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Real Division Algebras with a Left Unit Element that Satisfy Certain Identities

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Abstract. We study A, finite dimensional real division algebra with left unit e, satisfying: for all $x \in A$,
$(\mathbf{E1})\quad (x,x,x)=0, \quad (\mathbf{E2})\quad (x^2,x^2,x^2)=0, \quad (\mathbf{E3})\quad x^2e=x^2 \quad$ and $\quad(\mathbf{E4})\quad (xe)e=x$.
We show that:
- If A satisfies to (**E1**), then e is the unit element of A.
- (**E1**) $\Longrightarrow$ (**E2**) $\Longrightarrow$ (**E3**) $\Longrightarrow$ (**E4**).

In two-dimensional, we determine A satisfying $(\mathbf{Ei})_{i\in\{1,2,3,4\}}$. We have

A satisfies to	(E1)	(E2)	(E3)	(E4)
A isomorphic to	$\mathbb{R}$; $\mathbb{C}$	$\mathbb{R}$; $\mathbb{C}$; $^\star\mathbb{C}$	$\mathbb{R}$; $\mathbb{C}$; $^\star\mathbb{C}$	$\mathbb{R}$; $\mathbb{C}$; $^\star\mathbb{C}$; $\mathcal{L}(1,-1,\gamma,1)$

We show as well as (**E1**) $\Longrightarrow$ (**E2**) $\Longleftrightarrow$ (**E3**) $\Longrightarrow$ (**E4**).
We finally study the fused four-dimensional real division algebras satisfying $(\mathbf{Ei})_{i\in\{1,2\}}$. We have shown that those which verify (**E2**) are $\mathbb{H}$, $^\star\mathbb{H}$ and $\mathbb{C}\oplus\mathbb{B}$. and that $\mathbb{H}$ is the only fused algebra division with left unit satisfies to (**E1**).

2020 Mathematics Subject Classifications: 17A30, 17A35, 17A36

Key Words and Phrases: Division algebra, left unit, fused algebras and isomorphism of algebras.

1. Introduction

The discovery of the real algebra of Quaternions $\mathbb{H}$ by Hamilton in 1843, caused:
On the one hand to the study of real division algebras by great researchers mathematians and physicists. One of the fundamental results is kervair Milnor Bolt's theorem which states that the dimension of real finite-dimensional division algebra is 1, 2, 4 or 8 ([4], [12]). $\mathbb{R}$ is the only one-dimensional real division algebra. In two-dimensional, these algebras have been classified by Steven C. Althoen and Lawrence D. Kugler [3], reviewed

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by Marion Hübner and Holger P. Petersson [11]. Ana Lucia Cali and Michael Josephy classified the two-dimensional real division algebras with left unit [7]. The classification problem of finite-dimensional real division algebras opened in dimension four and eight. [1] have studied Real division algebras satisfying some identities. [13] have studied the Commuting maps and identities with inverses on alternative division rings and [5] studied Some results in the theory of linear non-associative algebras.
On the other hand, finite-dimensional Absolute Valued Algebras were classified by A. Calderon, A. Kaidi, C. Martin, A. Morales, M. Ramirez, and A. Rochdi [6]. A. Chandid and A. Rochdi have studied Absolute Valued Algebras satisfies $(x^i, x^j, x^k) = 0$ [8]. O. Diankha, A. Diouf, M.I. Ramirez and A. Rochdi have studied the absolute valued algebras with one sided unit satisfying $(x^2, x^2, x^2) = 0$ [9]. the latter have shown in absolute valued algebras having a left unit element of finite dimensional that $(E1) \Longrightarrow (E2) \Longleftrightarrow (E3) \Longrightarrow (E4)$.
It is with this in mind that we study, in this paper, these identities in the case of real division algebras with left unit of finite dimensional n. We obtained the same result if $n \in \{1, 2\}$. But if $n \in \{4, 8\}$ we have $(E1) \Longrightarrow (E2) \Longrightarrow (E3) \Longrightarrow (E4)$.

We study fused-algebras division with left unit satisfying to $(\mathbf{Ei})_{i\in\{1,2\}}$. we show that those which satisfy to (E2) are $\mathbb{H}$, $^\star\mathbb{H}$ and $\mathbb{C} \oplus \mathbb{B}$ with $\mathbb{B} = (\mathbb{R}^2, \bullet)$ is the real algebra whose multiplication table in the basis $\beta = \{u, v\}$ is given by

($\mathbb{B}$)

$\bullet$	u	v
u	u	$-v$
v	$c_{21}u - v$	$c_{22}u$

where c_{21}, c_{22} are real nombers thus that $c_{12}^2 < -4c_{22}$. We prove that $\mathbb{H}$ is the unique fused algebra division with left unit satisfies to (**E1**).

