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Restrained Global Defensive Alliances in Graphs

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Abstract. A *defensive alliance* in a graph G is a nonempty set of vertices $S \subseteq V(G)$ such that for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V(G) \setminus S)|$. A defensive alliance S is called *global* if every vertex in $V(G) \setminus S$ is adjacent to at least one member of the alliance S. In this paper, the concept of restrained global defensive alliance in graphs was introduced. In particular, a global defensive alliance S is a *restrained global defensive alliance* if the induced subgraph of $V \setminus S$ has no isolated vertex. Here, some properties of this alliance were identified, and its bounds were also determined. In addition, the restrained global defensive alliance number was also formulated, along with characterizations of some special classes of graphs, specifically complete, complete bipartite, and path graphs.

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Key Words and Phrases: Restrained domination, defensive alliance, global defensive alliance, restrained global defensive alliance

1. Introduction

An alliance refers to a gathering of individuals, organizations, or states aimed at achieving a common goal, mutual protection, or asserting dominance over those outside the alliance. For this reason, Kristiansen and colleagues explored and developed defensive and offensive alliances in the graphs [12].

In defensive alliances, the collaboration of nodes or entities achieved mutual security and protection. They established resilient networks capable of withstanding external influences and pressure. If these alliances were also dominating, then they are called global defensive alliances [11].

Global offensive alliances and global defensive alliances have been a focus of study among mathematics enthusiasts. Some of these studies include global offensive alliances in some special classes of graphs in 2011 by Cabahug and Isla [3], global defensive alliances in the lexicographic product of paths and cycles in 2020 by Barbosa, Dourado, and Da

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Silva [1], and global defensive k-alliances in directed graphs focusing on combinatorial and computational issues in 2020 by Mojdeh, Samadi, and Yero. Moreover, in 2022, Gaikwad and Maity studied globally minimal defensive alliances [9].

Domination in graphs is a growing area of research. Some recent studies on domination can be found in [4], [6], and [14]. On the other hand, in 1999, Hedetniemi and colleagues introduced the notion of a restrained dominating set wherein the subgraph induced by its complement has no isolated vertices [7]. Some studies related to restrained domination include fair restrained domination in graphs in 2020 by Enriquez [8] and restrained double Roman domination of a graph in 2022 by Mojdeh, Masoumi, and Volkmann [13].

Although global defensive alliance forms a defensive alliance that is also a dominating set, it could not guarantee an alliance where non-members were also adjacent to at least one non-member. To address this, a new type of alliance had to be formed. With this in mind, the authors decided to introduce restrained global defensive alliances in graphs. Using this alliance as a basis, the authors aim to contribute new insights to applications related to strategic interactions and mutual support within networks by establishing certain characterizations, developing formulas for the restrained global defensive alliance number, and determining some of its inherent properties on complete, complete bipartite, and path graphs.

2. Terminology and Notation

A graph G is a finite nonempty set $V(G)$ of objects called vertices (the singular is vertex) together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called edges. Here, $V(G)$ is the vertex set of a graph G while $E(G)$ is the edge set of graph G [5]. An edge joining a vertex to itself is called a *loop*. Two or more edges that join the same pair of distinct vertices are called *parallel edges*. If a graph has no loops and parallel edges then it is a *simple graph*. The *order* of a graph G refers to the number of vertices in G while the *size* of a graph G refers to the number of edges in G [5]. If uv is an edge of a graph G, then u and v are *adjacent vertices*. Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex v is called the *open neighborhood* of v(or simply the neighborhood of v) and is denoted by $N_G(v)$, or $N(v)$ if the graph G is understood. The set $N[v] = N(v) \bigcup \{v\}$ is called the *closed neighborhood* of v [5].

The *degree of a vertex* v in a graph G, denoted by deg v, is the number of vertices in G that are adjacent to v. The largest degree among the vertices of G is called the *maximum degree* of G, denoted by $\triangle(G)$, while the smallest degree among the vertices of G is called the *minimum degree* of G, denoted by $\delta(G)$ [5]. A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end-vertex* or a *leaf* [5].

