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Analysis and applications of the chaotic hyperbolic memristor model with fractional order derivative

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Abstract. This work presents a new five-dimensional fractional-order chaotic system that includes a feedback memristor. A Lorenz-Stenflo-based fractional-order chaotic system with five dimensions that contains feedback memristor dynamics is used to do this. This work focuses on chaotic models through the ABC fractional derivative, a concept we rigorously examine. We designed and used sophisticated numerical algorithms to model fractional-order dynamics with the utmost precision to understand this system's complex characteristics. These numerical approaches excel at non-integer order differentiation, which standard numerical methods struggle with. Our research uses fractional calculus to increase the system's complexity and robustness, revealing its hyperchaotic nature. We demonstrate that the system is chaotic and stable over fractional orders using eigenvalues, Lyapunov exponents, Kaplan-Yorke dimensions, maximal exponents, phase portraits, and equilibrium points. Our conclusions are strengthened by using these numerical systems, intended for this work, and analytical tools. We improve our understanding of fractional-order chaotic systems with our research. It illuminates how to improve electrical integrations and create more sophisticated nonlinear dynamical systems. This work establishes the approach and insights for future research, guiding the development of new systems with enhanced dynamical features

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Key Words and Phrases: Fractional model, Lorenz-Stenflo model, Hyperchaotic behavior, Nonlinear dynamical systems, Fractional derivative, Simulation.

1. Introduction

The history of non-integer order systems in science and technology,which has been heavily influenced by fractional derivatives, has had a significant impact on the emergence of chaotic systems [37].Fractional derivatives are used to explain and resolve integral equations as well as ordinary and partial differential equations as an extension of classical derivatives. Atangana-Baleanu, Caputo, Letnikov, Hadamard, Marchaud, Weyl, and Coimbra are just a few of the well-known definitions of fractional derivatives[38].

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Fractional derivatives also have a wide range of multidisciplinary applications. The consequences of the modified Chua's system explored by Tom et al. using fractional derivatives are only one example of the various current contributions of fractional derivatives in the field of fractional index chaotic systems[18].Recently, chaos dynamics control has been more well-liked due to a new trend of studying chaotic systems utilizing various mathematical and physical models, such as the Chen system with fractional order investigated by Chunguang and Guanrong[29].The study of chaotic regimes and the creation of low-power systems with rapid processing units both heavily rely on digital/analogue electronics and fractional index systems. We provide digital systems and their component viewpoints for a cubic nonlinear resistor system's fractional index, including adaptive controllers with FPGA implementation. The novel chaotic system introduced in this work is ingeniously built upon the rich foundation provided by the Lorenz-Stenflo model, a model that holds a prestigious place in the annals of chaos theory for its profound insights into the dynamics of complex systems. The original Lorenz system, from which the Lorenz-Stenflo model evolved, emerged as a cornerstone in the study of atmospheric convection, unveiling the intricate dance of determinism and unpredictability that characterises chaotic systems. This model's fame stems from its vivid illustration of the "butterfly effect," a metaphor that captures the essence of chaos, where minute variations in initial conditions can lead to dramatically divergent outcomes, underscoring the inherent unpredictability of certain natural systems. Expanding upon the groundwork laid by references [36], our investigation enriches the Lorenz-Stenflo framework by integrating the concept of fractional calculus, a mathematical approach that extends the notion of derivatives and integrals to non-integer orders. This extension into fractional-order systems introduces a new dimension of complexity and flexibility, allowing for a more nuanced exploration of dynamical behaviours that are closer to those observed in natural phenomena. Specifically, we employ a generalised Liouville-Caputo-type fractional derivative operator, marking a significant departure from traditional integer-order models. Through a meticulously formulated system of five fractional-order differential equations, we unveil a novel chaotic system that not only inherits the sensitive dependence on initial conditions characteristic of its predecessors but also exhibits a rich tapestry of dynamical features unique to fractional-order systems. This innovative approach to modelling chaos through fractional calculus opens up a plethora of possibilities for capturing the subtleties of real-world systems, where the effects of memory and hereditary properties are paramount. By venturing into the domain of fractional-order chaotic systems, we aim to bridge the gap between theoretical models and the complex dynamics observed in nature, providing a more accurate and versatile framework for understanding and harnessing the potential of chaos. This extended introduction sets the stage for a comprehensive exploration of the proposed system, highlighting its theoretical significance, practical implications, and the pioneering contributions it aims to make to the field of chaos theory and its applications. The integration of chaotic dynamics into the Lorenz-Stenflo model represents a significant stride in the exploration of complex systems within the realm of nonlinear dynamics and chaos theory. The Lorenz-Stenflo model, originally derived to describe atmospheric convection and later adapted to various physical phenomena, serves as a foundational framework for studying the intricate behaviour of dynamical systems. Recent advancements have focused on augmenting this classic model with memristive components and fractional-order calculus to introduce novel chaotic systems that exhibit a wider spectrum of dynamical behaviors. This fusion not only enriches the model's capability to mimic real-world processes with higher fidelity but also opens new avenues for applications in secure communications, computational neuroscience, and adaptive control systems. Studies leveraging the Lorenz-Stenflo model as a base have revealed that the incorporation of memristors and fractional derivatives can significantly enhance system complexity, leading to the discovery of unique attractors, bifurcation phenomena, and enhanced sensitivity to initial conditions. These contributions highlight the model's versatility and its potential for pioneering research in chaos theory, demonstrating a growing interest in developing more sophisticated and applicable chaotic systems. In the exploration of nonlinear dynamics and chaos theory, the integration of chaotic dynamics into established models like Lorenz-Stenflo has opened new avenues for understanding the intricate behaviours of complex systems. Recent studies have significantly contributed to this field by introducing memristive components and fractional-order calculus, thereby expanding the traditional frameworks used to describe such phenomena. For instance, research documented in [20, 22] delves into the applications of fractional calculus to enhance the dynamical range of chaotic systems. Similarly, the works represented by [8, 10], along with notable contributions from Hasan et al. [24] in the International Journal of Mathematical Engineering and Management Sciences and further studies by Abdoon et al. [2] in Mathematical Modelling of Engineering Problems, highlight the advanced mathematical techniques for solving nonlinear differential equations and their pivotal roles in chaos theory. These references collectively underscore the evolving complexity of modelling chaotic systems and the ongoing quest for models that more accurately reflect the multifaceted nature of the universe, thus paving the way for groundbreaking applications in science and engineering.

