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On a novel fractional calculus and its applications to well-known problems

A. Ait Brahim[1,*], A. El Hajaji[2], K. Hilal [1], J. El Ghordaf[1]

[1] *Laboratory of Applied Mathematics and Scientific Calculus, University of Sciences and Technology, Béni Mellal , Morocco.*
[2] *OEE Departement, ENCGJ, University of Chouaib Doukali, El jadida, Morocco*

Abstract. The objective of this study is to introduce three novel and comprehensive models utilizing the concept of conformable fractional calculus. These models encompass the fractional population growth model (FPGM), the fractional body cooling model (FBCM), and the fractional heat differential equation (FHDE). Firstly, using new results, a type of new conformable fractional derivative introduced in [5]:

$$(\mathcal{D}^{\alpha} H)(y) = \lim_{z \longrightarrow 0} \frac{H\left(y + z e^{(\alpha-1)y}\right) - H(y)}{z},$$

where $\alpha \in (0, 1]$ and H is a function. We obtain exact solutions of the considered models. The results feature excellent agreement of exact solutions. Finally, it is shown that the proposed method provides a more powerful mathematical tool for solving models in mathematical physics.

2020 Mathematics Subject Classifications: 26A33, 34A08, 65M70, 35R11, 92D25

Key Words and Phrases: New conformable fractional derivative, population grow model, body coolig model, fractional heat equation.

1. Introduction

Historically, the origins of fractional calculus began with the correspondence between L'Hopital and Leibniz [1, 2], where the former answered the question of whether the order of derivation in an expression could be of a fractional nature. Obviously, this question is self-contradictory, because the order of derivation indicates the frequency with which the differential operator is applied, that is, it counts the number of repetitions of the entire differential process. This origin was followed by further contributions by eminent mathematicians, including the European Euler, Lacroix, Laplace, etc. One of the most

*Corresponding author.
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Email addresses: abdessamad11117@gmail.com (A. Ait Brahim),
a_elhajaji@yahou.fr (A. El Hajaji), hilalkhalid2005@yahou.fr (K. Hilal),
elg_jalila@yahou.fr (J.El Ghordaf)

interesting, comprehensive, and well-founded the work is Riemman-Liouville [2], the corresponding fracture model starts from Fractional differential operators with two forms

$$\mathbf{D}_{c+}^{\alpha}H(z)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_c^z\frac{H(t)}{(z-t)^{\alpha+1-m}}dt, m-1<\alpha<m\in\mathbb{N}, \tag{1}$$

$$\mathbf{D}_{d-}^{\alpha}H(z)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_z^d\frac{H(t)}{(z-t)^{\alpha+1-m}}dt, m-1<\alpha<m\in\mathbb{N}. \tag{2}$$

Recently, the authors of [3] and [4] defined the classical fractional derivatives, supposedly conformable derivatives fractional derivatives, depending only on the fundamental limit definition derivation of Khalil et al. [3] presented a new derivation of H of order α, known as :

$$T_\alpha H(t)=\lim_{\varepsilon\longrightarrow 0}\frac{H\left(t+\epsilon t^{1-\alpha}\right)-H(t)}{\varepsilon}, \tag{3}$$

at 0 the fractional derivative is defined as $H^{(\alpha)}(0)=\lim_{t\longrightarrow 0^+}T_\alpha(H)(t)$.
Also, Katugampola [4] proposed a new container defined as

$$\mathbf{D}^{\alpha}(H)(t)=\lim_{\varepsilon\longrightarrow 0}\frac{H\left(te^{\varepsilon t^{-\alpha}}\right)-H(t)}{\varepsilon}, \tag{4}$$

with $t>0$ and $\alpha\in[0,1]$.
If H is differentiable in order α in some $([0,a], a>0$, and $\lim_{t\to 0^+}\mathbf{D}^{\alpha}(H)(t)$ exists, then we obtain :

$$\mathbf{D}^{\alpha}(H)(0)=\lim_{t\longrightarrow 0^+}\mathbf{D}^{\alpha}(H)(t). \tag{5}$$