For an integer $n \geq 1$, the path P_n is a graph of order n and size $n-1$ whose vertices can be labeled by $v_0, v_1, ..., v_{n-1}$ and whose edges are v_iv_{i+1} for $i = 0, 1, 2, ..., n-2$ [5]. A *complete graph* of order n, denoted by K_n, is graph with n vertices where in every pair of distinct vertices are adjacent [10]. An *empty graph* of order n is graph with n vertices where in every pair of distinct vertices are not adjacent [5]. A graph G is a *complete bipartite graph* if $V(G)$ can be partitioned into two sets A_1 and A_2 (called partite sets) so that uv is an edge of G if and only if $u \in A_1$ and $v \in A_2$. If $|A_1| = m$ and $|A_2| = n$, then this complete bipartite graph, denoted by $K_{m,n}$ (or $K_{n,m}$), has order $m + n$ and size mn. The complete bipartite graph $K_{1,n}$ is called a *star* [5].

A graph H is a *subgraph* of a graph G if the vertex set $V(H)$ of H is contained in the vertex set $V(G)$ of G and all edges of H are edges in G, i.e, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any vertex subset $S \subseteq V(G)$, the *induced subgraph* by S denoted by $\langle S\rangle_G$ contains all the edges of $E(G)$ whose extremities belong to S [2].

A set S of vertices of G is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as a minimum dominating set [5].

A *restrained dominating set* in a graph G is a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S as well as another vertex in $V(G) \setminus S$. In this case, the induced subgraph $\langle V(G) \setminus S\rangle$ has no isolated vertices. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G [7].

A *defensive alliance* in a graph G is a nonempty set of vertices $S \subseteq V(G)$ if for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V(G) \setminus S)|$. A defensive alliance S is called *global* if it effects every vertex in $V(G) \setminus S$, that is, every vertex in $V(G) \setminus S$ is adjacent to at least one member of the alliance S. In this case, S is a dominating set. The *global defensive alliance number* of G, denoted $\gamma_a(G)$, is the minimum size around all the global defensive alliances of G [11].

3. Results

This paper utilized the following terms to denote specific concepts: *ds* signified dominating set, *da* represented defensive alliance, *rds* stood for restrained dominating set, *gda* indicated global defensive alliance, and *rgda* denoted restrained global defensive alliance. Moreover, if G is a graph, its vertex set $V(G)$ and edge set $E(G)$ are denoted as V and E, respectively. Additionally, graphs considered is this paper are simple, finite, and undirected graphs.

Definition 1. *A restrained global defensive alliance of a graph $G = (V, E)$ is a set S of vertices of G that is a restrained dominating set and global defensive alliance. A set S with the least number of vertices is called a minimum restrained global defensive alliance. The cardinality of a minimum restrained global defensive alliance is called the restrained global defensive alliance number denoted by $\gamma_{ra}(G)$.*

Example 1. *In Figure 1, consider a set $S = \{v_0, v_1\}$ in $K_4 = (V, E)$. Notice that $V \setminus S = \{v_2, v_3\}$, and both v_2 and v_3 are adjacent to v_0. This means that S is a ds. Moreover, $\langle V \setminus S\rangle$ has no isolated vertices since v_2 is adjacent to v_3. This means that S is an rds. Now, it remains to show that S is a da. Observe that*

$$|N[v_0] \cap S| = |\{v_0, v_1\}| = 2 \geq 2 = |\{v_2, v_3\}| = |N(v_0) \cap (V \setminus S)|$$

and

$$|N[v_1] \cap S| = |\{v_0, v_1\}| = 2 \geq 2 = |\{v_2, v_3\}| = |N(v_1) \cap (V \setminus S)|.$$

Hence, S is a da. This implies that S is also a gda. Therefore, by Definition 1, S is an $rgda$.

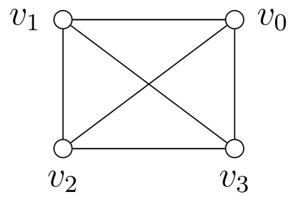

Figure 1: A Complete graph K_4.

Theorem 1. *Let $G = (V, E)$ be any graph of order $n \geq 1$. Then the set V is a restrained global defensive alliance in G. As consequence, $\gamma_{ra}(G) \leq n$.*

Proof. Let $G = (V, E)$ be any graph of order $n \geq 1$. Since V dominates itself and $V \setminus V$ is empty, it vacuously implies that V is an rds. For the same reason, notice that for every $v \in V$, $|N[v] \cap V| = |N[v]| \geq 0 = |\varnothing| = |N(v) \cap (V \setminus V)|$. Hence, V is a da in G. So, V is an $rgda$ in G.