In this research, novel chaotic attractors that are both in excellent agreement with one another and numerically represented using 2D phase portrait diagrams are described. Different mathematical techniques were used to examine the system dynamics for various values of the fractional order q. The system's stability is examined using equilibrium points, Eigenvalues, the Routh-Hurwitz stability criterion, Lyapunov exponents, and the Kaplan-Yorke dimension. The results reveal that the system is chaotic in nature and capable of producing random numbers. Additionally, phase portraits and three-dimensional graphs are used to graphically represent the system's characteristic curves.Finally,Lyapunov spectra and bifurcation graphs are shown, correspondingly, for the time domain and system parameters. The unique system was implemented in a circuit after a thorough mathematical examination of the system.

In the pursuit of advancing the study of chaotic systems, this research aims to delineate the unique characteristics and behaviours of a novel five-dimensional fractional-order chaotic system. This system distinguishes itself by incorporating the dynamics of a feedback memristor based on the Lorenz-Stenflo model. The primary objective of our investigation is to dissect the complexities introduced by fractional calculus in chaotic systems, thereby unveiling the hyperchaotic nature that emerges from such intricate dynamics. The introduction of fractional calculus not only intensifies the complexity but also augments the robustness against perturbations, offering a richer tapestry of dynamical behaviours to explore. The novelty of our approach lies in the integration of fractional calculus within the five-dimensional framework, a method not extensively explored in contemporary literature. This innovative melding of concepts is poised to yield insights into the stability and chaotic behaviour of higher-dimensional systems when subjected to fractional orders. By deploying a combination of analytical and numerical tools—specifically tailored for this research—we aim to thoroughly scrutinize the system's eigenvalues, Lyapunov exponents, Kaplan-Yorke dimension, maximal Lyapunov exponents, phase portraits, and equilibrium points. The detailed examination of these aspects under various fractional orders will contribute significantly to the understanding of the system's stability and chaos. This study is not merely an academic exercise but a step towards practical applications. By demonstrating the chaotic characteristics and stability of the system, we lay the groundwork for future advancements in electronic integration and the conception of more intricate nonlinear dynamical systems. As such, the outcomes of this research are anticipated to catalyse innovation in the realm of chaos theory and its applications across diverse scientific and engineering disciplines. One of the primary challenges lies in the computational domain. Future work could address these challenges by exploring the following avenues: The development of more advanced numerical algorithms that can reduce approximation errors and enhance the efficiency of simulations, making them more suitable for high-dimensional systems and real-time analysis, In-depth exploration of the interactions between memristive properties and fractional-order dynamics, potentially through the lens of other non-integer derivatives or through the adoption of alternative memristor models, Investigating the impact of system parameters on the robustness and sensitivity of the chaotic behaviours observed to better control and harness these dynamics for practical applications see [1, 3, 4, 26–28].

2. Basic Principles

Definition 1. *The Riemann Liouville integral (RLI) order of* $0 < \alpha < 1$ *and* $y(\tau)$ *is provided by [30]:*

$$D^{\alpha}y^{*}\left(t\right)=\frac{1}{\Gamma(n-\zeta)}\int_{0}^{t}\left(t-\tau\right)^{n-\zeta-1}y^{*n}\left(\tau\right)d\tau\ =I^{n-\zeta}y^{*n}\left(t\right),\quad t>0. \tag{1}$$

Definition 2. *The Riemann-Liouville fractional integral of order* $\zeta > 0$*, given by [6]:*

$$I_{a+}^{\zeta}y^{*}(t)=\frac{1}{\Gamma(\zeta)}\int_{a}^{t}(t-s)^{\zeta-1}y(s)ds,\quad t>a. \tag{2}$$

Definition 3. *For a function* $y\left(\tau\right)$ *Caputo derivative of order* $0 < \zeta < 1$ *is given by [19]:*

$$I^{\zeta}y\left(t\right)\ =\frac{1}{\Gamma\ (\zeta)}\int_{0}^{t}\left(t-\tau\right)^{\zeta-1}y\left(\tau\right)d\tau\ ,\quad t>0. \tag{3}$$

Definition 4. *The Mittag-Leffler function can be expressed as follows [13]:*

$$E_{\alpha}^{*}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\zeta k+1)}. \tag{4}$$

Definition 5. *The Lagrange's polynomial interpolation is defined by [17] as*

$$P_n^*(x)=\sum_{i=0}^{n} f\left(x_i^*\right) L_i(x),$$

where

$$L_i\left(x^*\right)=\prod_{j=0,j\neq i}^{n}\frac{x^*-x_i^*}{x_i^*-x_j^*}.$$

Definition 6. *The ABC operator, $y(t)$ in the RLI is given by [16]:*

$$ {}_{0}^{ABC}D_t^{\zeta} y(t)=\frac{B(\zeta)}{1-\zeta}\frac{d}{dt}\int_0^t y^*(\tau)\, E_{\zeta}\left(\frac{\zeta}{1-\zeta}(t-\tau)^{\alpha}\right)d\tau, \quad 0<\zeta<1. \tag{5}$$

Where $B^*(\zeta)$ satisfies the condition $B^*(1)=B^*(0)=1$.

