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Semi-Analytical Investigation for the ψ-Caputo Fractional Relaxation-Oscillation Equation Using the Decomposition Method

Mona Alsulami[1,*], Maryam Ahmed Alyami[1], Mariam Al-Mazmumy[1], Asrar Saleh Alsulami[1]

[1] *Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia*

Abstract. The relaxation-oscillation differential equation serves as the fundamental model governing the description of relaxation-oscillation processes with notable applications in fluid flow and oscillation dynamics. The present study seeks the help of the unswerving Adomian Decomposition Method (ADM) to construct a generalized recurrent scheme for the relaxation-oscillation differential equation conferred with the ψ-Caputo fractional derivative. Moreover, fractional-order derivatives are known for unearthing the hidden features that the classical integer-order derivatives are deficient in revealing. Thus, the outcomes acquired through this method when applied to certain ψ-Caputo fractional Cauchy problems prove to be precise and steadfast in contrast to those attained using previously studied methods for solving this equation.

2020 Mathematics Subject Classifications: 26A33, 34A08, 34K37

Key Words and Phrases: ψ-Caputo derivative, fractional relaxation-oscillation equation, Adomian decomposition method, semi-analytical method

1. Introduction

In recent times, fractional calculus has attracted much concern in favor of its capability to perfectly model real-life scenarios with profound insights. Lately, the theory of non-classical derivatives has been comprehensively employed in various science and technological engineering fields, counting rheology [19], anomalous diffusion [33], bioengineering [23], modeling of viscoelastic dampers [22] and others [32, 34]. During the theoretical development of fractional calculus, many fractional differential and integral operators emerged with specific motives and were used by contemporary researchers. Indeed, some of the most used fractional operators include the Riemann-Liouville [36], Caputo, Caputo-Hadamard,

*Corresponding author.
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Email addresses: mralsolami@uj.edu.sa (M. Alsulami), maalyami8@uj.edu.sa (M. Alyami), mhalmazmumy@uj.edu.sa (M. Al-Mazmumy), 2200531@uj.edu.sa (A.S.Alsulami)

Hadamard, Caputo-Erdelyi-Kober, [27], Hilfer [28], conformable [26] and Erdelyi-Kober [4] operators to mention a few. Furthermore, the concept of a fractional differentiation with regard to a given arbitrary function was initiated by Kilbas et al. [4] using the Riemann-Liouville fractional thought. Since then, a lot of definitions have followed suit, including the most used ψ-Caputo fractional derivative that was initiated by Almeida [5], extending the original concept of the Caputo fractional derivative to equally incorporate an arbitrary function ψ that satisfies certain impositions. [8] explored the uniqueness and existence results for the solutions of nonlinear Fractional Differential Equations (FDEs) that involve a ψ-Caputo derivative; this work plays an instrumental role in consolidating various fractional operators. In addition, recent studies on the ψ-Caputo differential operators suggest that FDEs featuring the ψ-Caputo fractional differentials offer greater flexibility and yield favorable results in various scenarios. Almeida, in a study focusing on world population growth, employed the ψ-Caputo derivative and showcased that the model's accuracy relies on carefully choosing the fractional operator. Additionally, opting for a suitable trial function is vital for accurately representing physical phenomena and improves the practical applicability of the approach from a standpoint [6, 8, 14].

Besides, various scientists have extensively employed the ψ-Caputo fractional derivative and investigated its diverse qualitative properties. In addition, the ψ-Caputo fractional derivatives serve as highly effective tools for modeling various real-world physical phenomena, showcasing their ability to reveal hidden features. As in [12] the experimental findings regarding the modeling of drug concentration levels in blood indicate that the ψ-Caputo method, employing a pure kernel function, yielded the highest performance. This was followed by a simple fractional approach, with the classical method performing last. Also, in this investigation [13], when the population's carrying capacity is significantly high and experiences restricted growth, the logistic and exponential approaches align. Nevertheless, these methods may not be appropriate for modeling the growth of such populations. In both scenarios, the ψ-Caputo method minimized error.

The literature presents various methods for the acquisition of both analytical and computation solutions for the defeenet type of ψ-FDEs, such as [31] introduce a wavelet method for the solution of linear and nonlinear ψ-Caputo fractional initial and boundary value problem, also paper [11] introduces a methodology for examining finite-approximate controllability in Hilbert spaces for linear/semilinear v-Caputo fractional evolution equations, in paper [10], the authors examine adequate conditions for both the existence and stability of solutions to a coupled system involving ψ-Caputo hybrid fractional derivatives. These derivatives have orders ranging from 1 to 2 and are subjected to Dirichlet boundary conditions. And in [38] They introduce a new Pharmacokinetic/ Pharmacodynamic (PK/PD) model specifically for the induction phase of anesthesia, which incorporates the ψ-Caputo fractional derivative. This model utilizes the Picard iterative process.

