**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 17, No. 4, 2024, 3772-3780
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Zero Forcing Domination in Some Graphs: Characterizations and Derived Formulas

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**Abstract.** In this paper, we initiate the study on zero forcing domination in a graph. We characterize zero forcing dominating sets in some special graphs and the join of two graphs, and we derive some formulas of the zero forcing domination using characterization results. Moreover, we present some relationships of this parameter with other known parameters in graph theory.

**2020 Mathematics Subject Classifications**: 05C69

**Key Words and Phrases**: Zero forcing set, zero forcing dominating set, zero forcing domination number

## 1. Introduction

The concept of zero forcing set was initially introduced in [4] as a bound for the minimum rank problem. Given a starting set of blue vertices, with all other vertices white, and a color-change rule, zero forcing is a graph propagation mechanism that increases the number of blue vertices. According to the zero forcing color-change rule, a blue vertex next to a single white neighbor can force that neighbor to also be blue. The notation u → w means that if u is a blue vertex and w is the solitary white vertex in N(u), then u

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5154

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compels w to be blue. With graph G as input, a zero forcing set of G is a subset of vertices from V(G) such that the color-change rule given B is applied iteratively if B is originally colored blue and the remaining vertices in G are white.

Apart from its use in the minimal rank problem, zero forcing sets have also proved interesting on their own. Numerous aspects and extensions of zero forcing have been examined. Among these are positive semi-definite zero forcing, a variant that bounds the minimum rank problem when the minimum is taken only over positive semi-definite matrices with graph G [3].

On the other hand, in the late 1950's and 1960's, the study on domination in graphs was developed. Beginning with C. Berge [1] in 1958, he referred to the domination number as the "coefficient of external stability" [13]. In 1962, O. Ore introduced the terms "dominating set" and "domination number". Domination in a graph has been on the topics studied by researchers recently. Different variants of this parameter have been established and some of these studies can be found in [2, 5–12, 14, 15].

In this study, we initiate the study of zero forcing domination in graphs. We believe this parameter and its results would serve as reference for future researchers who will study on concept related to zero forcing domination in a graph.

## 2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in $G$ of two vertices $u, v$ is the length of a shortest *$u$-$v$ path* in $G$. The greatest distance between any two vertices in $G$, denoted by $diam(G)$, is called the *diameter* of $G$.

Two vertices $x, y$ of $G$ are *adjacent*, or *neighbors*, if $xy$ is an edge of $G$. The *open neighborhood* of $x$ in $G$ is the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. The *closed neighborhood* of $x$ in $G$ is the set $N_G[x] = N_G(x) \cup \{x\}$. If $X \subseteq V(G)$, the *open neighborhood* of $X$ in $G$ is the set $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighborhood* of $X$ in $G$ is the set $N_G[X] = N_G(X) \cup X$.

A subset $S$ of $V(G)$ is a *dominating* of $G$ if for every $a \in V(G) \backslash S$, there exists $b \in S$ such that $d_G(a, b) = 1$. The minimum cardinality among all dominating sets of $G$, denoted by $\gamma(G)$, is called the *domination number* of $G$. Any dominating set with cardinality equal to $\gamma_h(G)$ is called a *$\gamma$-set* of $G$.

The color change rule states that a blue vertex adjacent to a single white neighbor can force its neighbor to blue. A zero forcing set for a graph G is a subset Z of V(G) such that if initially the vertices in Z are colored blue and the remaining vertices are colored white, the entire graph G may be colored blue by repeatedly applying the color-change rule. Furthermore, the zero forcing number of G, denoted by Z(G), is the minimum cardinality of zero forcing set of G.

A subset $Z$ of $V(G)$ is said to be a *zero forcing hop dominating* if $Z$ is both a zero forcing and a hop dominating in $G$. The minimum cardinality among all zero forcing hop dominating sets in $G$, denoted by $\gamma_{zh}(G)$, is called the *zero forcing hop domination number* of $G$. A zero forcing hop dominating set $Z$ with $|Z| = \gamma_{zh}(G)$, is called the minimum zero forcing hop dominating set of $G$ or a $\gamma_{zh}$-set of $G$.

