$ [10, Proposition 2.3]. We can easily see that the group $G_1 \underset{\varepsilon}{\times} G_2$ is abelian if and only if G_2 is abelian and $\varepsilon \in SZ^2(G_2, G_1)$. We know that $G_1 \underset{\varepsilon}{\times} G_2 = G_1 \times G_2$ if and only if $\varepsilon = 1$. But, it is possible for a central extension of G_1 by G_2 induced by a non-trivial 2-cocycle to be isomorphic to the direct product $G_1 \times G_2$ (see Proposition 3.2).

3. Preliminary results

Let $pr_i : G_1 \underset{\varepsilon}{\times} G_2 \to G_i$ be the *ith* canonical projection and $t_i : G_i \to G_1 \underset{\varepsilon}{\times} G_2$ be the *ith* canonical injection. Let φ be a group homomorphism from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$ and set $\varphi_{ij} = pr_i \circ \varphi \circ t_j$, where $1 \le i, j \le 2$. So we can write φ in the matrix form: $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$. Obviously, we see that pr_2 and t_1 are group homomorphisms, then φ_{21} is a group homomorphism. Furthermore, we have the following lemmas which we need in the sequel.

Lemma 3.1. *[10, Lemma 3.1] Let* $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ *be a group homomorphism from* $G_1 \underset{\varepsilon_1}{\times} G_2$ *to* $G_1 \underset{\varepsilon_2}{\times} G_2$*. Then*

$$\varphi(x, y) = (\varphi_{11}(x)\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x), \varphi_{22}(y)),\ \varphi_{21}(x)\varphi_{22}(y)) \tag{1}$$

for all $x \in G_1$*, and* $y \in G_2$*.*

Lemma 3.2. *[10, Lemma 3.2] Let* φ *be a set map from* $G_1 \underset{\varepsilon_1}{\times} G_2$ *to* $G_1 \underset{\varepsilon_2}{\times} G_2$*. Then* φ *is a group homomorphism if and only if*

$$\varphi(x,y) \underset{\varepsilon_2}{\bullet} \varphi(x',1) \quad = \quad \varphi(xx',y), \tag{2}$$

and

$$\varphi(x,y) \underset{\varepsilon_2}{\bullet} \varphi(1,y') \quad = \quad \varphi(x\varepsilon_1(y,y'),yy') \tag{3}$$

for all x, $x' \in G_1$, and y, $y' \in G_2$.

Definition 3.1. *Let G_2 be a group which acts trivially on an abelian group G_1. Let $\varepsilon \in Z^2(G_2, G_1)$. A map $\chi : G_1 \to G_1$ is called an ε-endomorphism of G_1, if*

$$\chi(x\varepsilon(y,y')) = \chi(x)\chi(\varepsilon(y,y'))$$

for all $x \in G_1$, and y, $y' \in G_2$. If in addition χ is a bijection, then it is said to be ε-automorphism.

The following lemma follows directly by using the 2-cocycle condition.

Lemma 3.3. *Let G_2 be a group which acts trivially on an abelian group G_1. Let $\varepsilon \in Z^2(G_2, G_1)$, $\delta \in Hom(G_1, G_2)$ and σ an ε-endomorphism of G_1. Then $\sigma \circ \varepsilon \in Z^2(G_2, G_1)$, $\delta \circ \varepsilon \in Z^2(G_2, G_2)$ and $\varepsilon \circ (\delta \times \delta) \in Z^2(G_1, G_1)$.*

From now, if $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ is a map from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$, then φ is defined by the formula (1). From the previous Lemmas, we get the following interesting result which will be frequently used in the sequel.

Proposition 3.1. *Let G_2 be a group such that the equivalence relation $(\sim)$ is trivial on $Z^2(G_2, G_2)$. Let $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be a map from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$. Then, φ is a group homomorphism if and only if*

(i) $\varphi_{21} \in Hom(G_1, C_{G_2}(\varphi_{22}(G_2)))$, $\varphi_{22} \in End(G_2)$ *and* φ_{11} *is an* ε_1*-endomorphism,*

(ii) $[\{1\} \times \varphi_{22}(G_2), \{1\} \times \varphi_{21}(G_1)] = 1$,

(iii) $Im(\varepsilon_1) \leq Ker(\varphi_{21})$ *and* $\varepsilon_2^{-1} \circ (\varphi_{21} \times \varphi_{21}) = \psi_{\varphi_{11}} \in B^2(G_1, G_1)$,

(iv) $(\varphi_{11} \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\varphi_{22} \times \varphi_{22})) = \psi_{\varphi_{12}} \in B^2(G_2, G_1)$,

where $\psi_{\varphi_{ij}}(y, y') = \varphi_{ij}(y)\varphi_{ij}(y')\varphi_{ij}(yy')^{-1}$ for all $1 \leq i, j \leq 2$ and y, $y' \in G_j$.