Moreover, the model of study in this manuscript is the relaxation-oscillation equation which models the dynamics of fluid flow and oscillation processes. The relaxation oscillator is an oscillator type reliant on the behavior of a given phenomenon to go back to an equilibrium state after experiencing a disturbance or distribution. In addition, this model stands as the fundamental equation governing the processes of relaxation and oscillation.

Thus, the equation that presides over the relaxation process is given by the following differential equation [17]

$$\frac{dy}{dx} + \lambda y(x) = f(x), \tag{1}$$

where λ is the material constant, while the function $f(x)$ represents strain rate. Moreover, when $f(x) = 0$, the equation admits the following exact solution $y(x) = Ce^{-\lambda x}$, with C serving as an arbitrary constant to be determined with the imposed initial data. In the same direction, the governing equation presiding over the oscillation process is considered as follows

$$\frac{d^2y}{dx^2} + \lambda y(x) = f(x), \tag{2}$$

where $\lambda = \frac{k}{m}$, k denotes the stiffness coefficient, while m represents the mass of the media. Equally, when $f(x) = 0$, the exact solution of the equation is found to be $y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$, where A and B are constants to be obtained from the constrained initial data.

Now, the concept of fractional calculus has allowed us to simultaneously get a sense of the relaxation and oscillation processes using one fractional differential equation with a varying fractional-order derivative. Thus, to simultaneously account for slow relaxation and damped oscillation processes, the two equations above are now combined to result in the fractional relaxation-oscillation model as follows

$$D^{\alpha}y(x) + \lambda y(x) = f(x), \qquad \alpha \in (0,2)\backslash\{1\}, \qquad x > 0, \tag{3}$$

where D^{α} is a given fractional-order derivative. In addition, the relaxation process requires the following initial condition

$$y(0) = y_0, \qquad \text{when} \qquad 0 < \alpha < 1, \tag{4}$$

while the oscillation process needs the following initial data

$$y(0) = y_0,\ y'(0) = y_1, \qquad \text{when} \qquad 1 < \alpha < 2. \tag{5}$$

In summary, considering the Initial-Value Problem (IVP) (3)-(5), one may see that for

- $0 < \alpha < 1$, the IVP (3) and (4) represents the relaxation process with attenuation power law,
- $1 < \alpha < 2$, the IVP (3) and (5) describes the damped oscillation process with viscoelastic intrinsic oscillator damping.

This model has been applied to various processes, including the description of cardiac pacemakers [17], predator-prey systems [15] and [24, 30] among others. In addition, as the fractional calculus plays a vital role in unraveling some of the salient features associated with certain physical models, different scientists have equally deployed various approaches to construct different numerical solutions for the fractional relaxation-oscillation model, including the application of residual power series method [9], block-by-block method [40] and the B-spline cubic wavelet collocation approach [18] to mention a few.

However, motivated by the given studies above and the quest for unraveling some hidden properties embedded in the fractional relaxation-oscillation equation, the need for ψ-Caputo fractional relaxation-oscillation IVP thus aroused, which is modeled to have the following expression

$$D_a^{\alpha,\psi} y(x) + \lambda y(x) = f(x), \quad \max\{n-1, \frac{1}{2}\} < \alpha < n, \ n \in \mathbb{N}, \quad a < x < b, \tag{6}$$

under the initial conditions

$$(\delta_\psi)^k y(a) = y_k, \quad k = 0, 1, \ldots, n-1, \tag{7}$$

where $D_a^{\alpha,\psi}$ is the fractional derivative defined using the ψ-Caputo fractional of order α, with both $\psi, f : [a, b] \to [0, 1]$, where ψ is a non-decreasing function in $C^n(I)$ such that $\psi(I) = [0, 1]$, and $\psi'(x) > 0$; while λ is a non-zero real constant.

Moreover, for $k = 0, 1, \ldots, n-1$,

$$(\delta_\psi)^k y(x) = \begin{cases} y(x), & \text{when } k = 0, \\ \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^k y(x), & \text{when } k = 1, 2, \ldots, n-1. \end{cases}$$

In addition, the problem (6)-(7) has been examined in [7] using the operational matrix of ψ-shifted Legendre polynomials and in [35] by using the ψ-Haar Wavelet operational matrix method. Similarly, when $0 < \alpha < 1$, the ψ-fractional IVP represents the relaxation process with power law attenuation; while when $1 < \alpha < 2$, the ψ-fractional IVP describes the damped oscillation development with viscoelastic intrinsic oscillator damping. In addition, we state some of the extensions of the governing model to have application in immersed spheres in fluid and coupled relaxation-oscillation equations in [39] and [21], respectively, among others, where the authors used variants of coupling between the Laplace transform and other mathematical infusions.