The studies cited in references [3, 4, 20] have significantly contributed to our understanding of the properties of derivatives with respect to the parameter α. These investigations have revealed that the derivatives of α adhere to fundamental rules such as the quotient rule and product rule, akin to their counterparts in traditional calculus. Notably, these findings evoke parallels with well-known theorems like Rolle's theorem and the mean value theorem, underscoring the robustness and applicability of the principles of calculus in the context of fractional calculus. By elucidating these parallels, these studies provide valuable insights into the behavior and characteristics of fractional derivatives, enriching our toolkit for analyzing and solving complex problems across various scientific disciplines.

The aim of this study is to leverage the outcomes derived in [5] and extend their application to physics, specifically in heat differential formulas, body cooling, and population growth models. These applications have previously been explored by Khalil et al. in [6], J. E. Napoles Valdes et al. in [19], and also in various references [20–23]. Our work builds upon these previous studies by applying simpler models using the novel results of the conformable fractional derivative outlined in [5].

2. New results of fractional calculus

From calculus we see that the derivative of the function H at a given point $z = d$ in its domain is defined by the following restriction:

$$H'(d) = \lim_{h \to 0} \frac{H(d+h) - H(d)}{h}. \tag{6}$$

We define H as a differentiable function at $z = d$ if its limit exists, denoted as $H'(d)$, furthermore, if H is differentiable at this point, it is also continuous at the same point.

Definition 1. *The new fractional derivative of H in order α is represented by*

$$(\mathbf{D}^\alpha H)(t) = \lim_{v \to 0} \frac{H\left(t + ve^{(\alpha-1)t}\right) - H(t)}{v}, \tag{7}$$

with , $\alpha \in [0,1]$ and $t > 0$.
On the other hand if H is differentiable of order α in $[0,a], a > 0$, and $\lim_{t \to 0^+} (\mathbf{D}^\alpha H)(t)$ exists, we have :

$$\mathbf{D}^\alpha H(0) = \lim_{t \longrightarrow 0^+} (\mathbf{D}^\alpha H)(t). \tag{8}$$

Theorem 1. *(see[5])*
If H is differentiable of order α at $t_0 > 0$, and $H : [0, +\infty) \longrightarrow \mathbb{R}$, then H is continuous at t_0.

Theorem 2. *(see[5])*
If H be α differentiable at a point $t > 0$, we have :
1) $\mathbf{D}^\alpha(aH + bH) = a(\mathbf{D}^\alpha H) + b(\mathbf{D}^\alpha H)$, *for all* $a, b \in \mathbb{R}$
2) $\mathbf{D}^\alpha(t^n) = ne^{(\alpha-1)t}t^{n-1}$ *for all* $n \in \mathbb{R}$
3) $\mathbf{D}^\alpha(\beta) = 0$, *for all constant* $H(t) = \beta$.
4) $\mathbf{D}^\alpha(HG) = H(\mathbf{D}^\alpha G) + G(\mathbf{D}^\alpha H)$.
5) $\mathbf{D}^\alpha(H/G) = H(\mathbf{D}^\alpha G) + G(\mathbf{D}^\alpha H)/G^2$.
6) If H is differentiable, then $(\mathbf{D}^\alpha H)(t) = e^{(\alpha-1)t}H'(t)$.

Theorem 3. *Let $\alpha \in (0,1]$, then:*
1) $\mathbf{D}^\alpha\left(\frac{1}{1-\alpha}e^{(1-\alpha)t}\right) = 1$.
2) $\mathbf{D}^\alpha\left(\sin\frac{1}{1-\alpha}e^{(1-\alpha)t}\right) = \cos\frac{1}{1-\alpha}e^{(1-\alpha)t}$.
3) $\mathbf{D}^\alpha\left(\cos\frac{1}{1-\alpha}e^{(1-\alpha)t}\right) = -\sin\frac{1}{1-\alpha}e^{(1-\alpha)t}$.
4) $\mathbf{D}^\alpha\left(e^{\frac{1}{1-\alpha}e^{(1-\alpha)t}}\right) = e^{\frac{1}{1-\alpha}e^{(1-\alpha)t}}$.