Proof. Indeed, evaluate the left hand side and right hand side of the formulas (2) and (3), we obtain

$$\varphi(x,y) \underset{\varepsilon_2}{\cdot} \varphi(x',1) = \varphi(xx',y)$$

$$\Leftrightarrow (\varphi_{11}(x)\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x),\varphi_{22}(y)),\ \varphi_{21}(x)\varphi_{22}(y)) \underset{\varepsilon_2}{\cdot} (\varphi_{11}(x'),\varphi_{21}(x'))$$
$$= (\varphi_{11}(xx')\varphi_{12}(y)\varepsilon_2(\varphi_{21}(xx'),\varphi_{22}(y)),\ \varphi_{21}(xx')\varphi_{22}(y))$$
$$\Leftrightarrow \varphi_{21}(x)\varphi_{22}(y)\varphi_{21}(x') = \varphi_{21}(xx')\varphi_{22}(y) \text{ and} \tag{4}$$
$$\varphi_{11}(x)\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x),\varphi_{22}(y))\varphi_{11}(x')\varepsilon_2(\varphi_{21}(x)\varphi_{22}(y),\varphi_{21}(x')) \tag{5}$$
$$= \varphi_{11}(xx')\varphi_{12}(y)\varepsilon_2(\varphi_{21}(xx'),\varphi_{22}(y)).$$

Setting $x = 1$ in the equations (4) and (5), we obtain

$$\varphi_{22}(y)\varphi_{21}(x') = \varphi_{21}(x')\varphi_{22}(y) \tag{6}$$

and

$$\varepsilon_2(\varphi_{22}(y),\varphi_{21}(x')) = \varepsilon_2(\varphi_{21}(x'),\varphi_{22}(y)). \tag{7}$$

That is, $[\{1\}\times\varphi_{22}(G_2),\{1\}\times\varphi_{21}(G_1)] = 1$. Now, combining the equations (4) and (6), we get that $\varphi_{21} \in Hom(G_1, C_{G_2}(\varphi_{22}(G_2)))$. Thus, using the 2-cocycle condition together with the equations (6) and (7), the equation (5) yields

$$\varphi_{11}(xx') = \varphi_{11}(x)\varphi_{11}(x')\varepsilon_2(\varphi_{21}(x),\varphi_{21}(x')) \tag{8}$$

which implies that $\varepsilon_2^{-1}\circ(\varphi_{21}\times\varphi_{21}) \in B^2(G_1,G_1)$.

On the other hand, we have that

$$\varphi(x,y) \underset{\varepsilon_2}{\cdot} \varphi(1,y') = \varphi(x\varepsilon_1(y,y'),yy')$$
$$\Leftrightarrow (\varphi_{11}(x)\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x),\varphi_{22}(y)),\ \varphi_{21}(x)\varphi_{22}(y)) \underset{\varepsilon_2}{\cdot} (\varphi_{12}(y'),\varphi_{22}(y'))$$
$$= (\varphi_{11}(x\varepsilon_1(y,y'))\varphi_{12}(yy')\varepsilon_2(\varphi_{21}(x\varepsilon_1(y,y')),\varphi_{22}(yy')),\ \varphi_{21}(x\varepsilon_1(y,y'))\varphi_{22}(yy'))$$
$$\Leftrightarrow \varphi_{21}(x)\varphi_{22}(y)\varphi_{22}(y') = \varphi_{21}(x\varepsilon_1(y,y'))\varphi_{22}(yy') \text{ and} \tag{9}$$
$$\varphi_{11}(x)\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x),\varphi_{22}(y))\varphi_{12}(y')\varepsilon_2(\varphi_{21}(x)\varphi_{22}(y),\varphi_{22}(y')) \tag{10}$$
$$= \varphi_{11}(x\varepsilon_1(y,y'))\varphi_{12}(yy')\varepsilon_2(\varphi_{21}(x\varepsilon_1(y,y')),\varphi_{22}(yy')).$$