However, motivated by the immense application of the governing model and the burning relevance of the ψ-Caputo fractional derivative, the present manuscript thus aimed at developing a technique for rapidly approximating solutions to a particular fractional relaxation-oscillation equation that involves the ψ-Caputo derivative. To achieve this, we employ the Adomian Decomposition Method (ADM), a robust and extensively utilized approach. Since its inception in 1980, ADM has been widely applied to solving both nonlinear and linear problems across diverse domains. Recently, it has gained significant traction as an effective tool for tackling a broad spectrum of Fractional Differential Equations (FDEs), comprising for instance, the study of FDEs with modification procedures of ADM [16], the examination of FDEs with augmented ADM [3], the analysis of fractional Bernoulli's equation via ADM [25] and the scrutinizing heat transfer process wing fractional including with the help of ADM [29] to mention but a few. The Adomian decomposition method has garnered growing interest in examining both its convergence and the stability of its solutions. Numerous studies have delved into investigating convergence and stability, as exemplified here, without reiterating the same information see [1, 2, 20, 37].

Lastly, the organization of the manuscript goes in the following pattern: In the first section, the introduction is given. In section 2, we present some preliminaries and essential definitions which are used to carry out our work. We derive an explicit scheme for treating ψ-Caputo fractional relaxation-oscillation models based on ADM in section 3. Section 4 presents several numerical examples showcasing the correctness and effectiveness of the devised approach. Finally, the last section provides some concluding notes.

2. Basic definitions

In the current section, we provide some essential definitions of ψ-fractional operators along with their fundamental properties.

Definition 1.
If $y : I \to \mathbb{R}$ is an integrable function, where $I = [a, b]$ and $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $\psi(x) \in C^n(I)$ upon which $\psi'(x) \neq 0\ \forall x \in I$. Then, the fractional integral and fractional derivatives of order $\alpha > 0$ of the function y with regard to another function ψ are defined as follows [5]

$$I_a^{\alpha,\psi} y(x) := \{\Gamma(\alpha)\}^{-1} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1+\alpha} y(t)dt, \tag{8}$$

and

$$\begin{aligned} D_{a^+}^{\alpha,\psi} y(x) &:= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{n-\alpha,\psi} y(x), \\ &= \{\Gamma(n-\alpha)\}^{-1} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-1-\alpha+n} y(t)dt, \end{aligned}$$

sequentially, with $n = 1 + [\alpha]$.

Definition 2.
Given the interval $I = [a, b]$ with $\alpha > 0$, $n \in \mathbb{N}$. The ψ-Caputo fractional derivative of order $\alpha > 0$ of the function y is defined as follows [5]

$$\begin{aligned} {}^C D_a^{\alpha,\psi} y(x) &= I_{a^+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n y(x), \\ &= \{\Gamma(n-\alpha)\}^{-1} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{n-\alpha-1} y_\psi^{[n]}(t)dt, \end{aligned}$$

where both the functions $\psi(x)$ and $y \in C^n(I)$ with the condition that ψ is increasing and $\psi'(x) \neq 0\ \forall x \in I$. Further,

$$\begin{cases} n = \alpha, & \textit{for} \quad \alpha \in \mathbb{N}, \\ n = 1 + [\alpha], & \textit{for} \quad \alpha \notin \mathbb{N}. \end{cases}$$

Remarkably, the definition above reduces to some well-known fractional operators upon choosing particular cases for the function $\psi(x)$ as follows [5]:

- $\psi(x) = x$ refers to the classical Riemann-Liouville and Caputo fractional operators,
- $\psi(x) = \ln(x)$ refers to the classical Hadamard and Caputo-Hadamard fractional operators.

In addition, considering $y(x) = (\psi(x) - \psi(a))^{\beta-1}$ where $\beta \in \mathbb{R}, \beta > n$, $\alpha > 0$, certain vita features for $\psi(x)$-fractional operators are thus deduced as follows [5]:

- $$^{C}D_{a^+}^{\alpha,\psi} y(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (\psi(x) - \psi(a))^{\beta-\alpha-1}, \tag{9}$$
- $$I_a^{\alpha,\psi} y(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (\psi(x) - \psi(a))^{\beta+\alpha-1}, \tag{10}$$
- $$I_a^{\alpha,\psi} \left(^{C}D_a^{\alpha,\psi} y(x) \right) = y(x) - \sum_{k=0}^{n-1} \frac{y_\psi^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k. \tag{11}$$

  Moreover, when $\alpha \in (0,1)$, the last result reduces to

$$I_a^{\alpha,\psi} \left(^{C}D_a^{\alpha,\psi} y(x) \right) = y(x) - y(a). \tag{12}$$

3. Adomian decomposition method for ψ-Caputo fractional relaxation-oscillation equation

The current section makes use of the standard ADM to derive a generalized recursive scheme for solving a fractional relaxation-oscillation equation endowed with the ψ-Caputo derivative. Thus, in doing so, let us again consider the IVP (6)-(7) for the fractional relaxation-oscillation equation involving the arbitrary function ψ as follows

$$D_a^{\alpha,\psi} y(x) + \lambda y(x) = f(x), \quad \max\{n-1, \frac{1}{2}\} < \alpha < n,\ n \in \mathbb{N}, \quad a < x < b, \tag{13}$$

under the following imposed initial conditions

$$(\delta_\psi)^k y(a) = y_k, \quad k = 0, 1, \ldots, n-1. \tag{14}$$

Therefore, to solve the IVP for ψ-Caputo relaxation-oscillation equation expressed in (13)-(14) by the application of ADM, we operate the operator $I_a^{\alpha,\psi}$ on the governing equation to get

$$I_a^{\alpha,\psi}[D_a^{\alpha,\psi} y(x)] + I_a^{\alpha,\psi}[\lambda y(x)] = I_a^{\alpha,\psi}[f(x)]. \tag{15}$$