Definition 7. *The memristor emulator is applied to our novel system which has a current, voltage relation as,*

$$I=W^*\left(v_0^*\right)v^*=\left[A-B\tanh\left(v_0^*\right)\right]v^*. \tag{6}$$

3. Numerical scheme for ABC

The goal of this section is to investigate chaotic models in the sense of the ABC fractional derivative. Central to this research is the rigorous examination of chaotic models through the lens of the ABC fractional derivative. To this end, we have developed and employed specialized numerical schemes capable of simulating the nuanced behaviors of our proposed fractional-order chaotic system with high fidelity. The numerical methods selected for this study are designed to address the non-integer order differentiation inherent to fractional calculus, which standard numerical approaches may not handle adequately.

These methods, including the ABC operator approach and the two-step Lagrange polynomial algorithm its renowned for their precision and stability when dealing with fractional differential equations. The former provides a direct discretization of the fractional derivative, while the latter offers an iterative refinement that ensures convergence to the true solution.

Moreover, the analysis of the system's dynamics is bolstered by the use of robust analytical tools. Eigenvalue analysis serves as the cornerstone for assessing the stability of the equilibrium points, offering a clear view of the system's response to infinitesimal perturbations. Concurrently, Lyapunov exponents provide a quantitative measure of the exponential divergence or convergence of nearby trajectories, a hallmark of chaotic behavior. The maximal Lyapunov exponent, in particular, is indicative of the presence of chaos when it is positive, while a spectrum of Lyapunov exponents gives a broader picture of the system's dynamical stability. These analytical instruments are invaluable for the characterization of chaos, as they allow for a detailed exploration of the system's response under varying initial conditions and parameter values.

The combination of these carefully chosen numerical schemes and analytical tools constitutes a robust framework for capturing the dynamic complexities of the proposed chaotic system. Through this dual approach, the study not only ensures the accurate portrayal of the system's behavior but also underscores the intricate interplay between stability and chaos in fractional-order environments, To expand on the given statement and provide more detail as requested, you might consider the following elaboration:

$$\begin{aligned} &{}_{0}^{ABC}D_{0}^{\zeta} v^{*}(t) = f\left(t, v^{*}(t)\right), \\ &v^{*}(0) = v_{0}^{*}. \end{aligned} \tag{7}$$

A fractional integral equation can be derived from the equation above

$$v^{*}(t) - v^{*}(0) = \frac{(1-\zeta) f(t, v^{*}(t))}{ABC(\zeta)} + \frac{\zeta}{\Gamma(\zeta+1)\times ABC(\zeta)} \int_{0}^{t} f(\tau, v^{*}(\tau))(t-\tau)^{\zeta-1} d\tau, \tag{8}$$

where $n = 0, 1, 2, 3 \ldots$, reformulated as

$$\begin{aligned} v^{*}(t_{n+1}) - v^{*}(0) &= \frac{(1-\zeta) f(t_n, v^{*}(t_n))}{ABC(\zeta)} \\ &+ \frac{\zeta}{ABC(\zeta)\times\Gamma(\zeta+1)} \int_{0}^{t_{n+1}} g\left(\tau, v^{*}(\tau)\right)\left(t_{n+1}-\tau\right)^{\zeta-1} d\tau \\ &= \frac{(1-\zeta) g(t_n, v^{*}(t_n))}{ABC(\zeta)} \\ &+ \frac{\zeta}{ABC(\zeta)\times\Gamma(\zeta)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} g(\tau, v^{*}(\tau))\left(t_{n+1}-\tau\right)^{\zeta-1} d\tau. \end{aligned} \tag{9}$$

The following can be approximated using two-step Lagrange polynomial interpolation:

$$\begin{aligned} P_{k}^{*}(\tau) &= \frac{(\tau - t_{k-1}) f\left(t_k, v^{*}(t_k)\right)}{t_k - t_{k-1}} - \frac{(\tau - t_k) f\left(t_{k-1}, v^{*}(t_{k-1})\right)}{t_k - t_{k-1}} \\ &= \frac{f\left(t_k, v^{*}(t_k)\right)(\tau - t_{k-1})}{h} - \frac{f\left(t_{k-1}, v\left(t_{k-1}\right)\right)(\tau - t_k)}{h} \\ &\simeq \frac{f\left(t_k, v_k^{*}\right)(\tau - t_{k-1})}{h} - \frac{f\left(t_{k-1}, v_{k-1}^{*}\right)(\tau - t_k)}{h}, \end{aligned} \tag{10}$$

$$\begin{aligned} v_{n+1}^{*} &= v_0^{*} + \frac{(1-\zeta)}{ABC(\zeta)} g(t_n, v^{*}(t_n)) \\ &+ \frac{\zeta}{ABC(\alpha)\times\Gamma(\zeta)} \sum_{k=0}^{n} \left(\frac{f(t_k, v_k^{*})}{h} \int_{t_k}^{t_{k+1}} (\tau + \ t_{k-1} t)(t_{n+1}-\tau)^{\zeta-1} d\tau \right. \\ &- \left. \frac{f(t_{k-1}, v_{k-1}^{*})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1}-\tau)^{\zeta-1} d\tau \right). \end{aligned} \tag{11}$$

For simplicity

$$A^*_{\zeta,k,1} = \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1} - \tau)^{\zeta-1} d\tau, \tag{12}$$

$$\begin{aligned}
A^*_{\zeta,k,2} &= \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+2} - \tau)^{\zeta-1} d\tau \\
A^*_{\zeta,k,1} &= h^{\zeta+1} \frac{(n+1-k)^{\zeta}(n-k+2+\zeta) - (n-k)^{\zeta}(n-k+2+2\zeta)}{\zeta(\zeta+1)} \\
A^*_{\zeta,k,2} &= (h^{\zeta+1}) \frac{(n+1-k)^{\zeta+1} - (n-k)^{\zeta}(n-k+1+\zeta)}{\zeta(\zeta+1)}.
\end{aligned} \tag{13}$$