Let $G$ and $H$ be any two graphs. The *join* of $G$ and $H$, denoted by $G+H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

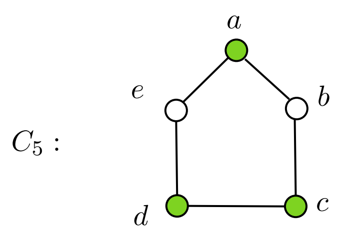
$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

## 3. Results

We begin this section by introducing the concept of zero forcing domination in a graph.

**Definition 1.** Let G be a graph. If B is zero forcing set and for every $u \in V(G) \setminus B$, there exists $v \in B$ such that $uv \in E(G)$, then $B$ is called a zero forcing dominating set of $G$. The minimum cardinality of a zero forcing dominating set of G, denoted by $\gamma_z(G)$, is called zero forcing domination number of $G$.

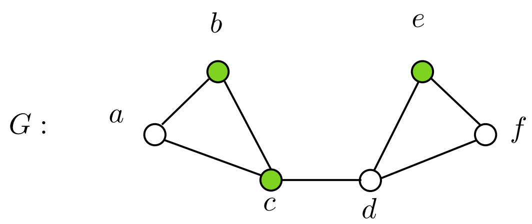
**Example 1.** Consider $C_5$ below:




Let $B = \{a, c, d\}$. Then $N_{C_5}[B] = V(C_5)$. Thus, $B$ is a dominating set of $C_5$. Observe that vertices $b$ and $e$ can be forced by $c$ and $d$, respectively. It follows that $B$ is a zero forcing set of $C_5$. Therefore, $B$ is a zero forcing dominating set of $C_5$. Moreover, it can be verified that $\gamma_z(C_5) = 3$.

**Remark 1.** *Let $G$ be a graph. Then the zero forcing domination and zero forcing hop domination parameter of $G$ are incomparable.*

To see this, consider the graph $G$ below:

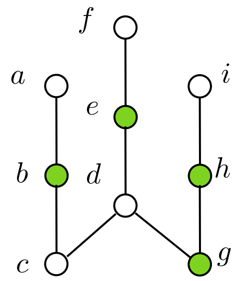



Let $Q = \{b, c, e\}$. Then $N_G[Q] = V(G)$. Thus, $Q$ is a dominating set of $G$. Moreover, observe that vertices $a, d$ and $f$ are forced by vertices $b, c$ and $e$. It follows that $Q$ is a zero forcing set of $G$. Since $\{c, d\}$ is not a zero forcing set of $G$, it follows that $Q = \{b, c, e\}$ is a minimum zero forcing dominating set of $G$. Hence, $\gamma_z(G) = 3$.

Now, let $S = \{a, c, d, e\}$. Then $S$ is a minimum zero forcing hop dominating set of $G$. Therefore, $\gamma_{zh}(G) = 4$.

Next, consider the graph $H$ below:

$H$ :




Let $P = \{c, d, g\}$. Then $N_G^2[P] = V(H)$, and so $P$ is a hop dominating set of $H$. Observe that vertices $b$ and $a$ are forced by $c$ and $b$, respectively, vertices $e$ and $f$ are forced by vertices $d$ and $e$, respectively, and vertices $h$ and $i$ are forced by vertices $g$ and $h$, respectively. Thus, $P$ is a zero forcing hop dominating set of $H$. Since $P$ is the minimum hop domnating set of $H$, it follows that $\gamma_{zh}(H) = 3$.

Now, let $D = \{b, e, h, g\}$. Then $D$ is a dominating set of $H$. Observe that vertices $d$ and $c$ are forced by vertices $g$ and $d$, respectively, and vetices $a, f$ and $i$ are forced by vertices $b, e$ and $h$, respectively. Thus, $D$ is a zero forcing dominating set of $H$. Since $\{b, e, h\}$ is not a zero forcing set, it follows that $D$ is a minimum zero forcing dominating set of $H$. Therefore, $\gamma_{zh}(H) = 4$.