In addition, when using the information expressed in (11), alongside utilizing the imposed initial conditions, the above ψ-fractional equation takes the following form

$$y(x)-\sum_{k=0}^{n-1}\frac{(\delta_\psi)^k y(a)}{k!}(\psi(x)-\psi(a))^k+I_a^{\alpha,\psi}[\lambda y(x)]=I_a^{\alpha,\psi}[f(x)], \tag{16}$$

or alternatively

$$y(x)=\sum_{k=0}^{n-1}\frac{(\delta_\psi)^k y(a)}{k!}(\psi(x)-\psi(a))^k-\lambda I_a^{\alpha,\psi}[y(x)]+I_a^{\alpha,\psi}[f(x)]. \tag{17}$$

Further, the standard ADM defines the solution $y(x)$ by the following infinite series

$$y(x)=\sum_{n=0}^{\infty}y_n(x), \tag{18}$$

which when substituted into (17) yields the following

$$\sum_{n=0}^{\infty}y_n(x)=\sum_{k=0}^{n-1}\frac{(\delta_\psi)^k y(a)}{k!}(\psi(x)-\psi(a))^k+I_a^{\alpha,\psi}[f(x)]-\lambda I_a^{\alpha,\psi}[\sum_{n=0}^{\infty}y_n(x)], \tag{19}$$

and admitting the generalized recurrent relation as follows

$$\begin{cases} y_0(x)=\sum_{k=0}^{n-1}\frac{(\delta_\psi)^k y(a)}{k!}(\psi(x)-\psi(a))^k+I_a^{\alpha,\psi}[f(x)],\\ y_n(x)=-\lambda I_a^{\alpha,\psi}[y_{n-1}(x)],\ n\geq 1. \end{cases} \tag{20}$$

Additionally, many researchers have formally shown that when an exact solution exists, the resulting series converges remarkably fast to the available exact solution. All of $y_n(x)$ are calculable, and $y(x)=\sum_{n=0}^{\infty}y_n(x)$. Indeed, as the series converges very quickly, the n-term partial sum

$$\phi_n(x)=\sum_{i=0}^{n-1}y_i(x), \tag{21}$$

serves as an exact solution with the closed-form solution taking the following form

$$y(x)=\lim_{n\to\infty}\phi_n(x)=\sum_{i=0}^{\infty}y_i(x). \tag{22}$$

4. Numerical applications

The current section assesses the competency of the proposed scheme for the solution of ψ-Caputo relaxation-oscillation equation via fractional IVPs involving various forms of strain rate functions $\psi(x)$. Besides, the section also attempts to graphically portray some of the acquired solutions for visualization.

Example 1. *Consider the IVP for ψ-fractional oscillator equation [7]*

$$D_0^{3/2,\psi} y(x) + \frac{2}{\Gamma(3/2)} y(x) = \frac{2}{\Gamma(3/2)} (\sqrt{\psi(x)})(1 + (\psi(x))^{3/2}), \quad x \in I = [0, 1], \tag{23}$$

subject to the following constrained initial condition

$$y(0) = y'(0) = 0. \tag{24}$$

Remarkably, the present ψ-fractional problem in (23)-(24) *has a unique exact solution as (cf [8])*

$$y^*(x) = (\psi(x))^2, \quad x \in I. \tag{25}$$

To solve the governing ψ-fractional IVP with the help of ADM, we operate both sides of (23) by the ψ-fractional integral $I_0^{3/2,\psi}$ to obtain

$$I_0^{3/2,\psi}\left[D_0^{3/2,\psi} y(x)\right] + I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} y(x)\right] = I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} (\sqrt{\psi(x)})(1 + (\psi(x))^{3/2})\right]. \tag{26}$$

Next, upon using the result of (11) and the initial conditions in (24), one obtains from the latter equation the following

$$y(x) + I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} y(x)\right] = I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} (\sqrt{\psi(x)})(1 + (\psi(x))^{3/2})\right], \tag{27}$$

or equally

$$y(x) = -I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} y(x)\right] + I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} (\sqrt{\psi(x)})(1 + (\psi(x))^{3/2})\right]. \tag{28}$$

Now, on deploying the ADM procedure, the last equation becomes

$$\sum_{n=0}^{\infty} y_n(x) = -I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} \sum_{n=0}^{\infty} y_n(x)\right] + I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} (\sqrt{\psi(x)})(1 + (\psi(x))^{3/2})\right], \tag{29}$$

which eventually leads to the acquisition of the resulting recurrent scheme as follows

$$\begin{cases} y_0(x) = I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} (\sqrt{\psi(x)})(1 + (\psi(x))^{3/2})\right], \\ y_n(x) = -I_0^{3/2,\psi}\left[\frac{2}{\Gamma(3/2)} y_{n-1}(x)\right], \quad n \geq 1. \end{cases} \tag{30}$$