Now, if no set $W \subset V$ is an $rgda$ in G, then $\gamma_{ra}(G) = |V| = n$. On the other hand, if there exist a set $W \subset V$ that is also an $rgda$ in G, then $\gamma_{ra}(G) < n$. Hence, $\gamma_{ra}(G) \leq n$. □

Theorem 2. *Let $G = (V, E)$ be a graph with leaf vertices. If $S \subseteq V$ is a restrained global defensive alliance in G, then S contains the leaf vertices of G.*

Proof. Let S be an $rgda$ in $G = (V, E)$. Assume that S does not contain all the leaf vertices of G. Then there must exist a leaf vertex $v \in V$ such that $v \notin S$. Suppose that v is adjacent to a vertex $a \in V$. This leads to the following cases:

Case 1: $a \notin S$.
Then no vertices in S can dominate v. This means that S is not a ds, a contradiction.

Case 2: $a \in S$.
Then $\langle V \setminus S \rangle$ contains an isolated vertex v. This means that S is not an rds, a contradiction.

Since neither of the cases holds, then $v \in S$. Therefore, every leaf vertex of G must be in S. □

Theorem 3. *Let $G = (V, E)$ be any graph of order n. Then $\gamma_{ra}(G) = 1$ if and only if G is a trivial graph.*

Proof. Let $\gamma_{ra}(G) = 1$. By Theorem 1 with $n = 1$, a graph containing a single vertex, trivial graph, is an *rgda*. If G is a trivial graph, then $\gamma_{ra}(G) = 1$. So, G can be a trivial graph.

Now, assume that G can also be a nontrivial graph with order $n \geq 2$. Then there must exist a singleton set $\{a\} \subset V$ that is an *rgda* in G. Observe,

Case 1 : G has order $n = 2$. Then $\langle V \setminus \{a\} \rangle$ is an isolated vertex. So, $\{a\}$ is not an *rds*, a contradiction.

Case 2 : G has an order $n \geq 3$. Then, knowing that G must be a *ds*, for every $a \in \{a\}$ implies

$$|N[a] \cap \{a\}| = |\{a\}| = 1 \ngeq n - 1 = |V \setminus \{a\}| = |N(a) \cap (V \setminus \{a\})|.$$

So, $\{a\}$ is not a *da*, a contradiction.

Since neither of the cases holds, G cannot be a nontrivial graph. Therefore, G must be a trivial graph.

Conversely, let $G = (V, E)$ be a trivial graph. By Theorem 1, V is an *rgda* in G. Since an empty set of G cannot dominate G, then V must be the minimum *rgda* in G. Therefore, $\gamma_{ra}(G) = |V| = 1$. □

Theorem 4. *Let $G = (V, E)$ be a graph with isolated vertices and $S \subseteq V$ be any restrained global defensive alliance in G. If v is an isolated vertex in G, then $v \in S$.*

Proof. Let S be an *rgda* in $G = (V, E)$ and $v \in G$ be an isolated vertex. Assume that $v \notin S$. Then $\langle V \setminus S \rangle$ contains an isolated vertex v. This means that S is not an *rds*, a contradiction. Hence, $v \in S$. □

Corollary 1. *Let $E_n = (V, E)$ be an empty graph of order $n \geq 1$. Then, $\gamma_{ra}(E_n) = n$.*

Theorem 5. *If $G = (V, E)$ is any graph with restrained global defensive alliance S, then $1 \leq |S| \leq n$.*

Proof. Let S be an *rgda* in $G = (V, E)$. By Theorem 1, $\gamma_{ra}(G) \leq n$. This means that $|S| \leq n$. Since S is necessarily a nonempty set, then $|S| \geq 1$. Therefore, $1 \leq |S| \leq n$. □

Theorem 6. *Let $K_n = (V, E)$ be a complete graph of order $n \geq 4$. Then $S \subseteq V$ is a restrained global defensive alliance if and only if the following holds:*

i. $|S| \geq \lceil \frac{n}{2} \rceil$;

ii. $|S| \neq n - 1$.

Proof. Let S be an *rgda* in $K_n = (V, E)$ of order $n \geq 4$. Assume that S does not satisfy *i* and *ii*. This means that either $|S| \ngeq \lceil \frac{n}{2} \rceil$ or $|S| = n - 1$.

Case 1 : Suppose that $|S| \ngeq \lceil \frac{n}{2} \rceil$. Since S is an *rgda* and at least one vertex is necessary to dominate K_n, S must not be empty. Then for every $v \in S$ implies

$$|N[v] \cap S| = |S| \ngeq \left\lceil \frac{n}{2} \right\rceil \leq |V| - |S| = |N(v) \cap (V \setminus S)|.$$

So, S is not a *da*, a contradiction. Therefore, $|S| \geq \lceil \frac{n}{2} \rceil$. This proves *i*.