3. Applications to physics

3.1. Model of population growth

Population growth models are essential tools in understanding the dynamics of populations over time. They are used in various fields such as biology, ecology, epidemiology, and sociology to predict and analyze changes in population size. We define the problem of population growth model by

$$\frac{dZ}{dt} = kZ(t), \quad Z(0) = N. \tag{9}$$

Here L is constant, N is the initial value of population and Z is the population function, and we can write the solution of this problem

$$Z(t) = Ne^{Lt}. \tag{10}$$

The following figure shows this solution :

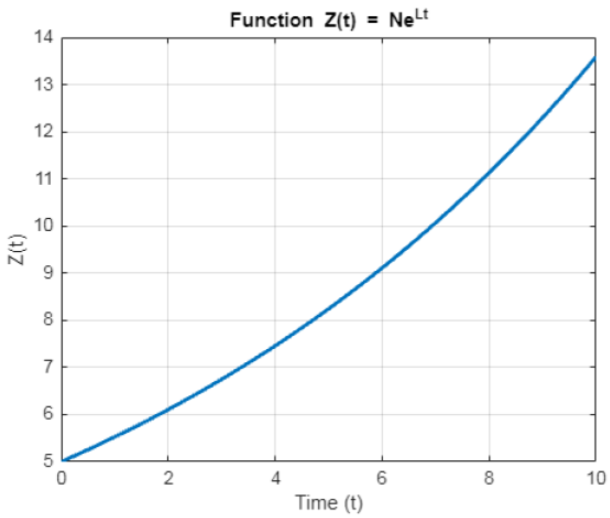

Figure 3.1.1 :Graph of differential equations solutions with ordinary derivative.

Now, we present this problem in the form of new conformable fractional :

$$Z^{(\alpha)}(t) = LZ(t), \quad Z(0) = N. \tag{11}$$

From property (6) theorem(2) , we have :

$$e^{(\alpha-1)t}Z'(t) = LZ(t). \tag{12}$$

Therefore, we can continue to use the following expression

$$Z'(t) = ke^{(\alpha-1)t}Z(t) \quad \text{or} \quad \frac{dZ(t)}{dt} = Le^{(\alpha-1)t}Z(t). \tag{13}$$

By applying the separation of variables, we obtain:

$$\frac{dZ(t)}{y(t)} = Le^{(\alpha-1)t}dt. \tag{14}$$

Now by integration we have :

$$\ln Z(t) = \frac{k}{\alpha - 1} e^{(\alpha-1)t} + B, \tag{15}$$

we obtain:

$$Z(t) = e^{B} e^{\frac{L}{\alpha-1} e^{(\alpha-1)t}}, \tag{16}$$

and we use the initial condition:

$$Z(t) = N e^{\frac{L}{\alpha-1} e^{(\alpha-1)t}}. \tag{17}$$

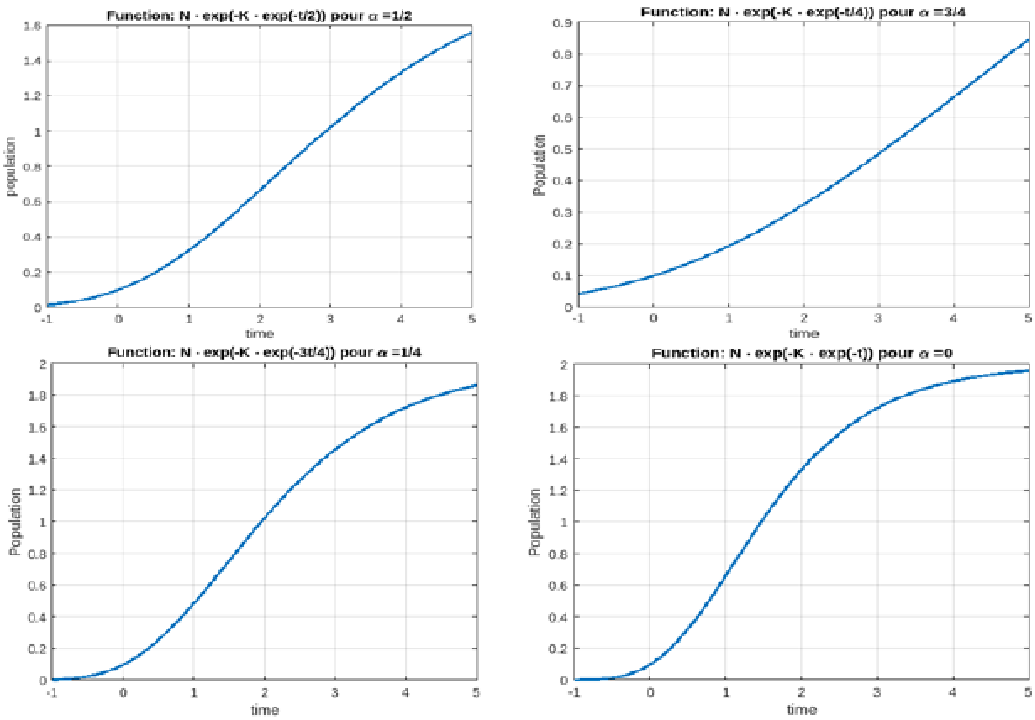

Figure 3.1.2 :Comparative solutions for several values of α.

In this example, we present the solution for different values of α: $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.75$ using the new conformable fractional calculus (see [5]), and we compare it with other methods like Khalil's conformable fractional derivative (see [3]) and ordinary derivatives.

3.2. Model of body cooling

As he shifted his sweater over time, Newton's law of cold was a body law designed to Body warming is proportional to the temperature difference between the body and its surroundings. The procedure is only a slight difference, since one must take into account the impossibility of temperature differences and the way the sweater transfers in the same way. This is equivalent to confirming that the heat transfer coefficient between the heat sink and the temperature difference is a constant.

We define the problem of Newton's cooling law states that the cooling by equation

$$\frac{dT}{dt} = -k\left(T(t) - T_c\right), T(0) = T_0. \tag{18}$$

In $t = 0$ we have the initial temperature T_0, and we obtain the solution :

$$T(t) = T_c + \left(T_0 - T_c\right) e^{-kt}. \tag{19}$$

The following figure shows this solution :

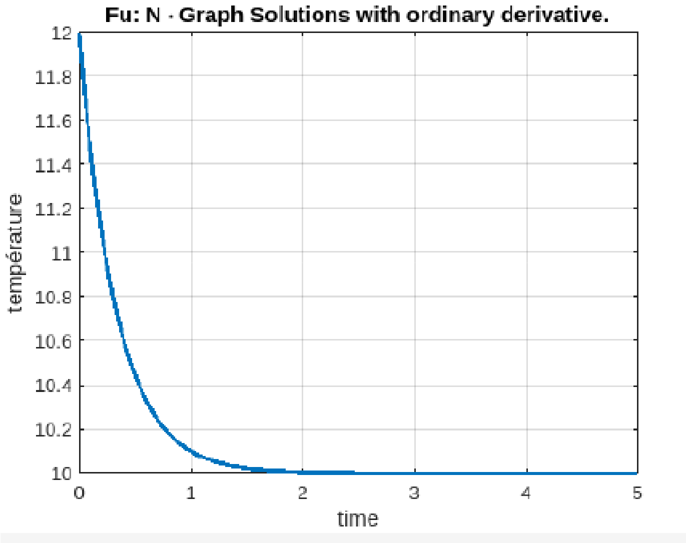

Figure 3.2.1 : Graph of differential equations solutions with ordinary derivative.