2. Notes and preliminary results

Let A be an arbitrary real algebra. Let x, y $e \in A$, we define :
$[x, y] = xy - yx$,
$(x, y, z) = (xy)z - x(yz)$,
and $id_A : A \longrightarrow A$ the identity application of A.
We define: $^\star\mathbb{A} := (\mathbb{A}, \odot)$ the real algebra whose vector space is the set A and the product $\odot$ is defined by $x \odot y = \overline{x}y$ $\forall x,\ y \in \mathbb{A}$ with $\mathbb{A}$ equal to either $\mathbb{C}$ or $\mathbb{H}$, and $x \mapsto \overline{x}$ mains standard involution. A(x) is a subalgebra of A generated by x. A is said:

- division if the operateros $Lx : A \to A, y \mapsto xy$ and $Rx : A \to A, y \mapsto yx$ are bijective, for all $x \in A$, $x \neq 0$.
- at third power-associative if $(x, x, x) = 0$ for all $x \in A$,.
- at (222) power-associative if $(x^2, x^2, x^2) = 0$ for all $x \in A$.
- at (121) power-associative if $(x, x^2, x) = 0$ for all $x \in A$.
- at power-commutative if any subalgebra generated by a single element is commutative.

We define the identities:

(**E1**) $(x, x, x) = 0$, (**E2**) $(x^2, x^2, x^2) = 0$, (**E3**) $x^2e = x^2$ and (**E4**) $(xe)e = x$.
$A \cong B$ if only if A and B are isomorphic.
We state now some preliminary results:

Lemma 1. *. Let A be a finite-dimensional real division algebra with left unit e. We have*
*1) It is obvious (**E1**) $\Longrightarrow$ (**E2**)*
*2) [8] have shown that (**E3**) $\Longrightarrow$ (**E4**)*
*3) [9] have shown that (**E2**) $\Longrightarrow$ (**E4**)*

Lemma 2. *Let A be a real division algebra of finite dimension $n \geq 2$ with left unit element e. Let $x \in A - \{0\}$. The following propositions are equivalent:*
(1) $xe \in \mathbb{R}e$,
(2) $x \in \mathbb{R}e$,
(3) $x^2 \in \mathbb{R}x$.

Proof.
(1) $\Longrightarrow$ (2) Suppose that $xe \in \mathbb{R}e$ and $x \notin \mathbb{R}e$. As $xe \in \mathbb{R}e$ so there exists $\alpha \in \mathbb{R} - \{0\}$ such as $xe = \alpha e$. Then $(x - \alpha e)(xe) = 0$ and as A is of division therefore $x = \alpha e$ absurd, hence the result.

(2) $\Longrightarrow$ (3) Suppose that $x \in \mathbb{R}e$, so there exists $\alpha \in \mathbb{R}$ such as $x = \alpha e \Longrightarrow x^2 = \alpha^2 e^2 = \alpha^2 e = \alpha(\alpha e) = \alpha x$ as a result $x^2 \in \mathbb{R}x$.

(3) $\Longrightarrow$ (1) Suppose that $x^2 \in \mathbb{R}x$, so there exists $\beta \in \mathbb{R} - \{0\}$ such as $x^2 = \beta x$ as a result $(x - \beta e)x = 0$ and dividing A gives us $x = \beta e \Longrightarrow xe = \beta e^2 = \beta e$, then $xe \in \mathbb{R}e$.

3. Two-dimensional real division algebras with left unit

Let A be a two-dimensional real algebra having a basis $B = \{e; u\}$ such that the products in the base are given by the multiplications **Table1** and **Table2**

.	e	u
e	e	u
u	$\alpha e + \beta u$	$\gamma e + \lambda u$

Table1

.	e	u
e	e	$\alpha e + \beta u$
u	u	$\gamma e + \lambda u$

Table2

.
with α, β, γ, $\lambda \in \mathbb{R}$. We will note respectively $\mathcal{L}(\alpha, \beta, \gamma, \lambda)$ and $\mathcal{R}(\alpha, \beta, \gamma, \lambda)$ algebras of to dimension 2 whose product in the base $\{e, u\}$ is given respectively by **Table1** and **Table2**.

[7] gives the following results: Theorem 1 and Proposition 1.