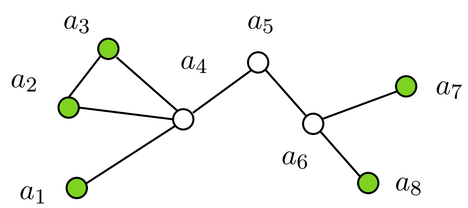
**Proposition 1.** *Let G be a graph. Then*

*(i) a dominating set D of G may not be a zero forcing set of G; and*

*(ii) a zero forcing set T of G may not be a dominating set of G;*

*Proof.* (i) Consider the G below:

$G$ :




Let $D = \{a_4, \ a_6\}$. Then $N_G\,[a_4] = \{a_1, \ a_2, \ a_3, \ a_4, \ a_5\}$ and $N_G\,[a_6] = \{a_5, \ a_6, \ a_7, \ a_8\}$. Thus, $N_G\,[D] = \ V(G)$, showing that $D$ is a dominating set of $G$. However, $D$ is not a zero forcing set of $G$ since vertices $a_4$ and $a_6$ cannot force any other vertices in $V(G) \setminus D$.

(ii) Consider $T = \{a_1, a_{2,}, a_3, a_7, a_8\}$. Then vertices $a_4,\ a_5$ and $a_6$ are forced by vertices $a_1,\ a_4$ and $a_5$, respectively. Thus, $T$ is a zero forcing set of $G$. However, $T$ is not dominating set of $G$ since $a_5$ cannot be dominated by any vertex in $T$.

**Proposition 2.** *Let G be a graph. Then*

*(i)* $Z(G) \leq \gamma_z(G)$;

*(ii)* $\gamma(G) \leq \gamma_z(G)$; *and*

*(iii)* $1 \leq \gamma_z(G) \leq |V(G)|$.

*Proof.* (i) Let $G$ be a graph and let $Q$ be a minimum zero forcing dominating set of $G$. Then $Q$ is a zero forcing set of $G$. Since $Z(G)$ is the minimum cardinality of a zero forcing set of $G$, it follows that $\gamma_z(G) = |Q| \geq Z(G)$.

(ii) Let $Q_1$ be minimum zero forcing domination of $G$. Then $Q_1$ is a dominating set of $G$. Since $\gamma(G)$ is the minimum cardinality of a dominating set of $G$, we have $\gamma_z(G) = |Q_1| \geq \gamma(G)$.

(iii) Since $\gamma(G) \geq 1$ for any graph $G$, by (ii), $\gamma_z(G) \geq 1$. Sine every zero forcing dominating set $P$ is always a subset of a vertex-set $V(G)$, we have $\gamma_z(G) \leq |V(G)|$. Therefore, $1 \leq \gamma_z(G) \leq |V(G)|$.

**Lemma 1.** *Let G be graph. Then* $Z(G) = 1$ *if and only if* $G = P_n$ *for all* $n \geq 1$.

**Theorem 1.** *Let G be a graph. Then* $\gamma_z(G) = Z(G) = 1$ *if and only if* $G = P_1$ *or* $G = P_2$.

*Proof.* Suppose that $\gamma_z(G) = Z(G) = 1$. Then by Lemma 1, $G = P_n\ \forall\ n \geq 1$. If $n \geq 4$, then $\gamma(P_n) \geq 2$. Since $\gamma_z(G) \geq \gamma(G)$for any graph $G$, $\gamma_z(G) \geq 2$, a contradiction. Assume that $n = 3$. Let $V(P_3) = \{a_1,\ a_2,\ a_3\}$. Note that $\{a_2\}$ is a dominating but not a zero forcing set in $P_3$ and $\{a_1\}$ or $\{a_3\}$ is a zero forcing but not a dominating set in $P_3$. Let $N = \{a_1, a_2\}$. Then $N$ is a minimum zero forcing dominating set $P_3$. Thus, $\gamma_z(P_3) = 2$, which is a contradiction. Therefore, either $n = 1$ or $n = 2$, that is $G = P_1$ or $G = P_2$.

