Methods of Generating Soft Topologies and Soft Separation Axioms

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Abstract. The paper develops a novel analysis of mutual interactions between topology and soft topology. It is known that each soft topology produces a system of crisp (parameterized) topologies. The other way round is also possible. Namely, one can generate a soft topology from a system of crisp topologies. Different methods of producing soft topologies are discussed by implementing two formulas. Then, the relationships between the resulting soft topologies are obtained. With the help of an example, it is demonstrated that one formula is more constructible than the other. Now, it is reasonable to ask which (topological) properties of a soft topology can be transferred to the set of crisp topologies, or the opposite. To address this question, we consider the standard separation axioms and show how well these axioms can be preserved when moving from a system of crisp topologies to the soft topology generated by it and contrariwise. Additionally, our findings extend and disprove some results from the literature.

2020 Mathematics Subject Classifications: 54A05, 54H99

Key Words and Phrases: soft topology, soft T_0, soft T_1, soft T_2, soft regular, soft normal, soft T_3, soft T_4

1. Introduction

In its modern version, the Weierstrass Extreme Value Theorem demonstrates that topological considerations can be useful in decision-making theory and economics, (see, [4]). Indeed, the development of topological structures helps to enhance other disciplines.

General topology is the mathematical branch of topology that concerns itself with the foundational set-theoretic notions and constructions. Motivated by the standard axioms of classical topological space, Shabir and Naz [28], and Çağman et al. [15], separately, introduced another branch of topology named "soft topology."

Soft topology is a combination of soft set theory and topology. It is focused on the construction of the system of all soft sets.

Soft sets were presented as a collection of relevant parameters to characterize a universe of possibilities. Soft set theory has been a fruitful area of study and connection with various disciplines since its establishment. Molodtsov [24], in 1999, originated the soft set theory as a mathematical tool for dealing with uncertainty which is free of the challenges related with other theories such as fuzzy set theory [32], rough set theory [26], and so on. In particular, the nature of parameter sets associated with soft sets provides a standardized foundation for modeling uncertain data. This leads to the rapid growth of soft set theory and soft topology in a short amount of time and provides various applications of soft sets in real life see [16, 17, 27, 30]).

There are various studies that have made significant contributions to the development of soft topology since its foundation in [15, 28]. A soft topological approach was then used to interpret the behavior of the most fundamental concepts in (general) topology. To be specific, soft compactness [14], soft connectedness [20], soft submaximality [2], soft extremal disconnectedness [13], soft clustering [9], soft simple extendedness [8], and soft nodecness [6], and congruence representations [11] of soft spaces.

Different methods of generating soft topologies on a common universal set were discussed in [3, 4, 19, 25, 29, 33].

Soft continuity of mappings has been widely generalized to diverse classes, including soft semi-continuity [21], soft β-continuity [31], soft $\mathcal{U}$-continuity [7], soft SD-continuity [10], and mappings of the Baire property [12].

Soft separation axioms are another significant aspect in the late development of soft topology; see for example [1, 23, 28].

Two remarkable formulas for generating soft topologies from a system of crisp topologies have been given by Terepeta [29]. One of the formulas (Formula 2) is said to generate a single set soft topology, while the other one generates a more general soft topology (Formula 1). Terepeta mainly applied Formula 2 to study the inheritance of soft separation axioms after the system of crisp topologies. Recently, Alcantud [3] proposed a slight extension of Formula 1 (we also call it Formula 1). He then employed such a formula to investigate the behavior of separability and second countability axioms between a system of crisp topologies and the soft topology generated by it. Recently, Alcantud [4] established crucial relationships between soft and fuzzy soft topologies. The work of Terepeta and Alcantud inspired us to attempt this research. Following their direction, we first apply the formulas to the system of crisp topologies taken from a soft topology in order to determine the connections between the obtained soft topologies and the original one. In addition, we use Formula 1 to verify how well the separation axioms are transferred between a system of crisp topologies and the soft topology generated by it. The latter statement extends the work of Terepeta (see, Section 2.1 in [29]), which is the main objective of this research.