Since φ_{21} is a group homomorphism, the equation (9) implies that

$$\varphi_{21}(\varepsilon_1(y,y')) = \varphi_{22}(y)\varphi_{22}(y')(\varphi_{22}(yy'))^{-1}$$

for all y, $y' \in G_2$. So, $\varphi_{21}\circ\varepsilon_1 \sim 1$. But, by the assumption, there is no nontrivial 2-cocycle in $Z^2(G_2,G_2)$ that is cohomologous to the trivial 2-cocycle. So, we have $\varphi_{21}\circ\varepsilon_1 = 1$ and then $\varphi_{22} \in End(G_2)$. Furthermore, by using the 2-cocycle condition, the equation (10) gives us

$$\varphi_{11}(x\varepsilon_1(y,y'))\varphi_{12}(yy') = \varphi_{11}(x)\varphi_{12}(y)\varphi_{12}(y')\varepsilon_2(\varphi_{22}(y),\varphi_{22}(y')). \tag{11}$$

But, the equation (8) yields $\varphi_{11}(x\varepsilon_1(y,y')) = \varphi_{11}(x)\varphi_{11}(\varepsilon_1(y,y'))$. Thus, the equation (11) is equivalent to

$$\varphi_{11}(\varepsilon_1(y,y'))\varphi_{12}(yy') = \varphi_{12}(y)\varphi_{12}(y')\varepsilon_2(\varphi_{22}(y),\varphi_{22}(y')),$$

which implies that

$$\varphi_{11}(\varepsilon_1(y,y'))(\varepsilon_2(\varphi_{22}(y),\varphi_{22}(y')))^{-1} = \varphi_{12}(y)\varphi_{12}(y')(\varphi_{12}(yy'))^{-1}.$$

Thus, the proof is completed.

Remark 3.1. *(i) Note that if G_2 is abelian then the assumption of the previous proposition and the condition $B^2(G_2,G_2)=1$ are equivalent.*

(ii) If $\varepsilon_2 \in SZ^2(G_2,G_1)$ then the second condition of the previous proposition is not required.

Definition 3.2. *The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are called upper isomorphic, if there exists an isomorphism $\varphi : G_1 \underset{\varepsilon_1}{\times} G_2 \longrightarrow G_1 \underset{\varepsilon_2}{\times} G_2$ leaving G_1 invariant.*

Remark 3.2. *It is possible for two central extensions to be isomorphic without being upper isomorphic. For example, consider the central extensions*

$$1 \to \mathbb{Z}_p^{\mathbb{N}} \overset{i_1}{\to} G \to \mathbb{Z}_p^{\mathbb{N}} \to 1$$

and

$$1 \to \mathbb{Z}_p^{\mathbb{N}} \overset{i_2}{\to} G \to \mathbb{Z}_p^{\mathbb{N}} \to 1$$

such that $G = \mathbb{Z}_p \times (\mathbb{Z}_{p^2})^{\mathbb{N}}$, $i_1(\mathbb{Z}_p^{\mathbb{N}}) = \{1\} \times (p\mathbb{Z}_{p^2})^{\mathbb{N}}$ and $i_2(\mathbb{Z}_p^{\mathbb{N}}) = \mathbb{Z}_p \times (p\mathbb{Z}_{p^2})^{\mathbb{N}}$. We have $i_1(\mathbb{Z}_p^{\mathbb{N}}) \cong i_2(\mathbb{Z}_p^{\mathbb{N}}) \cong \mathbb{Z}_p^{\mathbb{N}}$ and $G/i_1(\mathbb{Z}_p^{\mathbb{N}}) \cong G/i_2(\mathbb{Z}_p^{\mathbb{N}}) \cong \mathbb{Z}_p^{\mathbb{N}}$. Since $i_1(\mathbb{Z}_p^{\mathbb{N}}) = pG$ is a characteristic subgroup of G and $i_2(\mathbb{Z}_p^{\mathbb{N}}) \neq pG$, it follows that there is no automorphism of G sending $i_1(\mathbb{Z}_p^{\mathbb{N}})$ to $i_2(\mathbb{Z}_p^{\mathbb{N}})$.