Theorem 1. *Let A be a real division algebra of dimension two with left identity. Then A is isomorphic to precisely one of the following algebras:*
(Class I) $\mathcal{L}(0, \beta, -1, 0)$ for some $\beta > 0$;
(Class II) $\mathcal{L}(0, \beta, 1, 0)$ for some $\beta < 0$;
(Class III) $\mathcal{L}(1, \beta, \gamma, 0)$ for some $\beta \neq -1$ and γ with $4\beta\gamma < -1$;
(Class IV) $\mathcal{L}(1, -1, \gamma, 1)$ for some $\gamma > 0$;

Proposition 1. *β and $sgn\gamma$ are invariants among division algebras, that is, if $A = \mathcal{L}(\alpha, \beta, \gamma, \delta)$ and $A' = \mathcal{L}(\alpha', \beta', \gamma', \delta')$ are isomorphic division algebras, then $\beta = \beta'$ and $sgn\gamma = sgn\gamma'$*

Remark 1. *A be a finite-dimensional real division algebra with left unit e equals $A^{opp} = (A, \odot)$ with $x \odot y = yx$ be a finite-dimensional real division algebra with right unit e. Therefore the results on algebras with right unit are obtained by analogy the results of algebras with left unit.*

Lemma 3. *Let A be a two-dimensional real division algebra with left unit e. The following propositions are equivalent:*
(1) A satisfies to $(E4)$.
(2) A is isomorphic to either $\mathbb{C}$, $^{\star}\mathbb{C}$, $\mathcal{L}(1, -1, \gamma, 1)$ with $\gamma > 0$.

Proof.
(1) $\Rightarrow$ (2) According to the theorem 1, we have A is isomorphic to $\mathcal{L}(\alpha, \beta, \gamma, \lambda)$. We have
$$(ue)e = u \iff \alpha(1+\beta)e + (\beta^2 - 1)u = 0 \iff \begin{cases} \alpha(1+\beta) = 0 \\ \beta^2 = 1 \end{cases}$$
• If $\beta = 1$, then $\alpha = 0$, the theorem 1 shows that A is isomorphic to $\mathcal{L}(0, 1, -1, 0) \cong \mathbb{C}$.
• If $\beta = -1$, the theorem 1 shows that A isomorphic to $\mathcal{L}(0, -1, 1, 0) \cong {}^{\star}\mathbb{C}$. or $\mathcal{L}(1, -1, \gamma, 1)$ with $\gamma > 0$
(2) $\Rightarrow$ (1) Obvious.

Lemma 4. *Let A be a two-dimensional real division algebra with left unit e. The following propositions are equivalent:*
(1) A satisfies to $(E3)$
(2) A is isomorphic to either $\mathbb{C}$, $^{\star}\mathbb{C}$.

Proof.
(1) $\Rightarrow$ (2) According to the theorem 1, we have A is isomorphic to $\mathcal{L}(\alpha, \beta, \gamma, \lambda)$
• If $A \cong \mathcal{L}(0, \beta, -1, 0)$, we have $(e+u)^2e = (e+u)^2 \Rightarrow \beta = 1$ thus $A \cong \mathcal{L}(0, 1, -1, 0) \cong \mathbb{C}$.
• If $A \cong \mathcal{L}(0, \beta, 1, 0)$, we have $(e+u)^2e = (e+u)^2 \Rightarrow \beta = -1$ thus $A \cong \mathcal{L}(0, -1, 1, 0) \cong^{\star} \mathbb{C}$.
• If $A \cong \mathcal{L}(1, \beta, \gamma, 0)$ or $\mathcal{L}(1, -1, \gamma, 1)$, then A does not satisfy (E3).
Indeed $(e+u)^2e \neq (e+u)^2$.
(2) $\Rightarrow$ (1) Obvious.

Proposition 2. *Let A be a two-dimensional real division algebra with left unit e. The following propositions are equivalent:*
(1) A satisfies to (E2),
(2) A satisfies to (E3),
(3) A is isomorphic to either $\mathbb{C}$, $^{\star}\mathbb{C}$.

Proof.
(1) $\Longrightarrow$ (2) A satisfies to (E2), then A satisfies to (E4), the Lemma 3 shows that A isomorphic to either $\mathbb{C}$, $^{\star}\mathbb{C}$, $\mathcal{L}(1, -1, \gamma, 1)$. Therefore A isomorphic to either $\mathbb{C}$, $^{\star}\mathbb{C}$. Because

$\mathcal{L}(1,-1,\gamma,1)$ does not satisfy to (E2), indeed for this algebra $(u^2,u^2,u^2)\neq 0$. As a result A satisfies to (E3);

(2) $\Longrightarrow$ (3) Lemma 4 gives the result;

(3) $\Longrightarrow$ (1) Obvious.