By combining equations (12) and (13) and substituting in (11),

$$\begin{aligned}
v^*_{n+1} = v^*(1) &+ \frac{(1-\zeta)}{ABC(\zeta)} f\left(t_n, v^*\left(t_n\right)\right) + \frac{\zeta}{ABC(\zeta)} \\
&\sum_{j=0}^{n} \left(\frac{h^{\zeta} f\left(t_k, v^*_k\right)}{\Gamma(\zeta+1)} \Big((1+n-j)^{\zeta}\left(2+\zeta+n-k\right) + (j-n)^{\zeta}(2+n-k+2\zeta) \Big) \right. \\
&\left. - \frac{h^{\zeta} f\left(t_{j-1}, v^*_{j-1}\right)}{\Gamma(1+\zeta)} \Big((n-j+1)^{\zeta+1} + (j-n)^{\zeta}(n-j+1+\zeta) \Big) \right).
\end{aligned} \tag{14}$$

4. Model of a novel chaotic system

The created system mainly consists of a five-dimensional system with eleven terms and the following state variables:

$$\begin{cases}
{}_{0}^{ABC}D_t^{\zeta}\eta_1(t) = k\left(\alpha + \beta \tanh \eta_5\right)\eta_2 + a(\eta_2 + \eta_1) + r\eta_4, \\
{}_{0}^{ABC}D_t^{\zeta}\eta_2(t) = c\eta_1 - \eta_2 - \eta_1\eta_3, \\
{}_{0}^{ABC}D_t^{\zeta}\eta_3(t) = \eta_1\eta_2 - b\eta_3, \\
{}_{0}^{ABC}D_t^{\zeta}\eta_4(t) = -\eta_1 - a\eta_4, \\
{}_{0}^{ABC}D_t^{\zeta}\eta_5(t) = \eta_2,
\end{cases} \tag{15}$$

with initial conditions:
$\eta_1(0) = 0.2, \eta_2(0) = 0.2, \eta_3(0) = 0.2, \eta_4(0) = 0.2, \eta_5(0) = 0.2$
here, the values of constants are $a = 1, \alpha = 20, \beta = 0.02, c = 23, b = 0.7, \rho = 1.5, k = 4$

5. Analysis of a nonlinear chaotic dynamical system

In this section, we delve into the meticulous evaluation of equilibrium points for system (5) by considering varying fractional order values. This exploration is critical as equilibrium points serve as the cornerstone for understanding the long-term behavior of dynamical systems. By setting the right-hand side of the equation to zero, we embark on a systematic journey to solve the system, revealing the complex interplay between the system's inherent properties and the fractional order values.

Theorem.1 The equilibrium point $E_0$of the system(15) is LAS if all the eigenvaluesλ of the matrix J_E satisfy.

$$\alpha < \frac{2}{\pi}|\arg(\lambda_m)|, m = 1, \ldots, n, \tag{16}$$

Another basic lemma for the stability of fractional-order system(15)is also given as.

Lemma.1 Assume that defines continuous and differentiable function.Let t be a specific instant of time,then for any then

$$\begin{cases} 0 = k\left(\alpha + \beta \tanh \eta_5\right)\eta_2 + A(\eta_2 + \eta_1) + r\eta_4, \\ 0 = c\eta_1 - \eta_2 - \eta_1\eta_3, \\ 0 = \eta_1\eta_2 - b\eta_3, \\ 0 = -\eta_1 - a\eta_4, \\ 0 = \eta_2, \end{cases} \tag{17}$$

Solving above gives us the equilibrium points for integer and non integer or der i.e., $E\left(\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*, \eta_5^*\right)$, where all the calculated values of equilibrium points for different α values . Further the Jacobian matrix of (18) for equilibrium point $\alpha = 1$ is presented which is:

$$J_E = \begin{pmatrix} A & 2 + A & 0 & r & 0 \\ c & -1 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & 0 \\ -1 & 0 & 0 & -a & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

The calculated characteristic equation of Jacobian matrix at E is

$$A_0\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5 = 0, \tag{19}$$

From Eq.(19), the coeficients of the polynomial can be obtained, which are

$$\begin{aligned} &A_0 = 2, A_1 = a, A_2 = -3.0561A + 4c + 6.1591, \\ &A_3 = 2.5646ab + 4Ac + 15.095A + 117.8545b + 9.2433c + 11.8625, \\ &A_4 = 464.610Ab + 19.835Ac + 14.674A + 422.890b + 20.7693c \\ &\ + 6.3831, A_5 = 0. \end{aligned}$$

From (19), the calculated Eigenvalues are $\lambda_1 = 0, \lambda_{2,3} = 4.044 \pm 8.156j$. $\lambda_4 = -9.1137$ and $\lambda_5 = -5.247$ and the Jacobian matrix analysis for system (3.3) reveals three negative, one zero, and two complex eigenvalues, indicating the equilibrium points P_1 as unstable saddle-focus points. Utilizing the Routh-Hurwitz criterion for a deeper stability assessment, this mathematical approach bypasses direct eigenvalue calculation to confirm the system's instability, showcasing the blend of theoretical insight and analytical rigor in examining dynamical stability.

$$\Delta_1 = a_1, \Delta_2 = \begin{vmatrix} A_1 & A_3 \\ A_0 & A_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} A_1 & A_3 & A_5 \\ A_0 & A_2 & A_4 \\ 0 & A_1 & A_3 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} A_1 & A_3 & A_5 & A_7 \\ A_0 & A_2 & A_4 & A_6 \\ 0 & A_1 & A_3 & A_5 \\ 0 & A_0 & A_2 & A_4 \end{vmatrix}, \Delta_5 = \begin{vmatrix} A_1 & A_3 & A_5 & A_7 & A_9 \\ A_0 & A_2 & A_4 & A_6 & A_8 \\ 0 & A_1 & A_3 & A_5 & A_7 \\ 0 & A_0 & A_2 & A_4 & A_6 \\ 0 & 0 & A_1 & A_3 & A_5 \end{vmatrix}.$$

When the parameters $a = 9, b = 1.5, c = 11$ and $d = 4$ are plugged in Eq.(4.4) to calculate the Eq.(4.5) determinants, we get some negative and some positive values. The criterion suggests that if $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, and Δ_5 all eigenvalues are positive, the system is stable with no chaos. A negative eigenvalue, however, indicates potential for chaotic behavior.