Utilizing property (6) of theorem (2), we derive the novel conformable fractional differential formula for this particular problem:

$$\frac{d^{\alpha}T(t)}{dt^{\alpha}} = -k\left(T(t) - T_c\right), \tag{20}$$

so,

$$e^{(\alpha-1)t}T'(t) = -k\left(T(t) - T_c\right). \tag{21}$$

Then, we have:

$$\frac{dT(t)}{dt} = -e^{(1-\alpha)t}k\left(T(t) - T_c\right), \tag{22}$$

and

$$\frac{dT(t)}{T(t) - T_c} = -e^{(1-\alpha)t}k. \tag{23}$$

Therefore,

$$\text{Ln}\,(T(t) - T_c) = \frac{-ke^{(1-\alpha)t}}{1-\alpha} + C, \tag{24}$$

then,

$$T(t) = T_c + e^C e^{\frac{-ke^{(1-\alpha)t}}{1-\alpha}}, \tag{25}$$

with the initial condition, we have the final solution:

$$T(t) = T_c + (T_0 - T_c)\, e^{\frac{-ke^{(1-\alpha)t}}{1-\alpha}}. \tag{26}$$

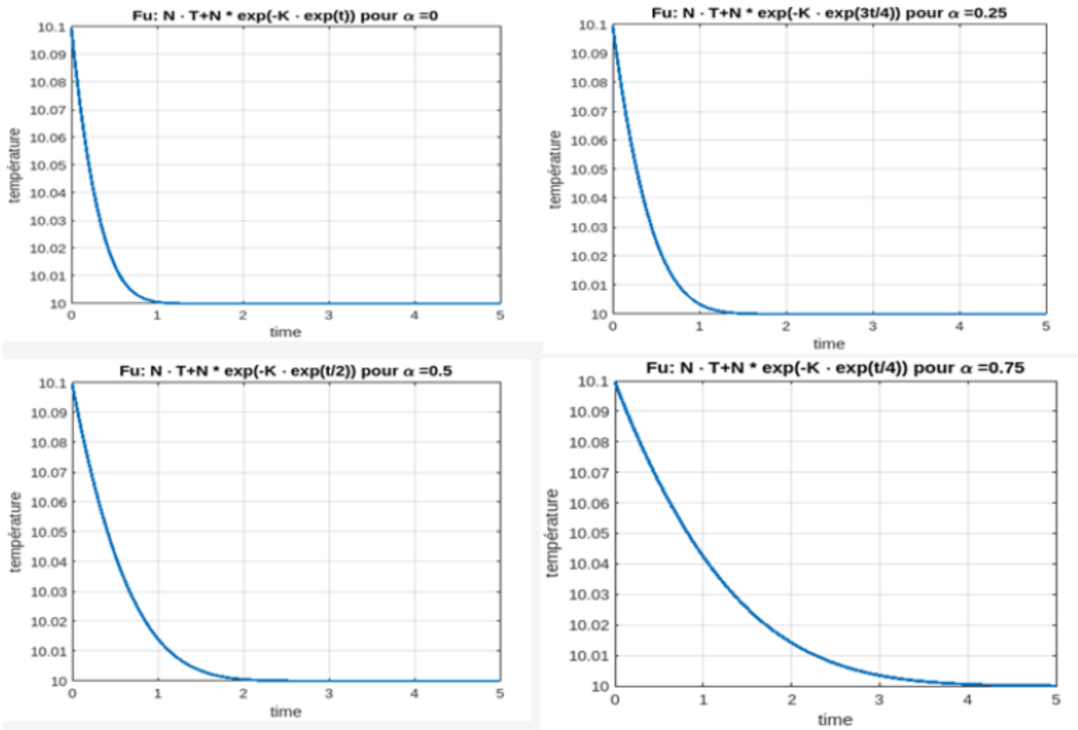

Figure 3.2.2 : Comparative solutions for several values of α.