Besides, when the above recurrent scheme is expressed, some of the few components from the scheme are obtained as follows

$$y_0(x) = \big(\psi(x)\big)^2 + \frac{128\big(\psi(x)\big)^{\frac{7}{2}}}{105\pi},$$

$$y_1(x) = -\frac{128\big(\psi(x)\big)^{\frac{7}{2}}}{105\pi} - \frac{4\big(\psi(x)\big)^5}{15\pi},$$
$$\vdots$$

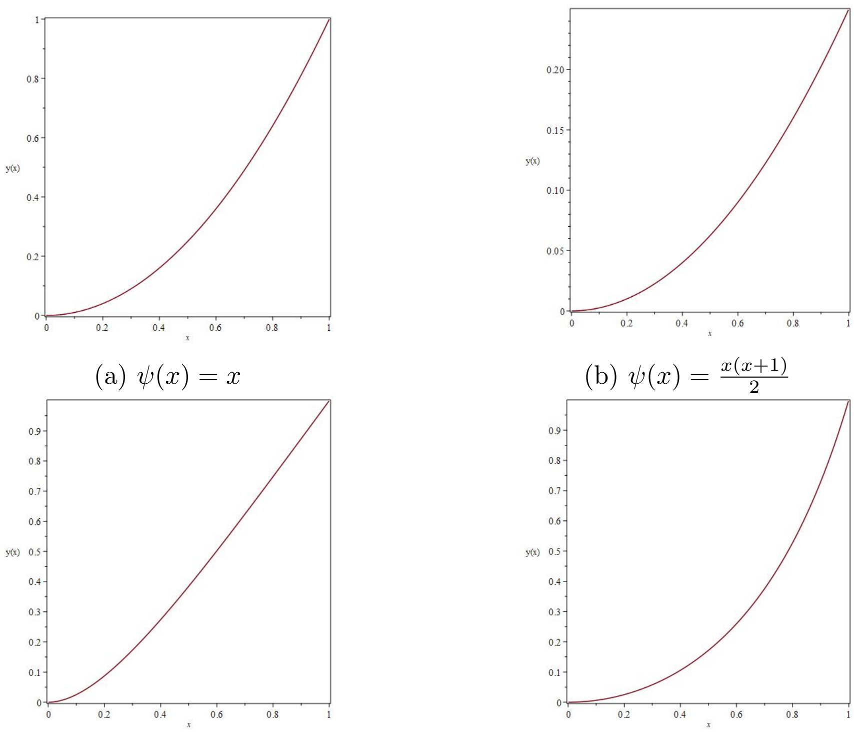
Remarkably, it is worth noting here that some noise terms arise in the recurrent solution of the present model, particularly, concerning $y_0(x)$ and $y_1(x)$ components as $\pm\big(\frac{128(\psi(x))^{\frac{7}{2}}}{105\pi}\big)$. Thus, when adding just these two components, the noise terms will eventually cancel out; thereby leaving behind the exact solution of the problem (23)-(24) as follows

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = \big(\psi(x)\big)^2. \tag{31}$$

Indeed, for certain special cases of interest concerning the choice of the function ψ, we present the following solution cases

$$\begin{cases} \text{when} \quad \psi(x) = x, \quad \text{then} \quad y(x) = x^2, \\ \text{when} \quad \psi(x) = \frac{x}{2}(x+1), \quad \text{then} \quad y(x) = \big(\frac{x}{2}(x+1)\big)^2, \\ \text{when} \quad \psi(x) = \ln((e-1)x+1), \quad \text{then} \quad y(x) = \big(\ln((e-1)x+1)\big)^2, \\ \text{when} \quad \psi(x) = \tan(\frac{\pi x}{4}), \quad \text{then} \quad y(x) = \big(\tan(\frac{\pi x}{4})\big)^2, \end{cases} \tag{32}$$

with the case of $\psi(x) = x$ featuring the state of the Caputo fractional derivative. Moreover, Almeida et al. [7] solved the linear version of the model expressed in (23)-(24) through the application of operational matrix by introduce shifted Legendre polynomial and obtained approximate solution. However, the ADM which is a semi-analytic method gives us a noise term which reveals the exact solution while using only 2 terms and that offers an advantage because it tends to converge to exact solution more effectively compared to numerical methods, Consequently, it leads to a reduction in computational complexity. In addition, we have shown in Figure 1, the graphical presentation of the acquired ψ-ADM solution for different kernels for the ψ-fractional-order IVP (23)-(24).



(a) $\psi(x) = x$ (b) $\psi(x) = \frac{x(x+1)}{2}$

(c) $\psi(x) = \ln((e-1)x+1)$ (d) $\psi(x) = \tan(\frac{\pi x}{4})$

Figure 1: ADM solutions with respect to the different kernels for the ψ-fractional IVP (23)-(24).