5.1. Lyapunov exponent

The determination of a system's Lyapunov exponents (LEs) is of paramount importance in discerning the presence of chaos inside the system.The Jacobian approach is among the methodologies used in the computation of Lyapunov exponents.Methods for computing time series-based Lyapunov exponents (LEs), such as Wolf's technique, may be used to determine the rate of divergence of nearby trajectories in a dynamical system. [15].The use of other methods is often seen in the computation of Lyapunov exponents for systems characterized by both integer and fractional orders.[39]. In this study, we used Wolf's algorithm to calculate LEs.

The Lyapunov exponents are calculated for system (5) with integer and fractional order $\zeta = 1$ and $\zeta = 0.99$ respectively. The values at integer order $\zeta = 1$ are $0.0796, 0, -0.838$ and at $\zeta = 0.99$ the values are $0.0520, -0.05266, -0.92940$. Lyapunov exponents are presented in Fig. 10a–b.

Figure 10a shows the Lyapunov exponents for $\zeta = 1$ while Fig. 10b illustrates the Lyapunov exponents for $\zeta = 0.99$. LEs at $\zeta = 1$ and $\zeta = 0.99$ show that the system (4) is chaotic because each of them consists of one positive, one negative and one which is almost zero.

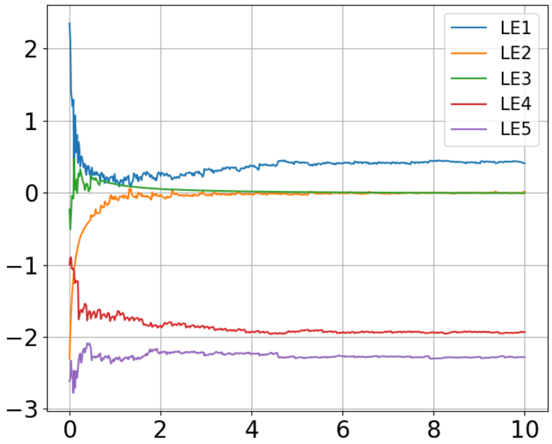

Figure 1: exponents of the system (4)

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6. The implementation of a circuit for a chaotic system

This segment of our research extends into the practical domain by detailing the intricate process of constructing analog circuits that embody our chaotic system, thereby allowing for empirical verification. Initially, beginning with the theoretical foundations laid out by system (1), we embarked on a methodical transformation of these abstract equations into tangible circuit representations. This translation from mathematical models to physical circuits necessitates meticulous consideration of the system's dynamical properties, ensuring that the resultant hardware not only captures the essence of the theoretical model but also faithfully reproduces its chaotic behavior. The conversion process involves selecting appropriate electronic components that mirror the system's parameters and operational amplifiers that enforce the nonlinearity central to chaotic dynamics. By doing so, we aim to bridge the gap between simulation and real-world application, providing a robust platform for observing and analyzing the nuanced behaviors of the chaotic system within a controlled experimental environment. In this part of the process, the analog circuits are constructed such that the system can be verified (1). In the first step of this process, we started with the system (1) and transformed it into circuit equations.

$$
\begin{aligned}
D^{0.95}\eta_1 &= \frac{1}{R_1C}\eta_2 - \frac{1}{R_2C}\eta_1 + \frac{1}{R_3C}\eta_4 + \frac{1}{R_4C}\left(\frac{R_{13}}{R_{14}} + \frac{R_{13}}{R_{15}}\eta_5^2\right)\eta_2 \\
D^{0.95}\eta_2 &= \frac{1}{R_5C}\eta_1 - \frac{1}{R_6C}\eta_1 - \frac{1}{R_7C}x\eta_1 \\
D^{0.95}\eta_3 &= \frac{1}{R_8C}\eta_1\eta_2 - \frac{1}{R_9C}\eta_3 \\
D^{0.95}\eta_4 &= -\frac{1}{R_{10}C}\eta_1 - \frac{1}{R_{11}C}\eta_4 \\
D^{0.95}\eta_5 &= \frac{1}{R_{12}C}\eta_5
\end{aligned}
\tag{20}
$$