The graphical representation of this solution of the new fractional derivative for the values of $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.75$ is compared with results obtained using the factor $T(t, \alpha) = e^{(\alpha-1)t}$ directly, as well as non-conformable fractional derivatives using the factor $T(t, \alpha) = t^{-\alpha}$ (see [19]).

3.3. Fractional Heat equation

The heat equation is a fundamental concept in physics and thermal engineering. It describes how the temperature in a material varies over time depending on the heat applied to it and its thermal properties. This equation is widely used in many fields such as thermodynamics, geophysics, meteorology, materials engineering, and many others.

Solving the heat equation makes it possible to predict the temporal evolution of the temperature in a given material. Solutions can be obtained analytically in some simple cases, but in most cases numerical methods are required to obtain accurate solutions. Techniques such as the finite difference method, finite element method or finite volume method are commonly used for this purpose.
The heat equation has many practical applications. For example :

- Modeling heat transfers in buildings to improve energy efficiency. Prediction of temperature distribution in materials during manufacturing processes.
- Study of heat propagation in soils for geothermal energy. Understanding of weather phenomena such as cloud formation and ocean currents.

The heat equation is an essential tool for understanding and predicting thermal phenomena in a wide variety of scientific fields and technological applications. Its simple but powerful mathematical formulation makes it a pillar of thermal modeling and contributes to many advances in science and engineering. Then form of fractional heat equation represented by :

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha z(x,t)}{\partial t^\alpha} = \frac{\partial^2 z(x,t)}{\partial x^2}, \tag{27}$$

$$z(0,t) = 0, \quad t > 0, \tag{28}$$

$$z(1,t) = 0, \quad t > 0, \tag{29}$$

$$\frac{\partial z(x,0)}{\partial t} = 0, \tag{30}$$

$$z(x,0) = F(x), \quad 0 < x < 1. \tag{31}$$

Consider the conformable fractional linear differential equations with constant coefficients using the findings presented in [20, 23]:

$$\frac{d^\alpha}{dy^\alpha}\frac{d^\alpha z}{dy^\alpha} \pm \mu^2 z = 0. \tag{32}$$

We use the equation $R^2 \pm \mu^2 = 0$, and associate to this equation, then we have $R = \pm\nu$, or $R = \pm\mu i$. Furthermore, according to theorem (3), we establish that

$$z = e^{\mp\frac{\mu}{1-\alpha}e^{(1-\alpha)t}}, \tag{33}$$

constitute two distinct solutions of the equation. Also, in the second case we obtain by properties (2) and (3) above (see theorem(3)), that

$$z_1 = \sin(\frac{\mu}{1-\alpha}e^{(1-\alpha)t}) \text{ and } z_2 = \cos(\frac{\mu}{1-\alpha}e^{(1-\alpha)t}). \tag{34}$$