Case 2 : Suppose that $|S| = n - 1$. Then there exists a unique vertex $a \in V$ such that $a \notin S$. This means that a is not adjacent to another vertex in $V \setminus S$. So, S is not an *rds*, a contradiction. Hence, $|S| \neq n - 1$. This proves *ii*.

Hence, *i* and *ii* must be true.

Conversely, let $S \subseteq V$ be a set in $K_n = (V, E)$, of order $n \geq 4$, that satisfies *i* and *ii*. By *i*, S is, necessarily, a nonempty set and for every $v \in S$,

$$|N[v] \cap S| = |S| \geq \left\lceil \frac{n}{2} \right\rceil \geq n - \left\lceil \frac{n}{2} \right\rceil \geq |V \setminus S| = |N[v] \cap (V \setminus S)|.$$

So, S is a *da*. Since every vertex in K_n is adjacent to one another, S is also a *ds*. By *ii*, $\langle V \setminus S \rangle$ does not contain an isolated vertex. Hence, S is an *rds*. Therefore, S is an *rgda* in K_n. □

Corollary 2. *Let $K_n = (V, E)$ be a complete graph of order $n \geq 1$. Then*

$$\gamma_{ra}(K_n) = \begin{cases} |V| & \text{if } n = 1, 2, 3\,; \\ \lceil \frac{n}{2} \rceil & \text{if } n \geq 4. \end{cases} \tag{1}$$

Proof. Let S be an *rgda* in $K_n = (V, E)$ with order $n \geq 1$.

Case 1 : $n = 1, 2, 3$.

Subcase 1 : $n = 1$. By Theorem 1, V is an *rgda* in K_1. Since K_1 has only one vertex, then V is the minimum *rgda* in K_1. Therefore, $\gamma_{ra}(K_1) = |V| = 1$.

Subcase 2 : $n = 2$. Notice that every vertex in V is a leaf vertex. By Theorem 2, $V \in S$. So, $\gamma_{ra}(K_2) = |V| = 2$.

Subcase 3 : $n = 3$.

If S is a singleton set, say, $S = \{a\}$ where $a \in V$, then $|N[a] \cap S| = |\{a\}| = 1 \ngeq 2 = |\{V \setminus \{a\}\}| = |N(a) \cap (V \setminus S)|$. So, S is not a *da*, a contradiction. Hence, $S \neq \{a\}$.

If S has two vertices, say $S = \{a, b\}$ where $a, b \in V$, then $\langle V \setminus S \rangle$ contains an isolated vertex $c \in V \setminus S$. So, S is not an *rds*, a contradiction. Hence, $S \neq \{a, b\}$.

If $S = V$, then by Theorem 1, V is an *rgda* in K_3.

Now, since $S = V$ is the only *rgda* in K_3, it is also the minimum *rgda* in K_3. Hence, $\gamma_{ra}(K_3) = |S| = |V| = 3$.

Case 2 : $v \geq 4$. By Theorem 6(i), $|S| \geq \lceil \frac{n}{2} \rceil$. This implies that the smallest value for $|S|$ is $\lceil \frac{n}{2} \rceil$. Therefore, $\gamma_{ra}(K_n) = \lceil \frac{n}{2} \rceil$.

□

Theorem 7. *Let $K_{m,n} = (V, E)$ be a complete bipartite graph with partite sets A_1 and A_2 such that $|A_1| = m$ and $|A_2| = n$ where $m, n \geq 2$. Then $S \subseteq V$ is a restrained global defensive alliance if and only if the following holds:*

i. $|S \cap A_1| \geq \lfloor \frac{m}{2} \rfloor$ *and* $|S \cap A_2| \geq \lfloor \frac{n}{2} \rfloor$;

ii. $|S \cap A_1| = m$ *if and only if* $|S \cap A_2| = n$.

Proof. Let S be a $rgda$ in $K_{m,n} = (V, E)$. Suppose that i and ii are false. Then either i or ii is not true.

Case 1 : i is false. Then either $|S \cap A_1| \ngeq \lfloor \frac{m}{2} \rfloor$ or $|S \cap A_2| \ngeq \lfloor \frac{n}{2} \rfloor$. Observe the following subcases:

Subcase 1 : $|S \cap A_1| \ngeq \lfloor \frac{m}{2} \rfloor$.
Since S is an $rgda$, at least one vertex in A_2 must exist to dominate all vertices in A_1. The same is true for the other partite set. So, $|S \cap A_1|$ and $|S \cap A_2|$ are both nonempty. Then for every $v \in S \cap A_2$,

$$\begin{aligned} |N[v] \cap S| &= |S \cap A_1| + |\{v\}| \\ &= |S \cap A_1| + 1 \\ &\ngeq m - |S \cap A_1| \\ &= |A_1 \setminus S| \\ &= |N(v) \cap V \setminus S|. \end{aligned}$$

So, S is not a da, a contradiction. Hence, $|S \cap A_1| \geq \lfloor \frac{m}{2} \rfloor$.