Proposition 3. *Let A be a two-dimensional real division algebra with left unit e. The following propositions are equivalent:*

(1) A is commutative,

(2) A is power-commutative ;

(3) A satisfies to (E1);

(4) A is isomorphic to $\mathbb{C}$.

(5) A is (121) power-associative.

Proof.
In [10], they proved in theorem 1 that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5). Show that

(3) $\Rightarrow$ (4) A satisfies to (E1), then satisfies to (E2). The proposition 2 shows that A is isomorphic to either $\mathbb{C}$, $^\star\mathbb{C}$. Consequently A is isomorphic to $\mathbb{C}$.

(4) $\Rightarrow$ (5) Obvious,

Corollary 1. *Let A be a n-dimensional real division algebra with left unit and $n\leq 2$. Let $\gamma\in\mathbb{R}$ with $\gamma>0$. We have*

A satisfies to	(E1)	(E2)	(E3)	(E4)
A isomorphic to	$\mathbb{R}$; $\mathbb{C}$	$\mathbb{R}$; $\mathbb{C}$; $^\star\mathbb{C}$	$\mathbb{R}$; $\mathbb{C}$; $^\star\mathbb{C}$	$\mathbb{R}$; $\mathbb{C}$; $^\star\mathbb{C}$; $\mathcal{L}(1,-1,\gamma,1)$

And we have the result $(E1)\Rightarrow(E2)\Leftrightarrow(E3)\Rightarrow(E4)$.

4. On finite-dimensional real division algebras with left unit.

We denote $Ae=\{x\in A, xe=x\}$ for all $x\in A$.

Lemma 5. *Let A be a finite-dimensional real division algebra with left unit e, satisfies to (**E3**). Then for all x and $y\in A$; $xy+yx\in Ae$.*

Proof.
Immediate consequence of equality $(x+y)^2e=(x+y)^2$.

Proposition 4. *Let A be a finite-dimensional real division algebra with left unit e, satisfies to (E1). The following propositions are equivalent:*

(1) A satisfies to (E4),

(2) e is the unit element.

Proof.
(1) $\Longrightarrow$ (2) Let $x\in A$, as A satisfies to (E1), then A checks the equation (2.1) of [10]. We have $[e^2,x]+[ex+xe,e]=0$. The fact that A satisfies to (E4) we have $x=xe$, as a result e is the unit element.

(2) $\Longrightarrow$ (1) Obvious.

Corollary 2. *Let A be a finite-dimensional real division algebra with left unit e, satisfies to (E1), then e is the unit element.*

Proof.
A satisfies to (E1), then satisfies to (E2), as a result A satisfies to (E4). The proposition 4 shows that e is the unit element.

Proposition 5. *Let A be a four-dimensional real division algebra with left unit e satisfies to (E2). The following propositions are equivalent:*
(1) A contains a central element
(2) A is power-commutative.

Proof.
(1) $\Longrightarrow$ (2) [1, Theorem 3] shows that A is power-commutative.
(2) $\Longrightarrow$ (1) A is power-commutative, then satisfies to (E1). Therefore satisfies to (E2) and to (E4). The proposition 4, shows that e is the unit element. Therefore e is a central element of A.

Lemma 6. *Let A be a finite-dimensional real division algebra with left unit e satisfies to (E2). If $x \in Ae$, then $x^2 \in Ae$.*

Proof.
Let $x \in Ae$, equality $(2,2)$ in [9] is verified, then
$(e^2, e^2, x^2) + (e^2, x^2, e^2) + (x^2, e^2, e^2) + (e^2, y, y) + (y, e^2, y) + (y, y, e^2) = 0$ (**a**)
with $y = x + xe = 2x$. A also satisfies to (E4), Thus
(**a**) $\Rightarrow (x^2, e, e) + (2x, 2x, e) = 0 \Rightarrow x^2e = x^2$. So $x^2 \in Ae$.

Proposition 6. *Let A be a finite-dimensional real division algebra with left unit e. We have the following result $(E2) \Longrightarrow (E3)$.*

Proof.
Let $x \in A$, A satisfies to (E2), equality $(2,2)$ in [9] is verified, then
$(e^2, e^2, x^2) + (e^2, x^2, e^2) + (x^2, e^2, e^2) + (e^2, y, y) + (y, e^2, y) + (y, y, e^2) = 0$ (**b**)
with $y = x + xe$, thus (**b**) $\Rightarrow x^2 - x^2e + y^2e - y^2 = 0$, as $y \in Ae \Rightarrow y^2 \in Ae \Rightarrow y^2e - y^2 = 0$, therefore we have $x^2 - x^2e = 0 \Rightarrow x^2e = x^2$.