In particular, suppose that $Z(G_1 \underset{\varepsilon_1}{\times} G_2) = Z(G_1 \underset{\varepsilon_2}{\times} G_2) = G_1$ or $(G_1 \underset{\varepsilon_1}{\times} G_2)' = (G_1 \underset{\varepsilon_2}{\times} G_2)' = G_1$. So each isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ between $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ leaves G_1 invariant and then $\varphi_{21} = 1$. So, the equation (9) implies that $\varphi_{22} \in End(G_2)$ and then the assumption of the previous proposition is not required. This case is covered by the following result which can be viewed as a consequence of [8, Theorem 3.7].

Proposition 3.2. *The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper isomorphic if and only if there exist $\sigma \in Aut(G_1)$ and $\rho \in Aut(G_2)$ such that*

$$(\sigma \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\rho \times \rho)) \in B^2(G_2, G_1).$$

Example 3.1. *Define a function $\varepsilon_0 : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ by $\varepsilon_0(i,j) = 0$ if $i+j < p$, and $\varepsilon_0(i,j) = 1$ if $i+j \geqslant p$ for all $0 \leqslant i,j < p$. This function is the 2-cocycle corresponding to the central extension $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ induced by the based section $\lambda : \mathbb{Z}_p \to \mathbb{Z}_{p^2}$ defined by $\lambda(i \ mod\ p) = i \ mod\ p^2$ for all $0 \leqslant i < p$. Now, let $\varepsilon \in Z^2(\mathbb{Z}_p,\mathbb{Z}_p)$ and $m_\varepsilon = \sum_{k=0}^{p-1} \varepsilon(k,1)$. By using the 2-cocycle condition, we can check that $\varepsilon - m_\varepsilon \varepsilon_0 \in B^2(\mathbb{Z}_p,\mathbb{Z}_p)$. So, putting $G_1 = G_2 = \mathbb{Z}_p$, $\sigma = id_{G_1}$ and $\rho = id_{G_2}$ in the preceding proposition, the groups $\mathbb{Z}_p \underset{\varepsilon}{\times} \mathbb{Z}_p$ and $\mathbb{Z}_p \underset{m_\varepsilon \varepsilon_0}{\times} \mathbb{Z}_p$ are upper isomorphic.*

In view of the preceding proposition, it is possible for a central extension induced by a non-trivial 2-cocycle to be isomorphic to the direct product of the two factors group. More precisely, for a non-trivial 2-cocycle $\varepsilon \in Z^2(G_2, G_1)$, it is easy to deduce that the groups $G_1 \underset{\varepsilon}{\times} G_2$ and $G_1 \times G_2$ are upper isomorphic if and only if there exists $\sigma \in Aut(G_1)$ such that $\sigma \circ \varepsilon \in B^2(G_2, G_1)$.

4. Central extensions with simple or purely non-abelian quotient group

Proposition 4.1. *Let G_2 be a simple non-abelian group which acts trivially on an abelian group G_1. The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are isomorphic if and only if they are upper isomorphic.*

Proof. The if direction is clear. For the converse, assume that $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are isomorphic by an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$. It follows from the equation $\varphi(x,1) \underset{\varepsilon_2}{\bullet} \varphi(1,y) = \varphi(1,y) \underset{\varepsilon_2}{\bullet} \varphi(x,1)$ that $[\varphi_{21}(x), \varphi_{22}(y)] = 1$ for all $x \in G_1$, and $y \in G_2$. Now let $g \in G_2$, there exists an element $(x,y) \in G_1 \underset{\varepsilon_1}{\times} G_2$ such that $\varphi(x,y) = (1,g)$, that is $g = \varphi_{21}(x)\varphi_{22}(y)$. So, for all $h \in G_1$, we have

$$\begin{aligned} g\varphi_{21}(h)g^{-1} &= \varphi_{21}(x)\varphi_{22}(y)\varphi_{21}(h)\varphi_{22}(y)^{-1}\varphi_{21}(x)^{-1} \\ &= \varphi_{21}(x)\varphi_{21}(h)\varphi_{21}(x)^{-1} \in \varphi_{21}(G_1). \end{aligned}$$

Thus, $\varphi_{21}(G_1)$ is a normal subgroup of G_2. As G_2 is simple non-abelian, then $\varphi_{21}(G_1)$ is either trivial or G_2. If $\varphi_{21}(G_1) = G_2$, then φ_{21} is an epimorphism and therefore G_2 is abelian, a contradiction. Hence, $\varphi_{21} = 1$ and then φ maps G_1 to itself, as required.