2. Preliminaries

Let X be an initial universe, $\mathcal{P}(X)$ be all subsets of X and Ω be a set of parameters. An ordered pair $(F, \Omega) = \{(\omega, F(\omega)) : \omega \in \Omega\}$ is said to be a soft set over X, where $F : \Omega \to \mathcal{P}(X)$ is a set value mapping. The family of all soft sets on X is represented by $S_\Omega(X)$. The soft set $(X, \Omega) \backslash (F, \Omega)$ (or simply $(F, \Omega)^c = (F^c, \Omega)$) is the complement of (F, Ω), where $F^c : \Omega \to \mathcal{P}(X)$ is given by $F^c(\omega) = X \backslash F(\omega)$ for each $\omega \in \Omega$. A soft subset (F, Ω) over X is called null, denoted by $\widetilde{\Phi}$, if $F(\omega) = \emptyset$ for each $\omega \in \Omega$ and is called absolute, denoted by $\widetilde{X}$, if $F(\omega) = X$ for each $\omega \in \Omega$. Notice that $\widetilde{X}^c = \widetilde{\Phi}$ and $\widetilde{\Phi}^c = \widetilde{X}$. It is said that (A, Ω) is a soft subset of (B, Ω) (written by $(A, \Omega) \widetilde{\subseteq} (B, \Omega)$, [22]) if $A(\omega) \subseteq B(\omega)$ for each $\omega \in \Omega$, and $(A, \Omega) = (B, \Omega)$ if $(A, \Omega) \widetilde{\subseteq} (B, \Omega)$ and $(B, \Omega) \widetilde{\subseteq} (A, \Omega)$. The union of soft sets $(A, \Omega), (B, \Omega)$ is represented by $(F, \Omega) = (A, \Omega) \widetilde{\cup} (B, \Omega)$, where $F(\omega) = A(\omega) \cup B(\omega)$ for each $\omega \in \Omega$, and intersection of soft sets $(A, \Omega), (B, \Omega)$ is given by $(F, \Omega) = (A, \Omega) \widetilde{\cap} (B, \Omega)$, where $F(\omega) = A(\omega) \cap B(\omega)$ for each $\omega \in \Omega$, see [5]. A soft point [28] is a soft set (F, Ω) over X in which $F(\omega) = \{x\}$ for each $\omega \in \Omega$, where $x \in X$, and is denoted by $(\{x\}, \Omega)$. It is said that a soft point $(\{x\}, \Omega)$ is in (F, Ω) (briefly, $x \in (F, \Omega)$) if $x \in F(\omega)$ for each $\omega \in \Omega$. On the other hand, $x \notin (F, \Omega)$ if $x \notin F(\omega)$ for some $\omega \in \Omega$. This implies that if $(\{x\}, \Omega) \widetilde{\cap} (F, \Omega) = \widetilde{\Phi}$, then $x \notin (F, \Omega)$.

Definition 1. *[28] A collection Σ of $S_\Omega(X)$ is said to be a soft topology on X if the following conditions are satisfied:*

(i) $\widetilde{\Phi}, \widetilde{X} \in \Sigma$;

(ii) If $(F_1, \Omega), (F_2, \Omega) \in \Sigma$, then $(F_1, \Omega) \widetilde{\cap} (F_2, \Omega) \in \Sigma$; and

(iii) If each $\{(F_i, \Omega) : i \in I\} \widetilde{\subseteq} \Sigma$, then $\widetilde{\bigcup}_{i \in I}(F_i, \Omega) \in \Sigma$.

Terminologically, we call (X, Σ, Ω) a soft topological space on X. The elements of Σ are called soft open sets in Σ (or simply, soft open sets when no confusion arises), and their complements are called soft closed sets in Σ (or shortly, soft closed sets).

We use (X, Σ, Ω) to mention a soft topological space. In addition, (F, Ω) and (G, Ω) are disjoint soft sets over X if $(F, \Omega) \widetilde{\cap} (G, \Omega) = \widetilde{\Phi}$.

Definition 2. *[15] A subcollection $\mathcal{B} \subseteq \Sigma$ is called a soft base for the soft topology Σ if each element of Σ is a union of elements of $\mathcal{B}$.*

Definition 3. *[15] Let Σ_1, Σ_2 be two soft topologies on X. It is said that Σ_2 is finer than Σ_1 (or Σ_1 is coarser than Σ_2) if $\Sigma_1 \widetilde{\subseteq} \Sigma_2$.*

Lemma 1. *[28] Let (X, Σ, Ω) be a soft topology on X. For each $\omega \in \Omega$, $\Sigma_\omega = \{F(\omega) : (F, \Omega) \in \Sigma\}$ is a crisp topology on X.*

Definition 4. *[2] Let $\mathcal{F} \widetilde{\subseteq} S_\Omega(X)$. The intersection of all soft topologies on X including $\mathcal{F}$ is called a soft topology generated by $\mathcal{F}$ and is referred to $T(\mathcal{F})$.*