Subcase 2 : $|S \cap A_2| \ngeq \lfloor \frac{n}{2} \rfloor$.
Using similar argument as Subcase 1, it follows that $|S \cap A_2| \geq \lfloor \frac{n}{2} \rfloor$.

Therefore, $|S \cap A_1| \geq \lfloor \frac{m}{2} \rfloor$ and $|S \cap A_2| \geq \lfloor \frac{n}{2} \rfloor$. This proves i.

Case 2 : ii is false. Then either $|S \cap A_1| = m$ and $|S \cap A_2| \neq n$ or $|S \cap A_2| = n$ and $|S \cap A_2| \neq m$.

Subcase 1 : $|S \cap A_1| = m$ and $|S \cap A_2| \neq n$.
This means that $A_1 \setminus S \subseteq V \setminus S$ is empty and $A_2 \setminus S \subseteq V \setminus S$ is nonempty. Since every vertex in A_2 is only adjacent to vertices in A_1, then every vertex $v \in A_2 \setminus S \subseteq V \setminus S$ is not adjacent to another vertex in $V \setminus S$. Hence, S is not an rds, a contradiction.

Subcase 2 : $|S \cap A_2| = n$ and $|S \cap A_1| \neq m$.
Since A_1 and A_2 are arbitrary, similar argument as Subcase 1 holds.

Therefore, $|S \cap A_1| = m$ if and only if $|S \cap A_2| = n$. This proves *ii*.

Conversely, let $S \subseteq V$ be a set in $K_{m,n} = (V, E)$ where $m, n \geq 2$ that satisfies *i* and *ii*. By *i*, $|S \cap A_1| \geq \left\lfloor \frac{2}{2} \right\rfloor = 1$ and $|S \cap A_2| \geq \left\lfloor \frac{2}{2} \right\rfloor = 1$, so, S is nonempty and a *ds*. Moreover, for every $v \in S \cap A_1$,

$$|N[v] \cap S| = |A_2 \cap S| + |\{v\}| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq n - \left\lfloor \frac{n}{2} \right\rfloor \geq |A_2 \setminus S| = |N(v) \cap (V \setminus S)|.$$

On the other hand, for every $w \in S \cap A_2$,

$$|N[w] \cap S| = |A_1 \cap S| + |\{w\}| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1 \geq m - \left\lfloor \frac{m}{2} \right\rfloor \geq |A_1 \setminus S| = |N(w) \cap (V \setminus S)|.$$

This means that S is a *da* in $K_{m,n}$. By *i* and *ii*, it is guaranteed that whenever $A_1 \setminus S$ is nonempty, $A_2 \setminus S$ is also nonempty. Since every vertex in A_1 is adjacent to A_2, then S is an *rds*. Therefore, S is a *rgds* in $K_{m,n}$. □

Corollary 3. *Let $K_{m,n} = (V, E)$ be a complete bipartite graph. If $m, n \geq 2$, then $\gamma_{ra}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let S be an *rgda* in $K_{m,n} = (V, E)$ where $m, n \geq 2$, and A_1 and A_2 be its partite sets such that $|A_1| = m$ and $|A_2| = n$. By Theorem 7(i), $|S \cap A_1| \geq \left\lfloor \frac{m}{2} \right\rfloor$ and $|S \cap A_2| \geq \left\lfloor \frac{n}{2} \right\rfloor$. Now, if $X_1 \subseteq S \cap A_1$ such that $|X_1| = \left\lfloor \frac{m}{2} \right\rfloor$ and $X_2 \subseteq S \cap A_2$ such that $|X_2| = \left\lfloor \frac{n}{2} \right\rfloor$, then $X_1 \cup X_2 = X \subseteq S$ is the minimum restrained global defensive alliance in $K_{m,n}$. Therefore,

$$\gamma_{ra}(K_{m,n}) = |X| = |X_1| + |X_2| = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

□

Theorem 8. *(Path Graph) Let $P_n = (V, E)$ be a path graph with $n \geq 2$. Then $S \subseteq V$ is a restrained global defensive alliance if and only if the following holds:*

i. the leaf vertices of P_n are in S;

ii. $\langle S \rangle$ has no isolated vertices that are not leaf vertices of P_n;

iii. $\langle V \setminus S \rangle$, where $S \subset V$, forms a class of P_2.