Lemma 7. *Let A be a four-dimensional real division algebra with left unit e. If there exists $u \in A$ such that the subalgebra of A generated by u, $A(u) := B$, is of two-dimensional. Then for all $v \in A - B$, $\{e, u, v, uv\}$ is a basis of A.*

Proof.
Let $v \in A - B$, we have e, u,, v and uv are linearly independent.
Suppose that $uv = \alpha e + \beta u + \gamma v$ with α, β, and $\gamma \in \mathbb{R}$.

$$\begin{aligned} uv = \alpha e + \beta u + \gamma v \quad &\Longrightarrow \quad uv - \gamma v = \alpha e + \beta u \\ &\Longrightarrow \quad (u - \gamma e)v = \alpha e + \beta u \end{aligned}$$

$$\Longrightarrow \quad (u-\gamma e)v \in B$$

As B is a subalgebra of A, it exists $v' \in B$ such that $(u-\gamma e)v' = \alpha e + \beta u$.
thus $(u-\gamma e)v = (u-\gamma e)v' \Longrightarrow v = v' \in B$ absurd. Therefore $\{e, u, v, uv\}$ is a basis of A.

Example 1. *The real division algebra A whose product in the basis $B = \{e, u, v, uv\}$ is given by:*

$.$	e	u	v	uv
e	e	u	v	uv
u	u	$-e$	uv	$-v$
v	v	$v-uv$	$-e$	$-e+u$
uv	uv	$v+2uv$	u	$-e-2u$

satisfies to (E3) and not to (E2). Because $((u+uv)^2, (u+uv)^2, (u+uv)^2) \neq 0$.
Consequently, lemma 1 and proposition 6, shows that $(E1) \Rightarrow (E2) \Rightarrow (E3) \Rightarrow (E4)$.

Proposition 7. *Let A be a real division algebra with left unit e of finite-dimensional $n \in \{2, 4, 8\}$, satisfies to (E2). Then there exists $u \in A - \mathbb{R}e$ such as $ue \notin \mathbb{R}e$ and $ue.u = -e$. We note that, if $ue \in \mathbb{R}e + \mathbb{R}u$, then $A(u)$ is isomorphic to either $\mathbb{C}$, $^\star\mathbb{C}$; Otherwise the dimension of $A(u)$ is greater than four.*

Proof.
A satisfies to (E2), then satisfies to (E4). The real division algebra $A' = (A, \odot)$ with $x \odot y = (xe)y$ contains e as unit element. [15] shows that there exists $u \in A - \mathbb{R}e$, shus that $u \odot u = -e \Longleftrightarrow ue.u = -e$.
Suppose that $ue \in \mathbb{R}e$, Lemma 2 shows $u \in \mathbb{R}e$ absurd, so $ue \notin \mathbb{R}e$.
• If $ue \in \mathbb{R}e + \mathbb{R}u$, then they exist α, $\beta \in \mathbb{R}$ shus that $ue = \alpha e + \beta u$. We have

$$(ue)e = u \Longrightarrow \alpha(1+\beta)e + (\beta^2-1)u = 0 \Longrightarrow (S1) \begin{cases} \alpha(1+\beta) = 0 \\ \beta^2 = 1 \end{cases} \quad \text{thus}$$

if $\beta = 1 \Rightarrow \alpha = 0$ then $ue.u = -e \Longrightarrow u^2 = -e$ therefore $A(u)$ is isomorphic to $\mathbb{C}$.
if $\beta = -1$, then $(u(ue))e = u(ue) \Longrightarrow \alpha = 0$, thus $ue = -u$ and
$ue.u = -e \Longrightarrow u^2 = e$ therefore $A(u)$ is isomorphic to $^\star\mathbb{C}$.
• If $ue \notin \mathbb{R}e + \mathbb{R}u$, then the elements e, u and ue are linearly independent and they belong $A(u)$ therefore $dim(A(u)) \geq 4$.

Proposition 8. *Let A be a four-dimensional real division algebra with left unit e. The following propositions are equivalent.*

(1) A satisfies to (E1)
(2) A contains a central element
(3) A is is power-commutative

Proof.
(1) $\Longrightarrow$ (2), Corollary 2 gives the reeults,
(2) $\Longleftrightarrow$ (3) Theorem 3 in [10],
(3) $\Longleftrightarrow$ (1) Obvious.