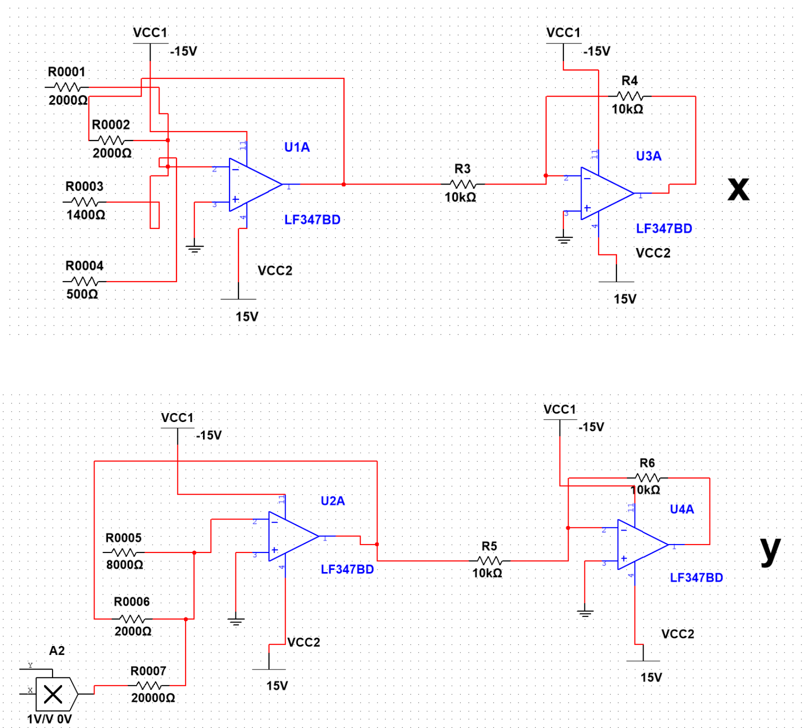

Figure 2: Circuit Realization of Chaotic System.

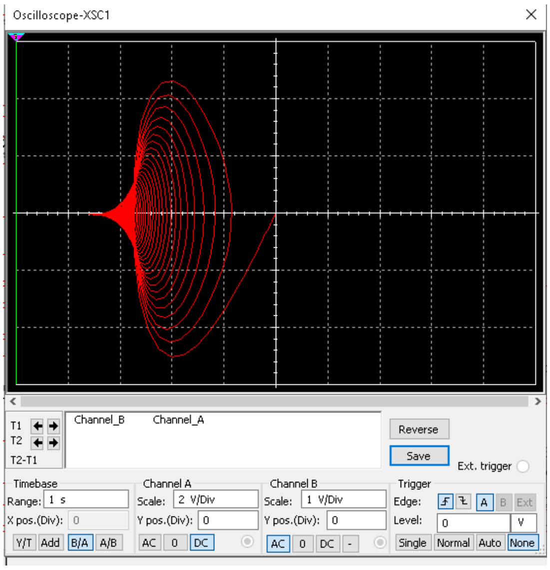

Figure 3: Circuit Realization of Chaotic System

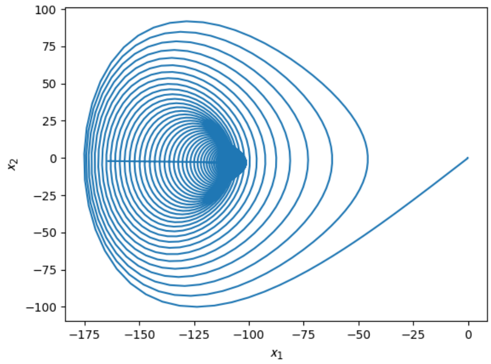

Figure 4: Numerical simulated and circuital generated results for system α=0.99.

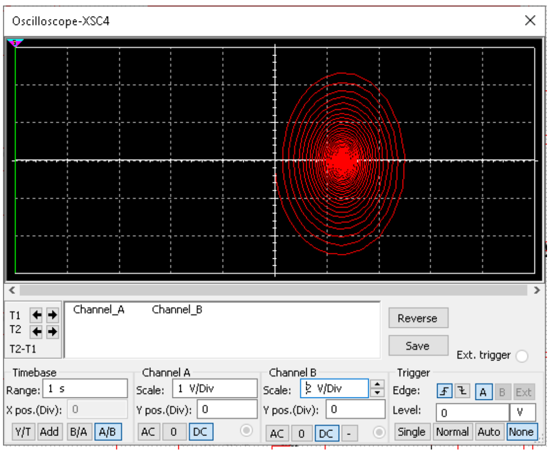

Figure 5: Numerical simulated and circuital generated results for system α=0.99.

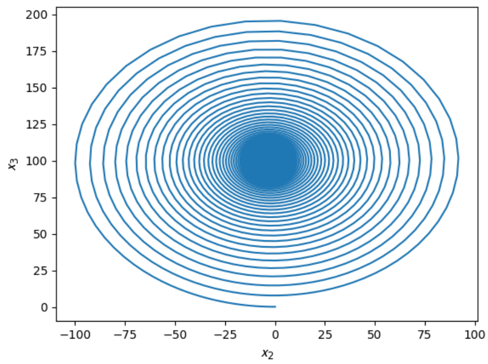

Figure 6: Numerical simulated and circuital generated results for system α=0.99.

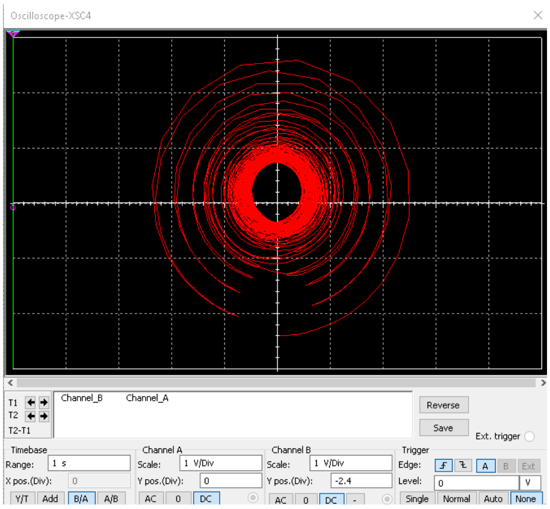

Figure 7: Numerical simulated and circuital generated results for system α=0.98.

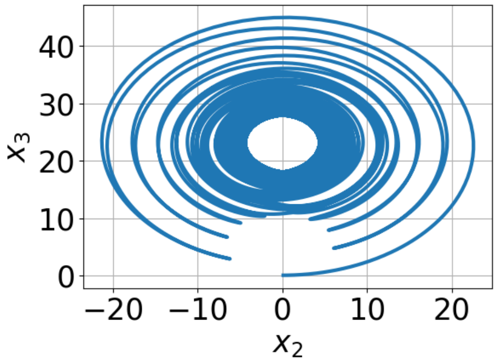

Figure 8: Numerical simulated and circuital generated results for system α=0.99.