Lemma 2. *[2, Lemma 3.5] Let Σ_1, Σ_2 be two soft topologies on X. The resulting soft topology $T(\Sigma_1 \widetilde{\cup} \Sigma_2)$ is identical to the soft topology $T(\mathcal{F})$ generated by $\mathcal{F} = \{(F_1, \Omega) \widetilde{\cap} (F_2, \Omega) : (F_1, \Omega) \in \Sigma_1, (F_2, \Omega) \in \Sigma_2\}$.*

Definition 5. *[28] A soft space (X, Σ, Ω) is called*

(i) *soft T_0 if for each $x, y \in X$ with $x \neq y$, there exist soft open sets $(U, \Omega), (V, \Omega)$ such that $x \in (U, \Omega), y \notin (U, \Omega)$ or $x \notin (V, \Omega), y \in (V, \Omega)$,*

(ii) *soft T_1 if for each $x, y \in X$ with $x \neq y$, there exist soft open sets $(U, \Omega), (V, \Omega)$ such that $x \in (U, \Omega), y \notin (U, \Omega)$ and $x \notin (V, \Omega), y \in (V, \Omega)$,*

(iii) *soft T_2 (soft Hausdorff) if for each $x, y \in X$ with $x \neq y$, there exist soft open sets $(U, \Omega), (V, \Omega)$ containing x, y respectively such that $(U, \Omega) \widetilde{\cap} (V, \Omega) = \widetilde{\Phi}$.*

(iv) *soft regular if for each soft closed set (F, Ω) and each soft point x with $x \notin (F, \Omega)$, there exist soft open sets $(U, \Omega), (V, \Omega)$ such that $x \in (U, \Omega), (F, \Omega) \widetilde{\subseteq} (V, \Omega)$ and $(U, \Omega) \widetilde{\cap} (V, \Omega) = \widetilde{\Phi}$.*

(v) *soft normal if for each soft closed sets $(F, \Omega), (D, \Omega)$ with $(F, \Omega) \widetilde{\cap} (D, \Omega) = \widetilde{\Phi}$, there exist soft open sets $(U, \Omega), (V, \Omega)$ such that $(F, \Omega) \widetilde{\subseteq} (U, \Omega), (D, \Omega) \widetilde{\subseteq} (V, \Omega)$ and $(U, \Omega) \widetilde{\cap} (V, \Omega) = \widetilde{\Phi}$.*

(vi) *soft T_3 if it is soft T_1 and soft regular.*

(vii) *soft T_4 if it is soft T_1 and soft normal.*

Lemma 3. *[23, Theorem 3.18] If (X, Σ, Ω) is a soft regular space, then $\Sigma_\omega = \Sigma_{\omega'}$ for each $\omega, \omega' \in \Omega$.*

3. Methods of generating soft topologies and their relationships

This section provides different methods of producing soft topologies via Formulas 1 & 2. An example is given which discusses the implementation of these formulas in detail. The relationships between the original soft topology and the soft topologies that are produced by Formulas 1 & 2.

Definition 6. *[3, 29] Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be a family of (crisp) topologies on a set X for some index set Ω. Then following procedures produce different soft topologies on X:*

$$(Formula\ 1) \qquad \mathcal{T}(\boldsymbol{\Sigma}) = \Big\{ \{(\omega, F(\omega)) : \omega \in \Omega\} \in S_\Omega(X) : F(\omega) \in \Sigma_\omega, \forall \omega \in \Omega \Big\},$$

$\mathcal{T}(\boldsymbol{\Sigma})$ is called a soft topology generated by $\boldsymbol{\Sigma}$. If for each $\omega, \omega' \in \Omega$, $\Sigma_\omega = \Sigma_{\omega'} = \Sigma$, then $\mathcal{T}(\boldsymbol{\Sigma}) = \mathcal{T}(\Sigma)$.

$$(Formula\ 2) \quad \hat{\mathcal{T}}(\Sigma_\omega) = \Big\{ \{(\omega, F(\omega)) : \omega \in \Omega\} \in S_\Omega(X) : F(\omega) = F(\omega') \in \Sigma_\omega, \forall \omega, \omega' \in \Omega \Big\},$$

$\hat{\mathcal{T}}(\Sigma_\omega)$ is called a single set soft topology generated by Σ_ω.

Definition 7. *Let (X, Σ, Ω) be a soft topological space. If $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ is the family of all crisp topologies from Σ, then $\mathcal{T}(\boldsymbol{\Sigma})$ is called the soft topology associated with Σ. Note that $\mathcal{T}(\boldsymbol{\Sigma})$ is called an extended soft topology in [25].*

Lemma 4. *Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be the family of all crisp topologies from (X, Σ, Ω). Then*

$$\Sigma \widetilde{\subseteq} \mathcal{T}(\boldsymbol{\Sigma}).$$

Proof. It can be concluded from the definition of soft sets and the soft topology generated by $\boldsymbol{\Sigma}$.