5. Fused algebras division with left unit satisfies to $(\mathbf{Ei})_{i\in\{1,2\}}$.

[2] gives the definition 1 and the theorem 2

Definition 1. *Let $A = (\mathbb{R}^2, \circ)$ and $B = (\mathbb{R}^2, \bullet)$ the two-dimensional real algebras with the following multiplication tables with respect to a basis $B = \{u, v\}$ of $\mathbb{R}^2$:*

(A)

$\circ$	u	v
u	$a_{11}u + b_{11}v$	$a_{12}u + b_{12}v$
v	$a_{21}u + b_{21}v$	$a_{22}u + b_{22}v$

(B)

$\bullet$	u	v
u	$c_{11}u + d_{11}v$	$c_{12}u + d_{12}v$
v	$c_{21}u + d_{21}v$	$c_{22}u + d_{22}v$

We define multiplication on the direct sum $A \oplus B$ by:

$$(a, b).(c, d) = (a \circ c - b \bullet d, a \circ d + b \bullet c)$$

setting $e_1 = (u, 0)$, $e_2 = (v, 0)$, $e_3 = (0, u)$ and $e_4 = (0, v)$. It is easy to verfy that the algebra $A \oplus B$ has the multiplication table

$(A \oplus B)$

$.$	e_1	e_2	e_3	e_4
e_1	$a_{11}e_1 + b_{11}e_2$	$a_{12}e_1 + b_{12}e_2$	$a_{11}e_3 + b_{11}e_4$	$a_{12}e_3 + b_{12}e_4$
e_2	$a_{21}e_1 + b_{21}e_2$	$a_{22}e_1 + b_{22}e_2$	$a_{21}e_3 + b_{21}e_4$	$a_{22}e_3 + b_{22}e_4$
e_3	$c_{11}e_3 + d_{11}e_4$	$c_{12}e_3 + d_{12}e_4$	$-c_{11}e_1 - d_{11}e_2$	$-c_{12}e_1 - d_{12}e_2$
e_4	$c_{21}e_3 + d_{21}e_4$	$c_{22}e_3 + d_{22}e_4$	$-c_{21}e_1 - d_{21}e_2$	$-c_{22}e_1 - d_{22}e_2$

We call this table a standard table for the A-based fused algebras $A \oplus B$. A is isomorphic to the subalgebra of pairs (a,0)

Note. In this part we work with A, B and $A \oplus B$ the algebras of definition 1.

Theorem 2. *A fused algebra $A \oplus B$ is a division algebra if and only if A and B are division algebras and in any standard table for $A \oplus B$*

$$(a_{11}b_{12} - b_{11}a_{12})(c_{11}d_{12} - d_{11}c_{12}) < 0$$

Proposition 9. *If a fused algebra $A \oplus B$ is a division with left unit e_1 satisfies to (E2), then A is isomorphic to either $\mathbb{C}$, $^\star\mathbb{C}$*

Proof.
A is an two-dimensional real division algebra with left unit u satisfies to $(E2)$, the proposition 2 shows that A is isomorphic to either $\mathbb{C}$, $^\star\mathbb{C}$.

Lemma 8. *Let $A = (\mathbb{R}^2, \circ)$ isomorphic to $\mathbb{C}$ and $\mathbb{B} = (\mathbb{R}^2, \bullet)$ the algebra whose multiplication table in the basis $\beta = \{u, v\}$*

$(\mathbb{B})$

$\bullet$	u	v
u	u	$-v$
v	$c_{21}u - v$	$c_{22}u$

with c_{21}, $c_{22} \in \mathbb{R}$ thus that $c_{12}^2 < -4c_{22}$.
Then $A \oplus \mathbb{B}$ is division algebra satisfies to (E2), not satisfies to (E3). And not isomorphic to $\mathbb{H}$

Proof.
The theorem 2, shows that $A \oplus \mathbb{B}$ is division algebra.

Now let $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \in A \oplus \mathbb{B}$ we have $(x^2, x^2, x^2) = 0$. So $A \oplus B$ satisfies to (E2). We have for $a = e_1 + e_2 + e_3 + e_4$, $(a, a, a) \neq 0$ and $(e_2, e_3, e_4) \neq 0$.

So $A \oplus \mathbb{B}$ not satisfies to (E1) and not isomorphic to $\mathbb{H}$.