In Figures 2 through 5, we delve into the intricate dynamics of a chaotic system through both circuit realization and numerical simulations, specifically focusing on the system with $\alpha = 0.99$. Figure 2 showcases the physical circuitry that underpins the chaotic behavior, serving as a tangible foundation for the theoretical models. Subsequently, Figures 2 to 5 present a detailed examination of the system's behavior through numerical simulations that corroborate the circuit-generated outcomes, illustrating the system's response at $\alpha = 0.99$. These figures collectively highlight the consistency between theoretical predictions and experimental observations, demonstrating the robustness of the chaotic system under study. The nuanced differences in the simulated results, despite the same α value, underline the sensitivity to initial conditions—a hallmark of chaotic systems. This alignment between experimental and simulated data not only validates our approach but also offers insights into the practical implementation and potential applications of such chaotic systems in real-world scenarios.

7. Conclusion

The exploration into a memristor fused five-dimensional fractional-order chaotic system delineates a significant advancement in the study of chaotic systems. Incorporating fractional calculus, this research unveils a more intricate and robust hyperchaotic behavior, verified through comprehensive simulations and analytical validations. The findings affirm the potential of fractional-order systems in enhancing the dynamics and stability of chaotic systems, as evidenced by the alignment between the circuit and numerical simulations. The study not only deepens the understanding of chaotic systems but also paves the way for future innovations in electronic circuit design and the broader application of chaos theory. As we embark on the next phase of research, the focus will shift towards the deployment and electronic integration of this innovative system, promising new avenues for exploration in the field of nonlinear dynamical systems. This work lays the groundwork for future research, highlighting the potential for further advancements in the domain of fractional-order chaotic systems and their practical applications.We proposed to compare numerical solutions with other ways [5, 33, 35] and use this method to solve novel fractional issues [7, 9, 11, 12, 14, 21, 23, 25, 31, 32, 34]

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References

[1] Mohamed A Abdoon. First integral method: a general formula for nonlinear fractional Klein-Gordon equation using advanced computing language. *American Journal of Computational Mathematics*, 5(02):127–134, 2015.

[2] Mohamed A Abdoon et al. Programming first integral method general formula for the solving linear and nonlinear equations. *Applied Mathematics*, 6(03):568, 2015.

[3] Mohamed A Abdoon and Faeza Lafta Hasan. Advantages of the Differential Equations for Solving Problems in Mathematical Physics with Symbolic Computation. *Mathematical Modelling of Engineering Problems*, 9(1), 2022.

[4] Mohamed A Abdoon and Faeza Lafta Hasan. Advantages of the Differential Equations for Solving Problems in Mathematical Physics with Symbolic Computation. *Mathematical Modelling of Engineering Problems*, 9(1), 2022.

[5] Ghada A Ahmed. On the fractional-order glucose-insulin interaction, journal = AIMS Mathematics. 8(7):15824–15843, 2023.

[6] Kilbas Anatoliĭ Aleksandrovich, Srivastava Hari M, and Trujillo Juan J. *Theory and applications of fractional differential equations.* Elsevier, 2006.

[7] Dalal Khalid Almutairi, Mohamed A Abdoon, Mohammed Berir, Rania Saadeh, and Ahmad Qazza. A numerical confirmation of a fractional seitr for influenza model efficiency. *Appl. Math*, 17(5):741–749, 2023.

[8] Abdulrahman BM Alzahrani, Mohamed A Abdoon, Mohamed Elbadri, Mohammed Berir, and Diaa Eldin Elgezouli. A comparative numerical study of the symmetry chaotic jerk system with a hyperbolic sine function via two different methods. *Symmetry*, 15(11):1991, 2023.

[9] Abdulrahman BM Alzahrani, Mohamed A Abdoon, Mohamed Elbadri, Mohammed Berir, and Diaa Eldin Elgezouli. A comparative numerical study of the symmetry

chaotic jerk system with a hyperbolic sine function via two different methods. *Symmetry*, 15(11):1991, 2023.

[10] Abdulrahman BM Alzahrani, Rania Saadeh, Mohamed A Abdoon, Mohamed Elbadri, Mohammed Berir, and Ahmad Qazza. Effective methods for numerical analysis of the simplest chaotic circuit model with atangana–baleanu caputo fractional derivative. *Journal of Engineering Mathematics*, 144(1):9, 2024.

[11] Abdulrahman BM Alzahrani, Rania Saadeh, Mohamed A Abdoon, Mohamed Elbadri, Mohammed Berir, and Ahmad Qazza. Effective methods for numerical analysis of the simplest chaotic circuit model with atangana–baleanu caputo fractional derivative. *Journal of Engineering Mathematics*, 144:9, 2024.

[12] Salem Mubarak Alzahrani, Rania Saadeh, Mohamed A Abdoon, Ahmad Qazza, Fathelrhman EL Guma, and Mohammed Berir. Numerical Simulation of an Influenza Epidemic: Prediction with Fractional SEIR and the ARIMA Model. *Appl. Math*, 18(1):1–12, 2024.

[13] Abdon Atangana and Dumitru Baleanu. New fractional derivatives with nonlocal and non-singular kerne. *arXiv preprint arXiv:1602.03408*, 2016.

[14] Abdon Atangana and Dumitru Baleanu. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *arXiv preprint arXiv:1602.03408*, 2016.

[15] Giancarlo Benettin, Luigi Galgani, Antonio Giorgilli, and Jean-Marie Strelcyn. Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. part 1: Theory. *Meccanica*, 15:9–20, 1980.