Lemma 5. *Let $\bar{\beta} = \{\beta_\omega : \omega \in \Omega\}$ be a family of bases for the topologies Σ_ω on X. Then $\mathcal{B}(\bar{\beta}) = \left\{\{(\omega, F(\omega)) : \omega \in \Omega\} \in S_\Omega(X) : F(\omega) \in \beta_\omega \cup \{\emptyset\}, \forall \omega \in \Omega\right\}$ is a base for a soft topology on X and $\mathcal{T}(\boldsymbol{\Sigma}) = T(\mathcal{B}(\bar{\beta}))$.*

Proof. By using Corollary 3 in [3] and simple modifications to the proof of Theorem 3 in [3], we can conclude the proof.

The following result is a straightforward generalization of Lemma 2, so the proof is omitted.

Lemma 6. *Let $\{\Sigma_\omega : \omega \in \Omega\}$ be a family of soft topologies on X. The resulting soft topology $T(\widetilde{\bigcup}_{\omega \in \Omega} \Sigma_\omega)$ is identical to the soft topology $T(\mathcal{F})$ generated by $\mathcal{F} = \{\widetilde{\bigcap}_{\omega_i=1}^{n}(F_{\omega_i}, \Omega) : (F_{\omega_i}, \Omega) \in \widetilde{\bigcup}_{\omega_i \in \Omega} \Sigma_{\omega_i}\}$.*

Lemma 7. *Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be a family of crisp topologies on X. Then*

$$T(\widetilde{\bigcup}_{\omega \in \Omega} \hat{\mathcal{T}}(\Sigma_\omega)) = \hat{\mathcal{T}}(T(\bigcup_{\omega \in \Omega} \Sigma_\omega)).$$

Proof. The Lemma 5 reduces the task of working with basic soft open sets rather than soft open sets. Let $(B_0, \Omega) \in T(\widetilde{\bigcup}_{\omega \in \Omega} \hat{\mathcal{T}}(\Sigma_\omega))$. Then $(B_0, \Omega) = \widetilde{\bigcap}_{i=1}^{n}(B_i, \Omega)$ for $(B_i, \Omega) \in \widetilde{\bigcup} \hat{\mathcal{T}}(\Sigma_\omega)$, and so $(B_0, \Omega) = \widetilde{\bigcap}_{i=1}^{n}(B_i, \Omega)$ such that $(B_i, \Omega) \in \hat{\mathcal{T}}(\Sigma_\omega)$ for some $\omega \in \Omega$. By Formula 2, one can detach Ω from (B_i, Ω) for $i = 0, 1, \cdots, n$, and get $B_0 = \bigcap_{i=1}^{n} B_i$, where $B_i \in \Sigma_\omega$ for some $\omega \in \Omega$. This implies that $B_0 = \bigcap_{i=1}^{n} B_i$ for $B_i \in \bigcup \Sigma_\omega$. Therefore, by Formula 2, $(B_0, \Omega) \in \hat{\mathcal{T}}(T(\widetilde{\bigcup} \Sigma_\omega))$. The reverse of the inclusion can be proved by a similar technique.

The following example shows how the techniques in Definition 1 and the relations in Lemmas $4-7$ can be used in practice:

Example 1. *Let $X = \{x_1, x_2, x_3\}$, $\Omega = \{\omega_1, \omega_2\}$. Consider the soft topology on X,*

$$\Sigma = \{\widetilde{\Phi}, (F_1, \Omega), (F_2, \Omega), (F_3, \Omega), (F_4, \Omega), \widetilde{X}\},$$

where

$(F_1, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \emptyset)\},$

$(F_2, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, X)\},$
$(F_3, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_3\})\},$ *and*
$(F_4, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \{x_3\})\}.$

The crisp topologies from Σ are

$$\Sigma_{\omega_1} = \{\emptyset, \{x_1\}, \{x_1, x_2\}, X\} \ and \ \Sigma_{\omega_2} = \{\emptyset, \{x_3\}, X\}.$$