Recall that a non-abelian group which has no non-trivial abelian direct factor is said to be purely non-abelian.

Theorem 4.1. *Let G_2 be a finite purely non-abelian group which acts trivially on a finite abelian group G_1. Let $\varphi \in \left\{ \begin{pmatrix} \sigma & \eta \\ \delta & \rho \end{pmatrix} \middle| \begin{array}{l} \sigma \in Aut(G_1), \eta \in Hom(G_2, G_1) \\ \delta \in Hom(G_1, G_2), \rho \in Aut(G_2) \end{array} \right\}$ where σ, δ and ρ satisfy the conditions:*

(i) $[\{1\} \times G_2, \{1\} \times \delta(G_1)] = 1$,

(ii) $\varepsilon_2 \circ (\delta \times \delta) = 1$ *and* $\delta \circ \varepsilon_1 = 1$,

(iii) $\sigma \circ \varepsilon_1 = \varepsilon_2 \circ (\rho \times \rho)$.

Then, φ is an isomorphism from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$.

Proof. Indeed, the map φ is defined by the formula (1). By Proposition 3.1, the map φ is clearly a group homomorphism. Now, assume that $\varphi(x,y) = 1$. So $\delta(x)\rho(y) = 1$

and then $\rho(y) = \delta(x^{-1})$, which implies that $\sigma(x)\eta(y)\varepsilon_2(\delta(x), \delta(x^{-1})) = 1$. So, the first equation of the condition (ii) ensures that $\sigma(x)\eta(y) = 1$, and then $x = \sigma^{-1}(\eta(y^{-1}))$. Hence, $\rho^{-1}(\delta(\sigma^{-1}(\eta(y)))) = y$. From the condition (i), we have $[G_2, \delta(G_1)] = 1$, that is $\delta(G_1) \le Z(G_2)$. Hence, we have $\Psi = \rho^{-1} \circ \delta \circ \sigma^{-1} \circ \eta \in Hom(G_2, Z(G_2))$, and then $Im\Psi \trianglelefteq G_2$. So, by Fitting's Lemma, we have $G_2 \cong Ker\Psi \times Im\Psi$ which contradicts to the fact that G_2 is purely non-abelian. Thus $y = 1$ and then $x = 1$. Therefore, the map φ is injective, and then it is an isomorphism.

Remark 4.1. *The previous proposition will not be true if G_2 is not purely non-abelian. Indeed, assume that G_1 is a direct factor of G_2 and let $\varphi = \begin{pmatrix} id_{G_1} & \varphi_{12} \\ \varphi_{21} & id_{G_2} \end{pmatrix}$ be a map from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$ where $\varphi_{12}(x) = \varphi_{21}(x) = x^{-1}$ for all $x \in G_1$. So, using the formula (1), we obtain $\varphi(x, x) = (\varepsilon_2(\varphi_{21}(x), \varphi_{21}(x^{-1})), 1)$. Hence, by using the first equation of the condition (ii), we get $\varphi(x, x) = (1, 1)$. Therefore, φ is not an isomorphism.*

5. Lower isomorphism problem for central extensions

Definition 5.1. *Let G_2 be a group which acts trivially on an abelian group G_1. The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are called lower isomorphic if there exists an isomorphism $\varphi : G_1 \underset{\varepsilon_1}{\times} G_2 \longrightarrow G_1 \underset{\varepsilon_2}{\times} G_2$ leaving G_2 invariant.*

Note that Remark 3.2 also shows that two central extensions can be isomorphic without being lower isomorphic. We now present the following main result of this section.

Theorem 5.1. *Let G_2 be a group such that the equivalence relation $(\sim)$ is trivial on $Z^2(G_2, G_2)$. If the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are lower isomorphic then there exist $\rho \in Aut(G_2)$, $\delta \in Hom(G_1, Z(G_2))$ and an ε_1-automorphism σ such that*

(i) $[\{1\} \times G_2, \{1\} \times \delta(G_1)] = 1$, $Im(\varepsilon_1) \le Ker(\delta)$,

(ii) $\varepsilon_2^{-1} \circ (\delta \times \delta) = \psi_\sigma \in B^2(G_1, G_1)$ *where* $\psi_\sigma(x, x') = \sigma(x)\sigma(x')\sigma(xx')^{-1}$ *for all* x, $x' \in G_1$,

(iii) $\varepsilon_2 \circ (\rho \times \rho) = \sigma \circ \varepsilon_1$.