Conversely, suppose that $G = P_1$. Then $\gamma_z(G) = 1 = Z(P_1)$. Similarly, when $G = P_2$, then $\gamma_z(G) = 1 = Z(P_2)$.

□

**Theorem 2.** *Let G be a graph. Then* $\gamma_z(G) = |V(G)| = Z(G)$ *if and only if* $G = \overline{K}_n$.

*Proof.* Suppose that $\gamma_z(G) = |V(G)| = Z(G)$. Then $V(G)$ is the minimum zero forcing set of $G$. Suppose there is a component $H$ of $G$ which is non-trivial. Let $V(H) = \{x_1, \ldots, x_k\}$, where $k \geq 2$. Then $V(G) \setminus \{x_1\}$ is a zero forcing set of $G$. Thus, $Z(G) \leq |V(G)| - 1$, a contradiction. Therefore, every component of $G$ is trivial, that is, $G = \overline{K}_n$.

Conversely, suppose that $G = \overline{K}_n$. Then $V(G)$ is the minimum dominating set of $G$, and so $\gamma(G) = |V(G)|$. Since $\gamma_z(G) \geq \gamma(G)$, it follows that $\gamma_z(G) = |V(G)|$. Clearly, $V(G)$ is the only zero forcing set of $G$. Consequently, $Z(G) = |V(G)| = \gamma_z(G)$. □

**Corollary 1.** *Let $G$ be a graph. Then $\gamma_Z(G) \leq |V(G)| - 1$ If and only if $G$ has non-trivial component.*

**Theorem 3.** *Let $n \geq 2$ be a positive integer and let $S$ be a zero forcing dominating set of $K_n$. Then $S$ is a minimum zero forcing dominating set of $K_n$ if and only if $|S| = n - 1$.*

*Proof.* Let $S$ be a minimum zero forcing dominating set of $K_n$ and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $|S| \leq n - 2$. Then there exist $u_i, u_j \in V(K_n)$ such that $u_i, u_j \notin S$ for some $i, j \in \{1, 2, \ldots, n\}$. Now, let $u_t \in S$, where $t \neq i, j$. Since the graph is complete, $u_t$ is adjacent to both $u_i$ and $u_j$. Thus, $u_t$ cannot force either $u_i$ or $u_j$. Since $u_t$ is arbitrary, it follows that $S$ is not a zero forcing set of $K_n$, which is a contradiction. Therefore, $|S| \geq n-1$. Since $S = V(K_n) \setminus \{v_k\}$ is a zero forcing dominating set of $K_n$ for some $k \in \{1, 2, \ldots, n\}$, the assertion follows.

Conversely, suppose that $|S| = n-1$, say $S = V(K_n) \setminus \{v_i\}$ for some $i \in \{1, 2, \ldots, n\}$. Then $S$ is a zero forcing dominating set of $K_n$. If $S$ is not the minimum, then there exists $T \subset S \subset V(K_n)$ such that $T$ is a zero forcing dominating set of $K_n$. Let $v_s, v_t \in V(K_n)$ such that $v_s, v_t \notin T$. Then any element of $T$ cannot force either $v_s$ or $v_t$, a contradiction. Hence, $S$ must be a minimum zero forcing dominating set of $K_n$.