5.1. Study of fused algebras with A is isomorphic to $\mathbb{C}$.

Lemma 9. *We have*

(1) $\mathbb{C} \oplus \mathcal{R}(0, d_{12}, 1, 0)$ *satisfies to (E2) if and only if* $d_{12} = -1$

(2) If $d_{12} \neq -1$ *and* $4d_{12}c_{22} < -1$*, then* $\mathbb{C} \oplus \mathcal{R}(1, d_{12}, c_{22}, 0)$ *not satisfies to (E2).*

(3) If $c_{22} > 0$*, then* $\mathbb{C} \oplus \mathcal{R}(1, -1, c_{22}, 1)$ *not satisfies to (E2).*

Proof.
Taking $u = 1$ and $v = i$.
(1) Suppose that $\mathbb{C} \oplus \mathcal{R}(0, d_{12}, 1, 0)$ satisfies to (E2).

Let $a = e_1 + e_2 + e_3 + e_4 \in \mathbb{C} \oplus \mathcal{R}(0, d_{12}, 1, 0)$, we have $(a^2, a^2, a^2) = 0 \Longrightarrow d_{12} = -1$.
Now if $d_{12} = -1$, we have $\mathbb{C} \oplus \mathcal{R}(0, d_{12}, 1, 0)$ is isomorphic to $\mathbb{H}$, then satisfies to (E2).
(2) Suppose that $d_{12} \neq -1$, $4d_{12}c_{22} < -1$ and $\mathbb{C} \oplus \mathcal{R}(1, d_{12}, c_{22}, 0)$ satisfies to (E2).

Let $b = e_2 + e_3 \in \mathbb{C} \oplus \mathcal{R}(1, d_{12}, c_{22}, 0)$, then $(b^2, b^2, b^2) \neq 0$ absurd.
(3) Suppose that $c_{22} > 0$, and $\mathbb{C} \oplus \mathcal{R}(1, -1, c_{22}, 1)$ Satisfies to (E2).

Let $c = e_1 + e_4 \in \mathbb{C} \oplus \mathcal{R}(1, -1, c_{22}, 1)$, then $(c^2, c^2, c^2) \neq 0$ absurd.

Lemma 10. *Let A be an algebra isomorphic to $\mathbb{C}$ and B be a two-dimensional real algebras with right unit . The following propositions are equivalent*

(1) $A \oplus B$ is division algebra satisfies to (E2),

(2) $A \oplus B$ is isomorphic to $\mathbb{H}$.

Proof.
(1) $\Rightarrow$ (2) We can take u the unit element of A and also the right unit of B. The theorem 2 shows that B be a two-dimensional real division algebra with right unit and $d_{12} < 0$. By analogy of the Theorem 1. B is isomorphic to either

$\mathcal{R}(0, d_{12}, 1, 0)$ with $d_{12} < 0$.

$\mathcal{R}(1, d_{12}, c_{22}, 0)$ with $d_{12} \neq -1$ and $4d_{12}c_{22} < -1$,

$\mathcal{R}(1, -1, c_{22}, 1)$ with $c_{22} > 0$.

The Lemma 9, shows that $B \cong \mathcal{R}(0, -1, 1, 0)$, therefore $A \oplus B \cong \mathbb{C} \oplus \mathcal{R}(0, -1, 1, 0)$ thus $A \oplus B$ is isomorphic to $\mathbb{H}$.
(2) $\Rightarrow$ (1) Obvious.

Theorem 3. *$A \oplus B$ be a division algebra with left unit e_1 satisfies to (E2). The following propositions are equivalent:*

(1) A is ismorphic to $\mathbb{C}$,

(2) $A \oplus B$ is isomorphic to either $\mathbb{H}$, $\mathbb{C} \oplus \mathbb{B}$.

Proof.
(1) ⇒ (2) We have B be a two-dimensional real division algebra, [14] shows that B contains a non-zero idempotent. Thus we can take u the unit element of A and the non-zero idempotent of B. The lemma 1 (3) shows that $A \oplus B$ satisfies to (E4) then $(e_4.e_1).e_1 = e_4 \quad \Longrightarrow \quad c_{21}(1 + d_{21}) = 0 \quad\quad and \quad d_{21}^2 = 1.$

• If $d_{21} = 1$, we have $c_{21} = 0$, thus B be an two-dimensional real division with right unit.The Lemma 10 shows that $A \oplus B$ is isomorphic to $\mathbb{H}$.