Applying the Formula 2, we obtain the following two soft topologies on X:

$$\hat{\mathcal{T}}(\Sigma_{\omega_1}) = \left\{ \widetilde{\Phi}, \{(\omega_1, \{x_1\}), (\omega_2, \{x_1\})\}, \{(\omega_1, \{x_1, x_2\}), (\omega_2, \{x_1, x_2\})\}, \widetilde{X} \right\}$$

$$= \left\{ \widetilde{\Phi}, (\{x_1\}, \Omega), (\{x_1, x_2\}, \Omega), \widetilde{X} \right\} \ (more \ compactly) \ and$$

$$\hat{\mathcal{T}}(\Sigma_{\omega_2}) = \left\{ \widetilde{\Phi}, \{(\omega_1, \{x_3\}), (\omega_2, \{x_3\})\}, \widetilde{X} \right\} = \{\widetilde{\Phi}, (\Omega, \{x_3\}), \widetilde{X})\}.$$

From Lemma 6, we can naturally generate a soft topology T on X by the union of $\hat{\mathcal{T}}(\Sigma_{\omega_1})$ and $\hat{\mathcal{T}}(\Sigma_{\omega_2})$. That is,

$$T\left(\widetilde{\bigcup}_{i=1}^{2} \hat{\mathcal{T}}(\Sigma_{\omega_i})\right) = \left\{ \widetilde{\Phi}, (G_1, \Omega), (G_2, \Omega), (G_3, \Omega), (G_4, \Omega), \widetilde{X} \right\},$$

where

$(G_1, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \{x_1\})\},$
$(G_2, \Omega) = \{(\omega_1, \{x_3\}), (\omega_2, \{x_3\})\},$
$(G_3, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, \{x_1, x_2\})\}, \ and$
$(G_4, \Omega) = \{(\omega_1, \{x_1, x_3\}), (\omega_2, \{x_1, x_3\})\}.$

The compact form of the above conclusion is

$$T\left(\widetilde{\bigcup}_{i=1}^{2} \hat{\mathcal{T}}(\Sigma_{\omega_i})\right) = \left\{ \widetilde{\Phi}, (\{x_1\}, \Omega), (\{x_3\}, \Omega), (\{x_1, x_2\}, \Omega), (\{x_1, x_3\}, \Omega), \widetilde{X} \right\}.$$

By applying the Formula 1, the next soft topology on X will be obtained.

$$\mathcal{T}(\boldsymbol{\Sigma}) = \mathcal{T}(\{\Sigma_{\omega_1}, \Sigma_{\omega_2}\}) = \left\{ \widetilde{\Phi}, (H_1, \Omega), (H_2, \Omega), \cdots, (H_{10}, \Omega), \widetilde{X} \right\},$$

where
$(H_1, \Omega) = \{(\omega_1, \emptyset), (\omega_2, X)\},$
$(H_2, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_3\})\},$
$(H_3, \Omega) = \{(\omega_1, X), (\omega_2, \emptyset)\},$
$(H_4, \Omega) = \{(\omega_1, X), (\omega_2, \{x_3\})\},$
$(H_5, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \emptyset)\},$
$(H_6, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, X)\},$

$(H_7, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \{x_3\})\},$
$(H_8, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, \emptyset)\},$
$(H_9, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, X)\}, \ and$
$(H_{10}, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, \{x_3\})\}.$

One can easily check that the above computations lead to the following observations:

- Σ *is a subcollection of* $\mathcal{T}(\mathbf{\Sigma})$.

- Σ *is independent of* $T\big(\widetilde{\bigcup}_{i=1}^{2} \hat{\mathcal{T}}(\Sigma_{\omega_i})\big)$.

- $T\big(\widetilde{\bigcup}_{i=1}^{2} \hat{\mathcal{T}}(\Sigma_{\omega_i})\big)$ *is independent of* $\mathcal{T}(\mathbf{\Sigma})$.

The relationships between the soft topologies on a common universe obtained by the methods described in this section is summarized in Diagram 1.
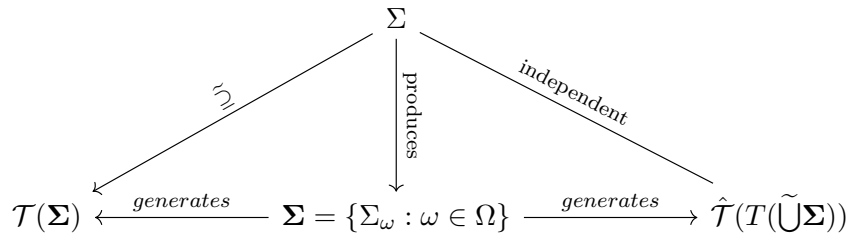

Figure 1: Relationships between different soft topologies

4. Non-uniqueness of soft topology $\mathcal{T}(\Sigma)$ associated with Σ

In this short section, we provide an example to witness that two different (even incomparable) soft topologies may have a common associated soft topology.