Proof. Suppose that $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are isomorphic by an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & 1 \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$. From Lemma 3.1, we have that

$$\varphi(x, y) = (\varphi_{11}(x)\varepsilon_2(\varphi_{21}(x), \varphi_{22}(y)),\ \varphi_{21}(x)\varphi_{22}(y))$$

for all $x \in G_1$, $y \in G_2$. Since φ is bijective, so is φ_{22}. By Proposition 3.1, the maps $\varphi_{21} \in Hom(G_1, Z(G_2))$, $\varphi_{22} \in Aut(G_2)$ and the ε_1-endomorphism φ_{11} satisfy the conditions (i)-(iii). So, it remains to show that φ_{11} is bijective. Let $g \in G_1$, since φ is surjective, there exists an element $(x,y) \in G_1 \underset{\varepsilon_1}{\times} G_2$ such that $\varphi(x,y) = (g,1)$, that is $\varphi_{11}(x)\varepsilon_2(\varphi_{21}(x), \varphi_{22}(y)) = g$ and $\varphi_{21}(x)\varphi_{22}(y) = 1$. So, $\varphi_{11}(x)\varepsilon_2(\varphi_{21}(x), \varphi_{21}(x^{-1})) = g$ and then, using the condition (ii), we have $\varphi_{11}(x) = g$. Therefore, φ_{11} is surjective. On the other hand, the map ψ defined by $\psi(x,y) = (x, y\varphi_{22}^{-1}(\varphi_{21}(x)^{-1}))$ is a bijection and we have $\varphi \circ \psi(x^{-1}, 1) = (\varphi_{11}(x^{-1})\varepsilon_2(\varphi_{21}(x^{-1}), \varphi_{21}(x)), 1)$ for all $x \in G_1$. Thus, the condition (ii) ensures that $\varphi \circ \psi(x^{-1}, 1) = (\varphi_{11}(x)^{-1}, 1)$ for all $x \in G_1$. Since $\varphi \circ \psi$ is injective, so is φ_{11}. Thus, the desired result follows directly by taking $\rho = \varphi_{22}$, $\sigma = \varphi_{11}$ and $\delta = \varphi_{21}$.

Remark 5.1. *The converse of the previous result holds if G_1 and G_2 are finite. Indeed, it suffices to show that φ is injective. Let $(x,y) \in G_1 \times_\varepsilon G_2$ such that $\varphi(x,y) = (1,1)$. Then, the equality $\varphi_{21}(x)\varphi_{22}(y) = 1$ implies that $\varphi_{22}(y) = \varphi_{21}(x^{-1})$. So*

$$\varphi_{11}(x)\varepsilon_2(\varphi_{21}(x), \varphi_{21}(x^{-1})) = 1.$$

Using the condition (ii), we get $\varphi_{11}(x^{-1})^{-1} = 1$ and then $x = 1$ since φ_{11} is injective. Since $\varphi_{21}(1) = 1$, it follows that $\varphi_{22}(y) = 1$ and then $y = 1$ since φ_{22} is injective. Therefore, φ is bijective and so it is a lower isomorphism by Proposition 3.1. As required.

Let $\varepsilon \in Z^2(G_2, G_1)$ be a non-trivial 2-cocycle. In view of the preceding theorem, the group $G_1 \underset{\varepsilon}{\times} G_2$ cannot be lower isomorphic to the direct product $G_1 \times G_2$.

Corollary 5.1. *Further to the assumption of the previous theorem, suppose that $B^2(G_1, G_1) = 1$. The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are lower isomorphic if and only if there exist $\rho \in Aut(G_2)$ and $\sigma \in Aut(G_1)$ such that $\varepsilon_2 \circ (\rho \times \rho) = \sigma \circ \varepsilon_1$.*

Proof. Indeed, suppose that the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are lower isomorphic. By Theorem 5.1, there exist $\rho \in Aut(G_2)$, $\delta \in Hom(G_1, Z(G_2))$ and an ε_1-automorphism σ satisfying the conditions (ii) and (iii). Since $B^2(G_1, G_1) = 1$, the condition (ii) implies that $\sigma \in Aut(G_1)$. Therefore, we conclude by using the condition (iii). For the converse, we can easily prove that the bijection φ defined by $\varphi(x,y) = (\sigma(x),\ \rho(y))$ is an isomorphism. Hence the corollary follows.