Proof. Let S be an *rgda* in $P_n = (V, E)$ with $n \geq 2$. Suppose that *i*, *ii*, and *iii* are false. Then either *i*, *ii* or *iii* is not true. Observe the following cases.

Case 1 : *i* is false.
It is known that P_n has leaf vertices. By Theorem 2, every leaf vertex must be in S. Hence, the leaf vertices of P_n must be in S. This proves *i*.

Case 2 : *ii* is false.
Then $\langle S\rangle$ has atleast one isolated vertex v that is not a leaf vertex of P_n. This implies that v is adjacent to two vertices $w, x \in V \setminus S$. So,

$$|N[v] \cap S| = |\{v\}| = 1 \not\geq 2 = |\{w, x\}| = |N(v) \cap (V \setminus S)|.$$

Hence, S is not a da, a contradiction. Thus, $\langle S\rangle$ has no isolated vertices that are not leaf vertices of P_n. This proves *ii*.

Case 3 : *iii* is false.
Then $S \subset V$ and $\langle V \setminus S\rangle$ does not form a class of P_2. This implies that there exists a component in $\langle V \setminus S\rangle$ that is not P_2. Observe the following cases:

Subcase 1 : If $\langle V \setminus S\rangle$ has a component P_1, then S is not an rds, a contradiction.

Subcase 2 : If $\langle V \setminus S\rangle$ has a component P_t where $3 \leq t \leq n$, then S can only dominate the leaf vertices of P_t. This means that S is not a ds, a contradiction.

Hence, $\langle V \setminus S\rangle$ forms a class of P_2. This proves *iii*.

Conversely, let $S \subseteq V$ be a set in $P_n = (V, E)$, with $n \geq 2$ that satisfies *i*, *ii*, and *iii*. Since P_n contains leaf vertices, by *i*, S is a nonempty set. By *i* and *iii*, the leaf vertices of P_n are in S and $\langle V \setminus S\rangle$ form a class of P_2. This implies that every vertex $v \in V \setminus S$ is adjacent to a vertex in S and another vertex in $V \setminus S$. Hence, S is an rds. In P_n, every vertex in V is adjacent to at most two vertices. So, by *i* and *ii*, for every $a \in S$ implies the following:

Case 1 : a is a leaf vertex

Subcase 1 : a is adjacent to another vertex in S.
Then $|N[a] \cap S| = 2 \geq 0 = |N(a) \cap (V \setminus S)|$.

Subcase 2 : a is not adjacent to another vertex in S.
Then $|N[a] \cap S| = 1 \geq 1 = |N(a) \cap (V \setminus S)|$.

Case 2 : a is not a leaf vertex

Subcase 1 : a is adjacent to one vertex in S.
Then $|N[a] \cap S| = 2 \geq 1 = |N(a) \cap (V \setminus S)|$.

Subcase 2 : a is adjacent to two vertices in S.
Then $|N[a] \cap S| = 3 \geq 0 = |N(a) \cap (V \setminus S)|$.

These implies that S is a da. Therefore, by *i*, *ii*, and *iii*, S is an $rgda$ in P_n. □

Lemma 1. *Let $P_n = (V, E)$, $n \equiv 0 \ (mod\ 4)$, be a path graph of order $n \geq 2$. Then $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-5}, v_{n-4}\}$ is a minimum restrained global defensive alliance in P_n.*

Proof. Let $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-5}, v_{n-4}\}$. Notice that all the leaf vertices of P_n are in S, that is, $v_0, v_{n-1} \in S$. So, Theorem 8 (i) is satisfied. Additionally, $\langle\{v_3, v_4, v_7, v_8, \ldots, v_{n-5}, v_{n-4}\}\rangle$ has no isolated vertices, so $\langle S\rangle$ has no isolated vertices that are not leaf. This means that Theorem 8 (ii) is satisfied. Moreover,
$\langle V \setminus S\rangle = \langle\{v_1, v_2, v_5, v_6, \ldots, v_{n-3}, v_{n-2}\}\rangle$ forms a class of P_2, hence, Theorem 8 (iii) is satisfied. This means that, by Theorem 8, S is an *rgda* in P_n.