□

**Corollary 2.** *Let $m$ be a positive integer. Then*

$$\gamma_Z(K_m) = \begin{cases} 1, & \text{if } m = 1 \\ m - 1, & \text{if } m \geq 2. \end{cases}$$

**Theorem 4.** *Let $G$ and $H$ be two non-complete graphs. Then $Q \subseteq V(G+H)$ is a zero forcing dominating set in $G+H$ if and only if $Q = Q_G \cup Q_H$ and satisfies one of the following:*

*(i) $Q_G = V(G)$ and $Q_H$ is a zero forcing set of $H$.*

*(ii) $Q_H = V(H)$ and $Q_G$ is a zero forcing set of $G$.*

*(iii) $Q_G = V(G) \setminus \{x\}$ and $Q_H$ is a zero forcing set of $H$ with $N_H[y] \cap (V(H) \setminus Q_H) = \emptyset$ for some $y \in Q_H$ and $x \in V(G)$.*

*(iv)* $Q_H = V(H) \setminus \{u\}$ *and* $Q_G$ *is a zero forcing set of* $G$ *with* $N_G[v] \cap (V(G) \setminus Q_G) = \emptyset$ *for some* $v \in Q_G$ *and* $u \in V(G)$.

*Proof.* Suppose that $Q = Q_G \cup Q_H$ is a zero forcing dominating set of $G+H$. Assume that $Q_G = V(G)$. If $Q_H = V(H)$, then (i) and (ii) follows. Suppose that $Q_H \neq V(H)$. If $Q_H$ is not a zero forcing set, then there exists two vertices $x, y \in V(H) \setminus Q_H$ such that $x$ and $y$ cannot be forced by any element in $Q_H$. Therefore, $Q = Q_G \cup Q_H$ is not a zero forcing set, a contradiction. Hence, $Q_H$ must be a zero forcing set of $H$, and so (i) holds. Similarly, (ii) holds.

Now, suppose that $Q_G \neq V(G)$ and $Q_H \neq V(H)$. Since $Q$ is zero forcing, either $|Q_G| = |V(G)| - 1$ or $|Q_H| = |V(H)| - 1$. Assume that $|Q_G| = |V(G)| - 1$. Since $Q$ is a zero forcing set of $G+H$, $Q_H$ must be a zero forcing in $H$. Now, suppose that for every element $w \in Q_H, N_H[w] \cap (V(G) \setminus Q_H) \neq \emptyset$. Since $|Q_G| = |V(G)| - 1$, and $H$ is non-complete graph, it follows that any element $w \in Q_H$ cannot force any element in $V(G) \setminus \{x\}$ and in $V(H) \setminus Q_H$, respectively, a contradiction. Hence (iii) holds. Similarly, (iv) holds.

Conversely, suppose that (i) holds. Since $Q_H$ is a zero forcing set in $H, Q = Q_G \cup Q_H$ is a zero forcing set in $G+H$. Thus, $Q$ is a zero forcing dominating set of $G+H$. Similarly, the assertion follows when (ii) holds.

Next, suppose that (iii) holds. Then $Q$ is a dominating set of $G+H$. Let $y \in Q$ such that $N_H[y] \cap (V(H) \setminus Q_H) = \emptyset$. Then $y$ forces vertex $x$ in $G$. Since $Q_H$ is a zero forcing set in $H$, it follows that $Q = Q_G \cup Q_H$ is a zero forcing set of $G+H$. Therefore, $Q$ is zero forcing dominating set of $G+H$. Similarly, the assertion holds, when (iv) is true. □

**Corollary 3.** *Let* $G$ *and* $H$ *be two non-complete graphs. Then*

$$\gamma_z(G+H) = min\{|V(G)| + Z(H),\ |V(H)| + Z(G\}.$$

## 4. Conclusion

The concept of zero forcing domination has been introduced and initially investigated in this study. Its bounds concerning other known parameters in graph theory have been established. Moreover, the zero forcing domination number of some graphs has been determined. In addition, characterizations of zero forcing dominating sets in some graphs were formulated and were used to solve the exact value of the parameter of these graphs. Analyzing graphs that aren't covered in this study may be intriguing and provide a different view

## Acknowledgements

The authors would like to thank Mindanao State University - Tawi-Tawi College of Technology and Oceanography, Korea University, and Ateneo de Davao University for funding this research.

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