Example 5.1. *Define functions ε_1, $\varepsilon_2 : (\mathbb{Z}_2 \times \mathbb{Z}_2)^2 \to \mathbb{Z}_2$ by $\varepsilon_1((h_1,h_2),(g_1,g_2)) = h_1g_1$ and $\varepsilon_2((h_1,h_2),(g_1,g_2)) = h_2g_2$. These functions are 2-cocycles corresponding to two inequivalent central extensions of $\langle(2,0)\rangle \cong \mathbb{Z}_2$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ whose the middle group is $\mathbb{Z}_4 \times \mathbb{Z}_2$. Note that ε_1 and ε_2 are the 2-cocycles induced by the based sections λ_1, $\lambda_2 : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_4 \times \mathbb{Z}_2$ defined by $\lambda_1(h \ mod\ 2, g \ mod\ 2) = (h \ mod\ 4, g \ mod\ 2)$ and $\lambda_2(h \ mod\ 2, g \ mod\ 2) = (g \ mod\ 4, h \ mod\ 2)$. Let $\rho \in Aut(\mathbb{Z}_2 \times \mathbb{Z}_2)$ defined by $\rho(g_1,g_2) = (g_2,g_1)$ and take $\sigma = id_{\mathbb{Z}_2}$, we can check easily that $\varepsilon_2 \circ (\rho \times \rho) = \sigma \circ \varepsilon_1$. Therefore, the groups $\mathbb{Z}_2 \underset{\varepsilon_1}{\times} (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $\mathbb{Z}_2 \underset{\varepsilon_2}{\times} (\mathbb{Z}_2 \times \mathbb{Z}_2)$ are lower isomorphic.*

6. Isomorphisms of central extensions with isomorphic factors group

Definition 6.1. *Let G_2 be a group which acts trivially on an abelian group G_1. Let $1 \leq i \leq 2$, the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are called (G_i)-isomorphic if there exists an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ between them such that $\varphi_{ii} = 1$.*

Proposition 6.1. *Let G_2 be an abelian group and suppose that the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (G_2)-isomorphic. Then, there exist an ε_1-endomorphism σ of G_1, an injective map $\eta : G_2 \to G_1$ and an epimorphism $\delta : G_1 \to G_2$ such that:*

(i) $\varepsilon_2^{-1} \circ (\delta \times \delta) = \psi_\sigma \in B^2(G_1, G_1),\ Im(\varepsilon_1) \leq Ker(\delta)$,

(ii) $\sigma \circ \varepsilon_1 = \psi_\eta \in B^2(G_2, G_1)$.

Proof. Let $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be a (G_2)-isomorphism from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$. So $\varphi_{22} = 1$, and therefore we can show easily that φ_{21} is surjective and φ_{12} is injective. Hence, by Proposition 3.1, the desired conditions follows directly by taking $\sigma = \varphi_{11}$, $\delta = \varphi_{21}$ and $\eta = \varphi_{12}$.

Note that if G_2 is non-abelian, then the condition $\delta \in Epi(G_1, G_2)$ implies that $\delta = 1$. Therefore, the previous result becomes a direct consequence of Proposition 3.2.

Corollary 6.1. *Suppose that G_1 and G_2 are two finite abelian groups with the same order. The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (G_2)-isomorphic if and only if $\varepsilon_1 = 1$ and there exists an isomorphism $\delta : G_1 \to G_2$ such that $\varepsilon_2^{-1} \circ (\delta \times \delta) \in B^2(G_1, G_1)$.*

Proof. Indeed, suppose that the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (G_2)-isomorphic. By the previous proposition, there exists an epimorphism $\delta : G_1 \to G_2$ such that $Im(\varepsilon_1) \leq Ker(\delta)$ and $\varepsilon_2^{-1} \circ (\delta \times \delta) \in B^2(G_1, G_1)$. But δ is in fact an isomorphism by the assumption, so we must have $\varepsilon_1 = 1$. Conversely, since $\varepsilon_2^{-1} \circ (\delta \times \delta) \in B^2(G_1, G_1)$, it follows that there exists a map $\sigma : G_1 \to G_1$ such that $\varepsilon_2^{-1}(\delta(x), \delta(x')) = \sigma(x)\sigma(x')\sigma(xx')^{-1}$ for all $x, x' \in G_1$. Let δ' be the inverse of δ. By the normalization condition, we have $\sigma(1) = 1$. So, the bijection φ defined by $\varphi(x, y) = (\sigma(x)\delta'(y),\ \delta(x))$ is clearly an isomorphism. As required.