Example 2. *Consider the soft topology on* X, $\Sigma = \{\widetilde{\Phi}, (F_1, \Omega), (F_2, \Omega), (F_3, \Omega), (F_4, \Omega), \widetilde{X}\}$ *given in Example 1, there* $X = \{x_1, x_2, x_3\}$, $\Omega = \{\omega_1, \omega_2\}$. *Let* $\Sigma' = \{\widetilde{\Phi}, (R_1, \Omega), (R_2, \Omega), (R_3, \Omega), (R_4, \Omega), \widetilde{X}\}$ *be another soft topology on* X, *where*

$(R_1, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \emptyset)\},$
$(R_2, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, X)\},$
$(R_3, \Omega) = \{(\omega_1, X), (\omega_2, \{x_3\})\}, \ and$
$(R_4, \Omega) = \{(\omega_1, \{x_1, x_2\}), (\omega_2, \{x_3\})\}.$

Then Σ *and* Σ' *are incomparable. Set* $\hat{\Sigma} = \Sigma \widetilde{\bigcup} \Sigma'$. *Therefore,* $\hat{\Sigma}$ *is finer than both* Σ *and* Σ'. *On the other hand,* Σ, Σ' *and* $\hat{\Sigma}$ *have the same family of crisp topologies* $\mathbf{\Sigma} = \{\Sigma_{\omega_1}, \Sigma_{\omega_2}\}$, *and thus they generate only one* $\mathcal{T}(\mathbf{\Sigma})$.

5. Separation axioms preservation between Σ and $\mathcal{T}(\Sigma)$

With the exception of $\mathcal{T}(\Sigma)$, a single set soft topology $\hat{\mathcal{T}}(\Sigma)$ generated by Σ inherits soft separation axioms after Σ, according to Terepeta [29]. In this section, we use $\mathcal{T}(\boldsymbol{\Sigma})$ to see how well separation axioms are preserved when moving from $\boldsymbol{\Sigma}$ to $\mathcal{T}(\boldsymbol{\Sigma})$ and vice versa.

Here, we start defining special types of soft sets that exist when constructing a soft topology by using $\mathcal{T}(\boldsymbol{\Sigma})$ for future usage.

Definition 8. *Let $G, H \subset X$ and let $\omega \in \Omega$, we define*

(i) *(F_ω^G, Ω) to be a soft set over X such that $F_\omega^G(\omega) = G$ and $F_\omega^G(\omega') = X$ for each $\omega' \neq \omega$.*

(ii) *(F_H^ω, Ω) to be a soft set over X such that $F_H^\omega(\omega) = H$ and $F_H^\omega(\omega') = \emptyset$ for each $\omega' \neq \omega$, (see, [3, Definition 5]).*

Note that $(F_\omega^G, \Omega)^c = (F_H^\omega, \Omega)$ if and only if $G^c = H$.

Theorem 1. *Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be a family of crisp topologies on X. If Σ_ω is a T_0-space for some $\omega \in \Omega$ then $\mathcal{T}(\boldsymbol{\Sigma})$ is a soft T_0-space.*

Proof. Suppose that Σ_ω is a T_0-space for some $\omega \in \Omega$. Let $x, y \in X$ with $x \neq y$. Then there exist open sets $U, V \in \Sigma_\omega$ such that $x \in U$, $y \notin U$ or $x \notin V$, $y \in V$. By Definition 8, there exist two corresponding soft sets $(F_\omega^U, \Omega), (F_\omega^V, \Omega) \in \mathcal{T}(\boldsymbol{\Sigma})$ for which $x \in (F_\omega^U, \Omega)$, $y \notin (F_\omega^U, \Omega)$ or $x \notin (F_\omega^V, \Omega)$, $y \in (F_\omega^V, \Omega)$. Thus, $\mathcal{T}(\boldsymbol{\Sigma})$ is soft T_0.

The following example shows that the converse of Theorem 1 is not true in general.