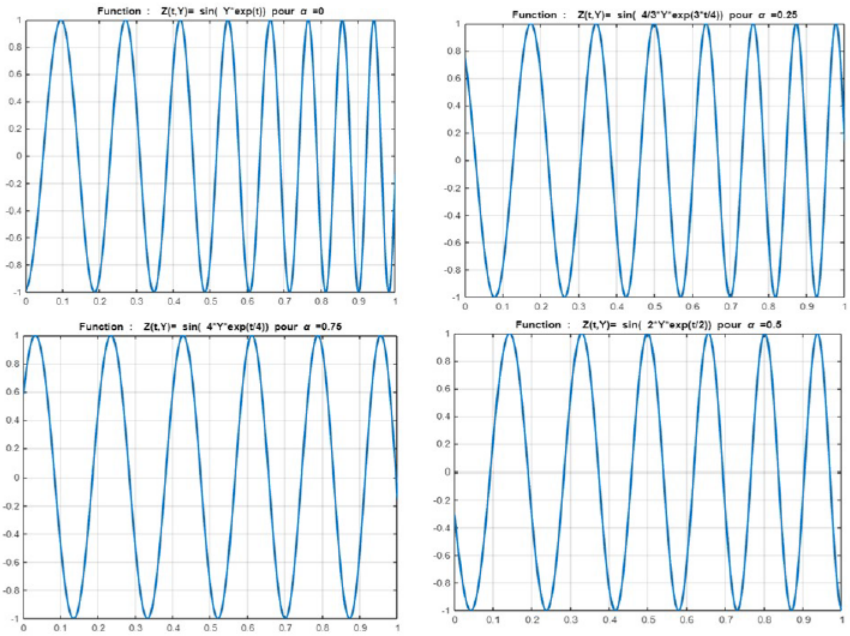

Figure 3.3.1 : Comparative solution z_1 for several values of α.

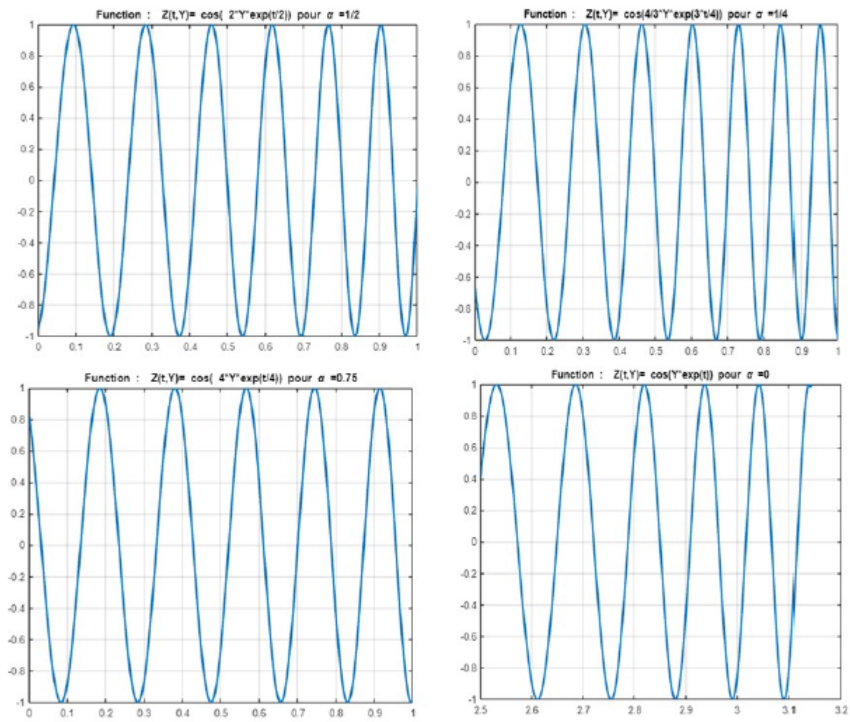

Figure 3.3.2 : Comparative solution z_2 for several values of α.

Let $z_1 = \sin(\frac{\mu}{1-\alpha}e^{(1-\alpha)t})$ and $z_2 = \cos(\frac{\mu}{1-\alpha}e^{(1-\alpha)t})$ are two independent solutions, we use the new fractional [5], for different value of α : $\alpha = 0$, $\alpha = 0.25$, $\alpha = 0.5$, and $\alpha = 0.75$, and we compare with ordinary derivative and Khalil conformable fractional [3].

Also, we have the form of new fractional differential formula of order 2α with constant coefficients represented by

$$\frac{d^\alpha}{dx^\alpha}\frac{d^\alpha z}{dx^\alpha} + a\frac{d^\alpha z}{dx^\alpha} + bz = h(x). \tag{35}$$

Can be write $\mathcal{D}^\alpha$ for $\frac{d^\alpha}{dx^\alpha}$, then we have :

$$\mathcal{D}^\alpha\left(\mathcal{D}^\alpha z\right) + a\mathcal{D}^\alpha z + bz = h(x). \tag{36}$$

To obtain the solution, we consider the equation $R^2 + aR + b = 0$. Utilizing the properties of the new conformable fractional derivative as outlined in [5], along with properties (2), (3), and (4) (refer to theorem (3)), we arrive at a theory akin to that of conventional linear differential equations.