[16] Jakob Bernoulli, Jean Bernoulli, Herman H Goldstine, Patricia Radelet-de Grave, David Speiser, et al. Die streitschriften von jacob und johann bernoulli: Variationsrechnung. *No Title*, 1991.

[17] Jean-Paul Berrut and Lloyd N Trefethen. Barycentric lagrange interpolation. *SIAM review*, 46(3):501–517, 2004.

[18] Donato Cafagna and Giuseppe Grassi. On the simplest fractional-order memristor-based chaotic system. *Nonlinear Dynamics*, 70:1185–1197, 2012.

[19] Michele Caputo and Mauro Fabrizio. A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation & Applications*, 1(2):73–85, 2015.

[20] Mohamed Elbadri, Mohamed A Abdoon, Mohammed Berir, and Dalal Khalid Almutairi. A numerical solution and comparative study of the symmetric rossler attractor with the generalized caputo fractional derivative via two different methods. *Mathematics*, 11(13):2997, 2023.

[21] Mohamed Elbadri, Mohamed A Abdoon, Mohammed Berir, and Dalal Khalid Almutairi. A numerical solution and comparative study of the symmetric rossler attractor with the generalized caputo fractional derivative via two different methods. *Mathematics*, 11(13):2997, 2023.

[22] Mohamed Elbadri, Mohamed A Abdoon, Mohammed Berir, and Dalal Khalid Almutairi. A Symmetry Chaotic Model with Fractional Derivative Order via Two Different Methods. *Symmetry*, 15(6):1151, 2023.

[23] Mohamed Elbadri, Mohamed A Abdoon, Mohammed Berir, and Dalal Khalid Almutairi. A Symmetry Chaotic Model with Fractional Derivative Order via Two Different Methods. *Symmetry*, 15(6):1151, 2023.

[24] Elsayed Elshoubary, Mohamed A Abdoon, Abdulgader Bahatheg, and Mousa Mubarak Albeladi. Stability analysis and numerical simulation of fractional model of Leishmaniasis. *Results in Nonlinear Analysis*, 6(2):9–17, 2023.

[25] Fathelrhman EL Guma, Ossama M Badawy, Mohammed Berir, and Mohamed A Abdoon. Numerical analysis of fractional-order dynamic dengue disease epidemic in sudan. *Journal of the Nigerian Society of Physical Sciences*, pages 1464–1464, 2023.

[26] Tareq Hamadneh, Amel Hioual, Rania Saadeh, Mohamed A Abdoon, Dalal Khalid Almutairi, Thwiba A Khalid, and Adel Ouannas. General methods to synchronize fractional discrete reaction–diffusion systems applied to the glycolysis model. *Fractal and Fractional*, 7(11):828, 2023.

[27] Faeza Hasan, Mohamed A Abdoon, Rania Saadeh, Mohammed Berir, and Ahmad Qazza. A new perspective on the stochastic fractional order materialized by the exact solutions of allen-cahn equation. *International Journal of Mathematical, Engineering and Management Sciences*, 8(5):912, 2023.

[28] Faeza L Hasan and Mohamed A Abdoon. The generalized (2+ 1) and (3+ 1)-dimensional with advanced analytical wave solutions via computational applications. *International Journal of Nonlinear Analysis and Applications*, 12(2):1213–1241, 2021.

[29] Ivo Petráš. *Fractional-order nonlinear systems: modeling, analysis and simulation.* Springer Science & Business Media, Ann Arbor, Michigan, 2011.

[30] Igor Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* Elsevier, 1998.

[31] Ahmad Qazza, Mohamed Abdoon, Rania Saadeh, and Mohammed Berir. A New Scheme for Solving a Fractional Differential Equation and a Chaotic System. *European Journal of Pure and Applied Mathematics*, 16(2):1128–1139, 2023.

[32] Rania Saadeh, Mohamed A. Abdoon, Ahmad Qazza, and Mohammed Berir. A Numerical Solution of Generalized Caputo Fractional Initial Value Problems. *Fractal And Fractional*, 7(4):332, 2023.

[33] Rania Saadeh, Mohamed A Abdoon, Ahmad Qazza, Mohammed Berir, Fathelrhman EL Guma, Naseam Al-kuleab, and Abdoelnaser M Degoot. Mathematical modeling and stability analysis of the novel fractional model in the caputo derivative operator: a case study. *Heliyon*, 2024.

[34] Rania Saadeh, Mohamed A. Abdoon, Ahmad Qazza, Fathelrhman EL Guma, Naseam Al-kuleab, and Abdoelnaser M. Degoot. Mathematical Modeling and Stability Analysis of the Novel Fractional Model in the Caputo Derivative Operator: A Case Study. *Heliyon*, E26611(1):1, 2024.

[35] Sayed Saber, Azza M Alghamdi, Ghada A Ahmed, and Khulud M Alshehri. Mathematical Modelling and optimal control of pneumonia disease in sheep and goats in Al-Baha region with cost-effective strategies. *AIMS Mathematics*, 7(8):12011–12049, 2022.

[36] L Stenflo. Generalized Lorenz equations for acoustic-gravity waves in the atmosphere. *Physica Scripta*, 53(1):83, 1996.

[37] Junwei Sun, Xingtong Zhao, Jie Fang, and Yanfeng Wang. Autonomous memristor chaotic systems of infinite chaotic attractors and circuitry realization. *Nonlinear Dynamics*, 94:2879–2887, 2018.

[38] Leimin Wang, Tiandu Dong, and Ming-Feng Ge. Finite-time synchronization of memristor chaotic systems and its application in image encryption. *Applied Mathematics and Computation*, 347:293–305, 2019.

[39] Shuang Zhou and Xingyuan Wang. Simple estimation method for the largest lyapunov exponent of continuous fractional-order differential equations. *Physica A: Statistical Mechanics and Its Applications*, 563:125478, 2021.