Example 2. *Consider the IVP for ψ-fractional relaxation-oscillation equation [35]*

$$D^{\alpha,\psi}y(x)+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}y(x) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\psi(x))^{\alpha}\left[1+(\psi(x))^{\alpha}\right],\ 0<\alpha\leq 1,\ x\in[0,1], \quad (33)$$

together with the following initial data

$$y(0) = 0. \quad (34)$$

The exact analytical solution for the model (33)-(34) *is established as follows*

$$y^{*}(x) = (\psi(x))^{2\alpha}, \quad x \in [0,1]. \quad (35)$$

To solve problem (33)-(34) with the aid the ADM, we operate the operator $I^{\alpha,\psi}$ on both sides of the governing equation to obtain

$$I^{\alpha,\psi}\left[D^{\alpha,\psi}y(x)\right]+I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}y(x)\right] = I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\psi(x))^{\alpha}\left[1+(\psi(x))^{\alpha}\right]\right]. \quad (36)$$

In addition, upon using (11), and the given initial condition, the latter equation becomes

$$y(x) = -I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}y(x)\right]+I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\psi(x))^{\alpha}\left[1+(\psi(x))^{\alpha}\right]\right],. \quad (37)$$

In addition, the standard ADM procedure necessitates the above equation to be expressed in series form as follows

$$\sum_{n=0}^{\infty} y_n(x) = -I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty} y_n(x)\right] + I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\psi(x))^{\alpha}\left[1+(\psi(x))^{\alpha}\right]\right], \quad (38)$$

which then reveals the following formal recursive relation

$$\begin{cases} y_0(x) = I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(\psi(x))^{\alpha}\left[1+(\psi(x))^{\alpha}\right]\right], \\ y_n(x) = -I^{\alpha,\psi}\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}y_{n-1}(x)\right], \quad n \geq 1. \end{cases} \quad (39)$$

Therefore, when the above recurrent scheme is expressed, some of the few components from the scheme are obtained as follows

$$\begin{aligned} y_0(x) &= (\psi(x))^{2\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}(\psi(x))^{3\alpha}, \\ y_1(x) &= -\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}(\psi(x))^{3\alpha} - \frac{\Gamma(2\alpha+1)^3}{\Gamma(\alpha+1)^2\Gamma(4\alpha+1)}(\psi(x))^{4\alpha}. \\ &\vdots \end{aligned}$$

Notably, noise terms $\pm\left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}(\psi(x))^{3\alpha}\right)$, arise between the components $y_0(x)$ and $y_1(x)$, upon which when cancelled gives the exact solution for the problem (33)-(34) as follows

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = (\psi(x))^{2\alpha}. \quad (40)$$

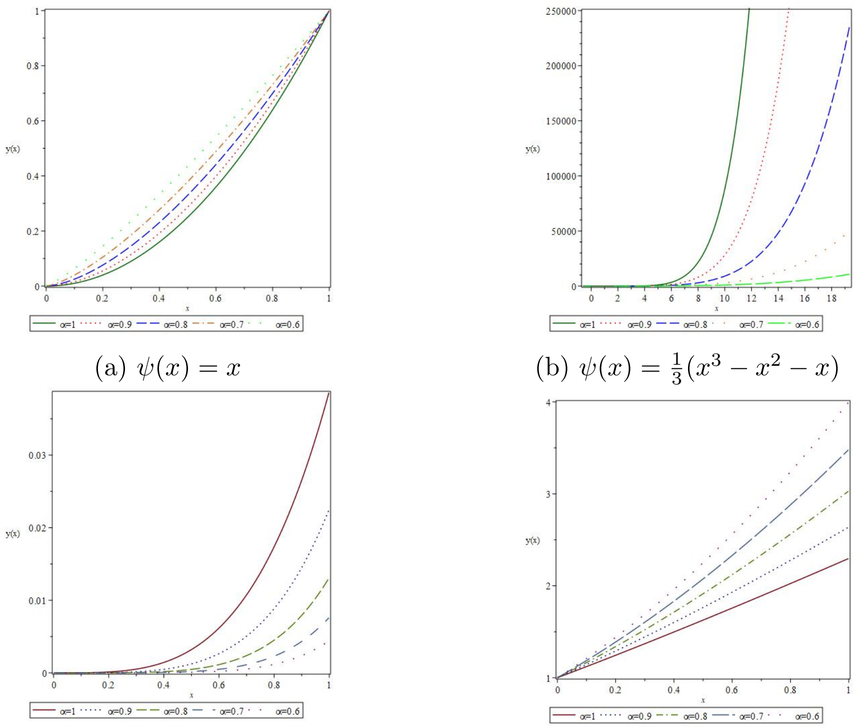
Accordingly, for certain special cases of interest concerning the choice of the function ψ, we present the following solution cases

$$\begin{cases} \text{when } \ \psi(x) = x, \ \text{ then } \ y(x) = (x)^{2\alpha}, \\ \text{when } \ \psi(x) = \frac{1}{3}(x^3 - x^2 - x), \ \text{ then } \ y(x) = \left(\frac{1}{3}(x^3 - x^2 - x)\right)^{2\alpha}, \\ \text{when } \ \psi(x) = \frac{x^3}{15}, \ \text{ then } \ y(x) = \left(\frac{x^3}{15}\right)^{2\alpha}, \\ \text{when } \ \psi(x) = x+1, \ \text{ then } \ y(x) = \left(x+1\right)^{2\alpha}, \end{cases} \quad (41)$$

with the case of $\psi(x) = x$ featuring the state of the Caputo fractional derivative.