Example 3. *Let $X = \{x_1, x_2, x_3\}$ and let $\Sigma_{\omega_1} = \{\emptyset, \{x_1\}, \{x_2, x_3\}, X\}$ and $\Sigma_{\omega_2} = \{\emptyset, \{x_3\}, \{x_1, x_2\}, X\}$ be crisp topologies on X indexed by $\Omega = \{\omega_1, \omega_2\}$. By using the Formula 1, the following soft topology on X will be obtained:*

$$\mathcal{T}(\boldsymbol{\Sigma}) = \mathcal{T}(\{\Sigma_{\omega_1}, \Sigma_{\omega_2}\}) = \left\{\widetilde{\Phi}, (H_1, \Omega), (H_2, \Omega), (H_3, \Omega), \cdots, (H_{14}, \Omega), \widetilde{X}\right\},$$

where

$(H_1, \Omega) = \{(\omega_1, X), (\omega_2, \emptyset)\}$,
$(H_2, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \emptyset)\}$,
$(H_3, \Omega) = \{(\omega_1, \{x_2, x_3\}), (\omega_2, \emptyset)\}$,
$(H_4, \Omega) = \{(\omega_1, \emptyset), (\omega_2, X)\}$,
$(H_5, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, X)\}$,
$(H_6, \Omega) = \{(\omega_1, \{x_2, x_3\}), (\omega_2, X)\}$,
$(H_7, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_1, x_2\})\}$,
$(H_8, \Omega) = \{(\omega_1, X), (\omega_2, \{x_1, x_2\})\}$,
$(H_9, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \{x_1, x_2\})\}$,

$(H_{10}, \Omega) = \{(\omega_1, \{x_2, x_3\}), (\omega_2, \{x_1, x_2\})\}$,
$(H_{11}, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_3\})\}$,
$(H_{12}, \Omega) = \{(\omega_1, X), (\omega_2, \{x_3\})\}$,
$(H_{13}, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \{x_3\})\}$, and
$(H_{14}, \Omega) = \{(\omega_1, \{x_2, x_3\}), (\omega_2, \{x_3\})\}$.

Since $x_1 \in (H_5, \Omega)$, $x_2, x_3 \notin (H_5, \Omega)$ and $x_3 \in (H_{14}, \Omega)$, $x_2 \notin (H_{14}, \Omega)$, then $\mathcal{T}(\boldsymbol{\Sigma})$ is soft T_0. On the other hand, neither of Σ_{ω_1} nor Σ_{ω_2} is T_0.

Theorem 2. *Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be a family of crisp topologies on X. If Σ_ω is a T_1-space for some $\omega \in \Omega$, then $\mathcal{T}(\boldsymbol{\Sigma})$ is a soft T_1-space.*

Proof. It is entirely analogous to the first part of the proof of Theorem 1.

The following example shows that the converse of Theorem 2 is not true in general. It also refutes Theorem 3.5 in [18]:

Example 4. *Let $X = \{x_1, x_2\}$ and let $\Sigma_{\omega_1} = \{\emptyset, \{x_1\}, X\}$ and $\Sigma_{\omega_2} = \{\emptyset, \{x_2\}, X\}$ be crisp topologies on X indexed by $\Omega = \{\omega_1, \omega_2\}$. By using the Formula 1, the following soft topology on X will be obtained:*

$$\mathcal{T}(\boldsymbol{\Sigma}) = \mathcal{T}(\{\Sigma_{\omega_1}, \Sigma_{\omega_2}\}) =$$
$$\left\{\widetilde{\Phi}, (H_1, \Omega), (H_2, \Omega), (H_3, \Omega), (H_4, \Omega), (H_5, \Omega), (H_6, \Omega), (H_7, \Omega), \widetilde{X}\right\},$$

where
$(H_1, \Omega) = \{(\omega_1, \emptyset), (\omega_2, X)\}$,
$(H_2, \Omega) = \{(\omega_1, X), (\omega_2, \emptyset)\}$,
$(H_3, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_2\})\}$,
$(H_4, \Omega) = \{(\omega_1, X), (\omega_2, \{x_2\})\}$,
$(H_5, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \emptyset)\}$,
$(H_6, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, X)\}$, and
$(H_7, \Omega) = \{(\omega_1, \{x_1\}), (\omega_2, \{x_2\})\}$.

Then (H_4, Ω) and (H_6, Ω) are soft open sets in $\mathcal{T}(\boldsymbol{\Sigma})$ such that $x_1 \in (H_6, \Omega)$, $x_2 \notin (H_6, \Omega)$ and $x_2 \in (H_4, \Omega)$, $x_1 \notin (H_4, \Omega)$. Thus, $\mathcal{T}(\boldsymbol{\Sigma})$ is soft T_1. On the other hand, neither of Σ_{ω_1} nor Σ_{ω_2} is T_1.