Now, suppose that $W \subset S$. Then there exists a vertex in S that is not in W. This leads to the following cases:

Case 1 : atleast one vertex in $\{v_0, v_{n-1}\}$ is not in W.
Then Theorem 8 (i) is not satisfied, so W is not an *rgda* in P_n.

Case 2 : atleast one vertex in $\{v_3, v_4, v_7, v_8, \ldots, v_{n-5}, v_{n-4}\}$ is not in W.
Then there exists a class in $\langle V \setminus W\rangle$ that is not P_2, so Theorem 8 (iii) is not satisfied. Hence, W is not an *rgda* in P_n.

Therefore, S is a minimum *rgda* in P_n. □

Lemma 2. *Let $P_n = (V, E)$, $n \equiv 1 \ (mod\ 4)$, be a path graph of order $n \geq 2$. Then $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-6}, v_{n-5}\} \cup \{v_{n-2}\}$ is a minimum restrained global defensive alliance in P_n.*

Proof. Let $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-6}, v_{n-5}\} \cup \{v_{n-2}\}$. Notice that all the leaf vertices of P_n are in S, that is, $v_0, v_{n-1} \in S$. So, Theorem 8 (i) is satisfied. Additionally, $\langle\{v_3, v_4, v_7, v_8, \ldots, v_{n-6}, v_{n-5}\}\rangle$ has no isolated vertices and v_{n-2} is adjacent to v_{n-1}, so $\langle S\rangle$ has no isolated vertices that are not leaf. This means that Theorem 8 (ii) is satisfied. Moreover, $\langle V \setminus S\rangle = \langle\{v_1, v_2, v_5, v_6, \ldots, v_{n-4}, v_{n-3}\}\rangle$ forms a class of P_2, hence, Theorem 8 (iii) is satisfied. This means that, by Theorem 8, S is an *rgda* in P_n.

Now, suppose that $W \subset S$. Then there exists a vertex in S that is not in W. This leads to the following cases:

Case 1 : atleast one vertex in $\{v_0, v_{n-1}\}$ is not in W.
Then Theorem 8 (i) is not satisfied, so W is not an *rgda* in P_n.

Case 2 : atleast one vertex in $\{v_3, v_4, v_7, v_8, \ldots, v_{n-6}, v_{n-5}\} \cup \{v_{n-2}\}$ is not in W.
Then there exists a class in $\langle V \setminus S\rangle$ that is not P_2, so Theorem 8 (iii) is not satisfied. Hence, W is not an *rgda* in P_n.

Therefore, S is a minimum *rgda* in P_n. □

Lemma 3. *Let $P_n = (V, E)$, $n \equiv 2 \ (mod\ 4)$, be a path graph of order $n \geq 2$. Then $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-3}, v_{n-2}\}$ is a minimum restrained global defensive alliance in P_n.*

Proof. Let $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-3}, v_{n-2}\}$. Notice that all the leaf vertices of P_n are in S, that is, $v_0, v_{n-1} \in S$. So, Theorem 8 (i) is satisfied. Additionally, $\langle\{v_3, v_4, v_7, v_8, \ldots, v_{n-3}, v_{n-2}\}\}\rangle$ has no isolated vertices, so $\langle S\rangle$ has no isolated vertices that are not leaf. This means that Theorem 8 (ii) is satisfied. Moreover,
$\langle V \setminus S\rangle = \langle\{v_1, v_2, v_5, v_6, \ldots, v_{n-5}, v_{n-4}\}\rangle$ forms a class of P_2, hence, Theorem 8 (iii) is satisfied. This means that, by Theorem 8, S is an *rgda* in P_n.

Now, suppose that $W \subset S$. Then there exists a vertex in S that is not in W. This leads to the following cases:

Case 1 : atleast one vertex in $\{v_0, v_{n-1}\}$ is not in W.
Then Theorem 8 (i) is not satisfied, so W is not an *rgda* in P_n.

Case 2 : atleast one vertex in $\{v_3, v_4, v_7, v_8, \ldots, v_{n-3}, v_{n-2}\}$ is not in W.
Then there exists a class in $\langle V \setminus S\rangle$ that is not P_2, so Theorem 8 (iii) is not satisfied. Hence, W is not an *rgda* in P_n.