Consequently, Sunthrayuth et al. [35] equally solved the linear version of the ψ-fractional model expressed using the ψ-Haar wavelet operational matrix method, allows us to obtain an approximate solution. However, the ADM which is a semi-analytic method gives us a noise term which reveals the exact solution while using only 2 terms and that offers an advantage because it tends to converge to exact solution more effectively compared to numerical methods, Consequently, it leads to a reduction in computational complexity.

In addition, we report in Figure 2 ADM solutions for different choices of α and ψ for the governing ψ-fractional IVP (33)-(34).



(a) $\psi(x) = x$ (b) $\psi(x) = \frac{1}{3}(x^3 - x^2 - x)$

(c) $\psi(x) = \frac{x^3}{15}$ (d) $\psi(x) = x + 1$

Figure 2: ADM solutions for different choices of α and ψ for the ψ-fractional IVP (33)-(34).

Example 3. *Consider the IVP for ψ-fractional relaxation-oscillation equation [35]*

$$D^{\alpha,\psi}y(x)+y(x) = 1-4\psi(x)+5(\psi(x))^2-\frac{4}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha}+\frac{10}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha}, \quad (42)$$

where $0 < \alpha \leq 1$, together with the following initial data

$$y(0) = 1. \quad (43)$$

In addition, the exact analytical solution of the present model (42)-(43) *can be found to be*

$$y^*(x) = 1 - 4\psi(x) + 5(\psi(x))^2, \quad x \in [0, 1]. \quad (44)$$

To solve problem (42)-(43) with the help of ADM, we operate the fractional operator $I^{\alpha,\psi}$ on both sides of the governing equation to obtain

$$I^{\alpha,\psi}\Big[D^{\alpha,\psi}y(x)\Big]+I^{\alpha,\psi}\Big[y(x)\Big] = I^{\alpha,\psi}\Big[1-4\psi(x)+5(\psi(x))^2-\frac{4}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha}+\frac{10}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha}\Big].$$

Equally, when using (11) and applying the related initial condition, one obtains

$$y(x) = 1-I^{\alpha,\psi}\Big[y(x)\Big]+I^{\alpha,\psi}\Big[1-4\psi(x)+5(\psi(x))^2-\frac{4}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha}+\frac{10}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha}\Big].$$

Now, on using ADM process, the above equation is re-expressed as follows

$$\sum_{n=0}^{\infty} y_n(x) = 1-I^{\alpha,\psi}\Big[\sum_{n=0}^{\infty} y_n(x)\Big]+I^{\alpha,\psi}\Big[1-4\psi(x)+5(\psi(x))^2-\frac{4}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha}+\frac{10}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha}\Big],$$

that then leads to the recurrent scheme for the model as follows

$$\begin{cases} y_0(x) = 1 + I^{\alpha,\psi}\Big[1 - 4\psi(x) + 5(\psi(x))^2 - \frac{4}{\Gamma(2-\alpha)}(\psi(x))^{1-\alpha} + \frac{10}{\Gamma(3-\alpha)}(\psi(x))^{2-\alpha}\Big], \\ y_n(x) = -I^{\alpha,\psi}\Big[y_{n-1}(x)\Big], \quad n \geq 1. \end{cases} \tag{45}$$

As proceed, some of the components for the solution of the model are obtained from the above ADM schemes as follows

$$\begin{aligned} y_0(x) &= \frac{(\psi(x))^{\alpha}}{\Gamma(\alpha+1)} - \frac{4(\psi(x))^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{10(\psi(x))^{\alpha+2}}{\Gamma(3+\alpha)} - 4(\psi(x)) + 5(\psi(x))^2 + 1, \\ y_1(x) &= -\frac{(\psi(x))^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4(\psi(x))^{2\alpha+1}}{\Gamma(2+2\alpha)} - \frac{10(\psi(x))^{2\alpha+2}}{\Gamma(3+2\alpha)} - \frac{(\psi(x))^{\alpha}}{\Gamma(\alpha+1)} + \frac{4(\psi(x))^{\alpha+1}}{\Gamma(2+\alpha)} - \frac{10(\psi(x))^{\alpha+2}}{\Gamma(3+\alpha)}, \vdots \end{aligned}$$