Now, let's delve into the discussion of our heat equation (27). We will employ the method of separation of variables. Assume $z(x,t) = M(x)R(t)$. Substituting this into the differential equation yields:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha R(t)}{dt^\alpha}M(x) = R(t)\frac{d^2 M(x)}{dx^2}, \tag{37}$$

then, we obtain:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha R(t)}{dt^\alpha}/R(t) = \frac{d^2 M(x)}{dx^2}/P = \beta, \tag{38}$$

for some constant. Then:

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha R(t)}{dt^\alpha} - \beta R = 0, \quad \text{ and } \frac{d^2 M(x)}{dx^2} - \beta M = 0. \tag{39}$$

We consider the formula :

$$\frac{d^2 M(x)}{dx^2} - \beta M = 0. \tag{40}$$

As is well known, there are three cases for the values of β to be considered. $\beta = 0, \beta = -\nu^2$ and $\beta = \nu^2$. Conditions (28) and (29) forces

$$\nu = n\pi \text{ and } M_n(x) = c_n \sin(n\pi x). \tag{41}$$

Through equations (32) and (34), we obtain:

$$R(t) = b_1\cos(\frac{n\pi}{1-\alpha}e^{(1-\alpha)t}) + b_2\sin(\frac{n\pi}{1-\alpha}e^{(1-\alpha)t}). \tag{42}$$

And the condition (30) now gives $b_2 = 0$, and $R(t) = b_1\cos(\frac{n\pi}{1-\alpha}e^{(1-\alpha)t})$, then, using equation (41), we obtain :

$$z(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(\frac{n\pi}{1-\alpha}e^{(1-\alpha)t}), \tag{43}$$

using condition (31) we find that a_n is the nth Fourier coefficient of the function $h(x)$.

Finally, we can consider the following form of the new fractional heat differential equation:

$$\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\alpha z(x,t)}{\partial x^\alpha} = \frac{\partial z}{\partial t}, \tag{44}$$

with conditions:

$$\begin{aligned} &z(x,0) = h(x), \quad 0 < x < 1, \\ &z(0,t) = 0, \quad t > 0, \\ &z(1,t) = 0, \quad t > 0. \end{aligned}$$

In this context, equation (27) can be effectively applied, enabling the solution of problem (44) with the provided boundary conditions in a manner akin to solving the ordinary heat equation. This approach considers properties (2), (3), and (4) (refer to theorem (3)).

Remark 1. *One should note that if we set $\alpha = 1$, problem (44) will become the original problem treated by the fractional Black-Scholes equation (see [8]), a significant equation in financial mathematics. Utilizing this solution of the heat equation, we deduce the solution of the Black-Scholes model.*

4. Conclusion

In conclusion, this informative article has shed light on the novel concept of conformable derivatives and their definitions, unveiling a rich landscape of possibilities in mathematical analysis and applications across various scientific domains. By delving into the fundamental properties of adaptive differential operators, we've unlocked new avenues for understanding complex phenomena in physics and beyond.

The inclusion of graphical representations has not only enhanced our comprehension of the proposed fractional differential equations but has also demonstrated their remarkable resemblance to solutions obtained using traditional derivatives, underscoring the efficacy of these novel mathematical tools.

Looking ahead, the insights gleaned from our exploration, particularly in the realm of the fractional heat equation, hold promise for addressing significant challenges in financial modeling. Leveraging these outcomes to tackle the fractional Black-Scholes model represents a natural progression, offering potential breakthroughs in pricing exotic options and managing risk in modern financial markets.

As we continue to push the boundaries of mathematical innovation and its applications, the journey of discovery sparked by conformable derivatives promises to be both exciting and fruitful, paving the way for deeper insights into the intricacies of natural phenomena and the complexities of financial systems.

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