Proposition 6.2. *Let G_2 be a group such that the equivalence relation $(\sim)$ is trivial on $Z^2(G_2, G_2)$. Suppose that the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (G_1)-isomorphic. Then, there exist $\rho \in End(G_2)$, a surjective map $\eta : G_2 \to G_1$ and a monomorphism $\delta : G_1 \to G_2$ such that*

(i) $\varepsilon_1 = 1$,

(ii) $\varepsilon_2 \circ (\delta \times \delta) = 1$ *and*

(iii) $\varepsilon_2^{-1} \circ (\rho \times \rho) = \psi_\eta \in B^2(G_2, G_1)$.

Proof. Let $\varphi = \begin{pmatrix} 1 & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be an isomorphism from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$. Hence, by taking $\rho = \varphi_{22}$, $\eta = \varphi_{12}$ and $\delta = \varphi_{21}$, the conditions (ii) and (iii) follow directly from Proposition 3.1. Notice that $\varphi(x, y) = (\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x), \varphi_{22}(y)),\ \varphi_{21}(x)\varphi_{22}(y))$ for all $x \in G_1$, $y \in G_2$. So, $\varphi(x, 1) = (1,\ \varphi_{21}(x))$ and then φ_{21} is injective. But, by the condition (iii) of Proposition 3.1, we have $Im(\varepsilon_1) \leq Ker(\varphi_{21})$, which implies that $\varepsilon_1 = 1$. On the other hand, let $g \in G_1$, then there exists $(x, y) \in G_1 \underset{\varepsilon_1}{\times} G_2$ such that $\varphi(x, y) = (g, 1)$. This gives us $\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x), \varphi_{21}(x^{-1})) = g$. So the condition (ii) ensures that $\varphi_{12}(y) = g$ and then φ_{12} is surjective. As required.

Now, we derive the following consequence.

Corollary 6.2. *Further to the assumption of the previous proposition, suppose that G_1 and G_2 are two finite abelian groups with the same order. The groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (G_1)-isomorphic if and only if $\varepsilon_1 = 1$ and there exists an isomorphism $\delta : G_1 \to G_2$ such that $\varepsilon_2 \circ (\delta \times \delta) = 1$.*

Proof. Indeed, using the assumptions, the only if direction comes immediately from the previous result. Conversely, let δ' be the inverse of δ. Define a bijective map φ between $G_1 \times G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ given by $\varphi(x, y) = (\delta'(y), \delta(x))$, for all $x \in G_1$, $y \in G_2$. Since $\varepsilon_2 \circ (\delta \times \delta) = 1$, it is easy to check that φ is a group homomorphism, and therefore it is a group isomorphism.

7. Conclusions and future problems

As mentioned in the introduction, our choice to focus on the isomorphism problem for extensions with abelian kernel group is partially motivated by the Jordan–Hölder Theorem. Our other works of particular relevance are [8–11]. We study in [11] the isomorphism problem for split extensions, and as an application, we determine how isomorphism of split extensions and conjugacy of the images of the corresponding actions are related. As is well known, the identity element of the second cohomology group corresponds to the split extensions. This motivates us to consider also the non-identity different elements, which give rise to the construction of non-split extensions. So, in [8–10], we study the isomorphism problem for non-split abelian extensions in some special cases. We mainly deal with isomorphisms leaving one of the two factors or even both invariant. We characterize such isomorphisms in various situations with some assumptions on the quotient and the kernel group. In this paper, by using a similar approach, we give a further contribution to this topic. More precisely, we complete the work with the isomorphism problem for central extensions in different special cases. So most of the results obtained here considered as a generalization of those in [9, 10].

After the outcomes obtained in this work, one may naturally try to obtain similar results with suitable properties for G_1 and G_2 other than previously studied. One may also would like to study the isomorphism problem for non-split extensions with non-abelian kernel. This is of course the most important future problem on this topic. In our future works, we will continue with the isomorphism problem for extensions with abelian kernel group and try to generalize the results into any groups G_1 and G_2. We also would like to choose another groups extension such as Zappa-Szep product to apply similar results.

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