Accordingly, upon cancelling the noise terms $\pm\left(\frac{(\psi(x))^{\alpha}}{\Gamma(\alpha+1)} - \frac{4(\psi(x))^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{10(\psi(x))^{\alpha+2}}{\Gamma(3+\alpha)}\right)$ between the components $y_0(x)$ and $y_1(x)$, and justify that the remaining terms of $y_0(x)$ that satisfies the imposed initial data, one obtained the resultant exact solution of the model as follows

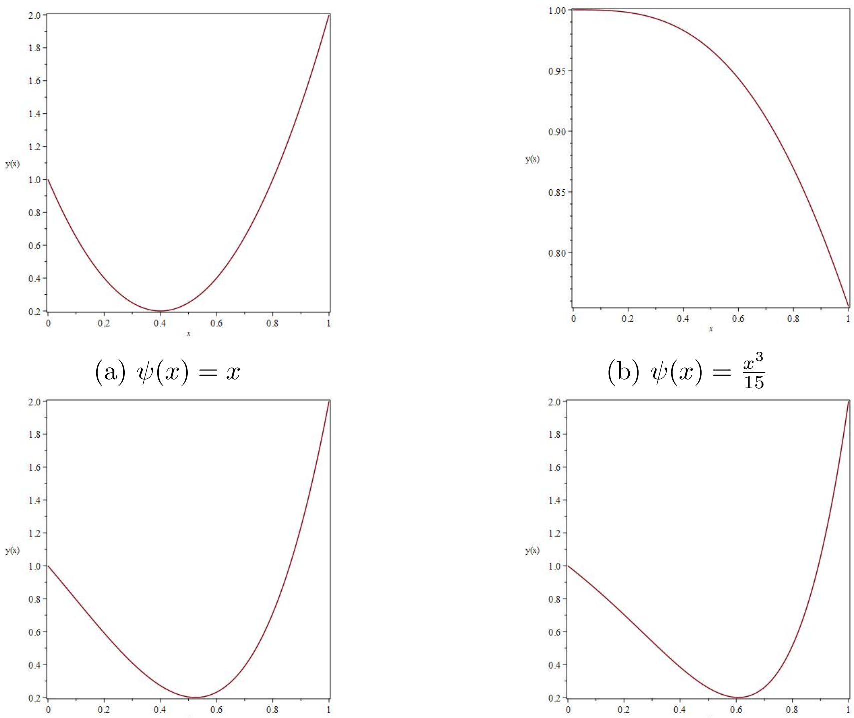
$$y(x) = \sum_{n=0}^{\infty} y_n(x) = 1 + 5(\psi(x))^2 - 4(\psi(x)). \tag{46}$$

In the same vein, for certain special cases of interest with regard to the choice of the function ψ, we present the following solution cases

$$\begin{cases} \text{when } \psi(x) = x, \text{ then } y(x) = 1 - 4x + 5x^2, \\ \text{when } \psi(x) = \frac{x^3}{15}, \text{ then } y(x) = 1 - 4\big(\frac{x^3}{15}\big) + 5\big(\frac{x^3}{15}\big)^2, \\ \text{when } \psi(x) = \frac{1}{2}(x^2+x), \text{ then } 1 - 4(\frac{1}{2}(x^2+x)) + 5(\frac{1}{2}(x^2+x))^2, \\ \text{when } \psi(x) = \frac{1}{3}(x^3+x^2+x), \text{ then } y(x) = 1 - 4(\frac{1}{3}(x^3+x^2+x)) + 5(\frac{1}{3}(x^3+x^2+x))^2, \end{cases} \tag{47}$$

with the case of $\psi(x) = x$ featuring the state of the Caputo fractional derivative.

Conspicuously, Sunthrayuth et al. [35] similarly solved the linear version of the present model with the help of the ψ-Haar wavelet operational matrix method, allows us to obtain an approximate solution. On the other hand,the ADM which is a semi-analytic method gives us a noise term which reveals the exact solution while using only 2 terms and that offers an advantage because it tends to converge to exact solution more effectively compared to numerical methods, Consequently, it leads to a reduction in computational complexity; see Figure 3 for the graphical depiction of the obtained ADM solutions for different choices of ψ function for the ψ-fractional IVP (42)-(43).



(a) $\psi(x) = x$ (b) $\psi(x) = \frac{x^3}{15}$

(c) $\psi(x) = \frac{1}{2}(x^2 + x)$ (d) $\psi(x) = \frac{1}{3}(x^3 + x^2 + x)$

Figure 3: ADM solutions for different choices of ψ for the ψ-fractional IVP (42)-(43).

5. Conclusion

In the present study, the standard ADM has been adopted to derive a generalized recurrent scheme for the solution of fractional relaxation-oscillation equation involving ψ-Caputo derivative. Indeed, the study showcased the accuracy and efficiency of the adopted methodology by testing it with several numerical examples.Our method offers an advantage in that it tends to converge towards the exact solution more effectively compared to numerical methods, thereby reducing the computational workload. Hence, the study ends by recommending that the implored method should be extended to more complex physical models of real-life application.

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