• If $d_{21} = -1$, the $A \oplus B$ satisfies to (E3), the lemme 5 gives

. $(e_2e_3 + e_3e_2)e_1 = e_2e_3 + e_3e_2 \Longrightarrow d_{12} = -1$

. $(e_2e_4 + e_4e_2)e_1 = e_2e_4 + e_4e_2 \Longrightarrow d_{22} = 0$

The division of $A \oplus B$ gives $c_{12}^2 < -4c_{22}$
Let $a = c_{12}e_1 + e_2 + e_3$, $(a^2, a^2, a^2) = 0 \Longrightarrow c_{12} = 0$. So $A \oplus B$ is isomorphic to $\mathbb{C} \oplus \mathbb{B}$.
(2) ⇒ (1) Obvious..

5.2. Study of fused algebras with A is isomorphic to $^\star\mathbb{C}$.

Lemma 11. *Let A be an algebra isomorphic to $^\star\mathbb{C}$ and B be a two-dimensional real algebras with right unit . The following propositions are equivalent*

(1) $A \oplus B$ is division algebra satisfies to (E2),

(2) $A \oplus B$ is isomorphic to $^\star\mathbb{H}$.

Proof.
(1) ⇒ (2) We can take u the left unit element of A and also the right unit element of B. the theorem 2 shows that B be a two-dimensional real division algebra with right unit and $d_{12} < 0$. we have $(e_4e_3 + e_3e_4)e_1 = e_4e_3 + e_3e_4 \Longrightarrow d_{12} = -1$ and $e_4^2e_1 = e_4 \Longrightarrow d_{22} = 0$
By analogy of the Theorem 1. B is isomorphic to $\mathcal{R}(0, -1, 1, 0)$.
Therefore $A \oplus B \simeq^\star \mathbb{C} \oplus \mathcal{R}(0, -1, 1, 0)$ isomorphic to $^\star\mathbb{H}$.
(2) ⇒ (1) Obvious.

Theorem 4. *$A \oplus B$ be a division algebra with left unit e_1 satisfies to (E2). The following propositions are equivalent:*

(1) A is isomorphic to $^\star\mathbb{C}$,

(2) $A \oplus B$ is isomorphic to $^\star\mathbb{H}$.

Proof.
(1) ⇒ (2) B be a two-dimensional real division alegbra and $d_{12} < 0$. [14] shows that B contains a non-zero idempotent. Thus we can take u the left unit element of A and the non-zero idempotent of B. $A \oplus B$ satisfies to (E4), we have

$$(e_4.e_1).e_1 = e_4 \quad \Longrightarrow \quad c_{21}(1 + d_{21}) = 0 \quad\quad et \quad d_{21}^2 = 1.$$

• If $d_{21} = 1$, then $c_{21} = 0$, thus B be a two-dimensional real division alegbra with right unit. The Lemma 11, shows that $A \oplus B$ is isomorphic to $^\star\mathbb{H}$.

• If $d_{21} = -1$, we have $(e_4e_3 + e_3e_4)e_1 = e_4e_3 + e_3e_4 \Longrightarrow d_{12} = 1$, absurd because $d_{12} < 0$.
(2) ⇒ (1) Obvious.

5.3. Fused algebras division with left unit satisfies to (E1) and to (E2).

Theorem 5. *$A \oplus B$ be a division algebra with left unit e_1. The following propositions are equivalent:*

(1) $A \oplus B$ satisfies to (E2),

(2) $A \oplus B$ is isomorphic to either $\mathbb{H}$, $^\star\mathbb{H}$, $\mathbb{C} \oplus \mathbb{B}$.

Proof.
(1) $\Longrightarrow$ (2) $A \oplus B$ satisfies to (E2), the proposition 9 shows that A is isomorphic to eihter $\mathbb{C}$, $^\star\mathbb{C}$.

• If $A \cong \mathbb{C}$, the theorem 3, shows that $A \oplus B$ is isomorphic to either $\mathbb{H}$, $\mathbb{C} \oplus \mathbb{B}$.

• If $A \cong^\star \mathbb{C}$, the theorem 4 shows that $A \oplus B$ is isomorphic to $^\star\mathbb{H}$.

Therefore $A \oplus B$ is isomorphic to either $\mathbb{H}$, $^\star\mathbb{H}$, $\mathbb{C} \oplus \mathbb{B}$.

(2) $\Longrightarrow$ (1) Obvious.

Corollary 3. *$A \oplus B$ be a division algebra with left unit e_1. We have*

$A \oplus B$ satisfies to	*(E1)*	*(E2)*
$A \oplus B$ isomorphic to	$\mathbb{H}$	$\mathbb{H}$, $^\star\mathbb{H}$, $\mathbb{C} \oplus \mathbb{B}$

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