Therefore, S is a minimum *rgda* in P_n. □

Lemma 4. *Let $P_n = (V, E)$, $n \equiv 3$ $(\,mod\ 4)$, be a path graph of order $n \geq 2$. Then $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-4}, v_{n-3}\} \cup \{v_{n-2}\}$ is a minimum restrained global defensive alliance in P_n.*

Proof. Let $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-4}, v_{n-3}\} \cup \{v_{n-2}\}$. Notice that all the leaf vertices of P_n are in S, that is, $v_0, v_{n-1} \in S$. So, Theorem 8 (i) is satisfied. Additionally, $\langle\{v_3, v_4, v_7, v_8, \ldots, v_{n-4}, v_{n-3}\}\rangle$ has no isolated vertices and v_{n-2} is adjacent to v_{n-1}, so $\langle S\rangle$ has no isolated vertices that are not leaf. This means that Theorem 8 (ii) is satisfied. Moreover, $\langle V \setminus S\rangle = \langle\{v_1, v_2, v_5, v_6, \ldots, v_{n-6}, v_{n-5}\}\rangle$ forms a class of P_2, hence, Theorem 8 (iii) is satisfied. This means that, by Theorem 8, S is an *rgda* in P_n.

Now, suppose that $W \subset S$. Then there exists a vertex in S that is not in W. This leads to the following cases:

Case 1 : atleast one vertex in $\{v_0, v_{n-1}\}$ is not in W.
Then Theorem 8 (i) is not satisfied, so W is not an *rgda* in P_n.

Case 2 : atleast one vertex in $\{v_3, v_4, v_7, v_8, \ldots, v_{n-4}, v_{n-3}\} \cup \{v_{n-2}\}$ is not in W.
Then there exists a class in $\langle V \setminus S\rangle$ that is not P_2, so Theorem 8 (iii) is not satisfied. Hence, W is not an *rgda* in P_n.

Therefore, S is a minimum *rgda* in P_n. □

Corollary 4. *If $P_n = (V, E)$ is a path graph of order $n \geq 1$, then*

$$\gamma_{ra}(P_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \ (\,mod\ 4) \\ \frac{n+1}{2} & n \equiv 1 \ (\,mod\ 4) \\ \frac{n+2}{2} & n \equiv 2 \ (\,mod\ 4) \\ \frac{n+3}{2} & n \equiv 3 \ (\,mod\ 4) \end{cases} \tag{2}$$

.

Proof. Let $P_n = (V, E)$ be a path graph of order $n \geq 2$. Observe the following cases:

Case 1 : $n \equiv 0 \pmod 4$. Let $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-5}, v_{n-4}\}$. By Lemma 1, S is a minimum *rgda* in P_n. Therefore,

$$\begin{aligned}\gamma_{ra}(P_n) &= |S| \\ &= |\{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-5}, v_{n-4}\}| \\ &= 2 + \frac{n-4}{2} \\ &= \frac{n-4+4}{2} \\ &= \frac{n}{2}.\end{aligned}$$

Case 2 : $n \equiv 1 \pmod 4$. Let

$$S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-6}, v_{n-5}\} \cup \{v_{n-2}\}.$$

By Lemma 2, S is a minimum *rgda* in P_n. Therefore,

$$\begin{aligned}\gamma_{ra}(P_n) &= |S| \\ &= |\{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-6}, v_{n-5}\} \cup \{v_{n-2}\}| \\ &= 2 + \frac{n-5}{2} + 1 \\ &= \frac{4+n-5+2}{2} \\ &= \frac{n+1}{2}.\end{aligned}$$

Case 3 : $n \equiv 2 \pmod 4$. Let $S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-3}, v_{n-2}\}$. By Lemma 3, S is a minimum *rgda* in P_n. Therefore,

$$\begin{aligned}\gamma_{ra}(P_n) &= |S| \\ &= |\{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-3}, v_{n-2}\}| \\ &= 2 + \frac{n-2}{2} \\ &= \frac{4+n-2}{2} \\ &= \frac{n+2}{2}.\end{aligned}$$

Case 4 : $n \equiv 3 \pmod 4$. Let

$$S = \{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-4}, v_{n-3}\} \cup \{v_{n-2}\}.$$

By Lemma 4, S is a minimum $rgda$ in P_n. Therefore,

$$\begin{aligned}\gamma_{ra}(P_n) &= |S| \\ &= |\{v_0, v_{n-1}\} \cup \{v_3, v_4, v_7, v_8, \ldots, v_{n-4}, v_{n-3}\} \cup \{v_{n-2}\}| \\ &= 2 + \frac{n-3}{2} + 1 \\ &= \frac{4+n-3+2}{2} \\ &= \frac{n+3}{2}.\end{aligned}$$

This completes the proof. □

Remark 1. *If P_1 is a path graph of order 1, then $\gamma_{ra}(P_1) = 1$.*

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