Theorem 3. *Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be a family of crisp topologies on X. Then Σ_ω is a T_2-space for each $\omega \in \Omega$ if and only if $\mathcal{T}(\boldsymbol{\Sigma})$ is a soft T_2-space.*

Proof. Assume that Σ_ω is T_2 for each $\omega \in \Omega$. Let $x, y \in X$ with $x \neq y$. Then, for each ω, there exist open sets $U(\omega), V(\omega) \in \Sigma_\omega$ such that $x \in U(\omega)$, $y \in V(\omega)$ and $U(\omega) \cap V(\omega) = \emptyset$. Set $(U, \Omega) = \{(\omega, U(\omega)) : \omega \in \Omega\}$ and $(V, \Omega) = \{(\omega, V(\omega)) : \omega \in$

$\Omega\}$. Then, $(U, \Omega), (V, \Omega) \in \mathcal{T}(\boldsymbol{\Sigma})$ such that $x \in (U, \Omega)$, $y \in (V, \Omega)$ and $(U, \Omega) \widetilde{\bigcap} (V, \Omega) = \{(\omega, U(\omega) \cap V(\omega)) : \omega \in \Omega\} = \widetilde{\Phi}$. Hence, $\mathcal{T}(\boldsymbol{\Sigma})$ is soft T_2.

Conversely, let $x, y \in X$ with $x \neq y$. Suppose that $\mathcal{T}(\boldsymbol{\Sigma})$ is soft T_2. Then, there exist soft open sets $(G, \Omega), (H, \Omega)$ such that $x \in (G, \Omega)$, $y \in (H, \Omega)$ and $(G, \Omega) \widetilde{\bigcap} (H, \Omega) = \widetilde{\Phi}$. This means that for each $\omega \in \Omega$, $x \in G(\omega)$, $y \in H(\omega)$ and $G(\omega) \cap H(\omega) = \emptyset$. Thus, Σ_ω is T_2 for each $\omega \in \Omega$.

Corollary 1. *If (X, Σ, Ω) is a soft T_2-space, then Σ_ω is T_2 for each $\omega \in \Omega$.*

Proof. It is an immediate consequence of Lemma 4 and Theorem 3.

Notice that Theorem 3 and Corollary 1 generalize (part of) Theorem 4 in [29] and Proposition 17 in [28], respectively.

Lemma 8. *[29, Theorem 3] Let Σ be a (crisp) topology on a set X. A soft set (F, Ω) is soft closed in $\mathcal{T}(\Sigma)$ if and only if $(F, \Omega) = \{(\omega, F(\omega)) : F^c(\omega) \in \Sigma\}$.*

Theorem 4. *Let $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ be a family of crisp topologies on X. If $\mathcal{T}(\boldsymbol{\Sigma})$ is a soft regular space, then Σ_ω is a regular space for each $\omega \in \Omega$.*

Proof. Let $\omega \in \Omega$. Take $x \in X$ and $F(\omega)$ be a closed set in (X, Σ_ω) such that $x \notin F(\omega)$. The Definition 6 and Lemma 3 tell us that soft regularity of $\mathcal{T}(\boldsymbol{\Sigma})$ guarantees the equality of $\mathcal{T}(\boldsymbol{\Sigma}) = \mathcal{T}(\Sigma)$. Set $(F, \Omega) = \{(\omega, F(\omega)) : F^c(\omega) \in \Sigma\}$. By Lemma 8, (F, Ω) is soft closed in $\mathcal{T}(\Sigma)$ along with $x \notin (F, \Omega)$. Since $\mathcal{T}(\Sigma)$ is soft regular, then there exist soft open sets $(U, \Omega), (V, \Omega)$ in $\mathcal{T}(\Sigma)$ such that $x \in (U, \Omega), (F, \Omega) \widetilde{\subseteq} (V, \Omega)$ and $\widetilde{\Phi} = (U, \Omega) \widetilde{\bigcap} (V, \Omega) = \{(\omega, U(\omega) \cap V(\omega)) : \omega \in \Omega\}$. This implies that $x \in U(\omega)$, $F(\omega) \subseteq V(\omega)$ and $U(\omega) \cap V(\omega) = \emptyset$ for each $\omega \in \Omega$. Since $U(\omega), V(\omega) \in \Sigma_\omega$, then Σ_ω is regular for each $\omega \in \Omega$.

Corollary 2. *If (X, Σ, Ω) is a soft regular space, then Σ_ω is regular for each $\omega \in \Omega$.*

Proof. It can be concluded from Lemma 4 and Theorem 4.

Remark 1. *We shall mention that it is observed in Remark 3.23 (2') [23] that if (X, Σ, Ω) is a soft T_3 space, then Σ_ω is T_3 for each $\omega \in \Omega$. This conclusion is more general than Corollary 2, but it cannot be followed from any of our results due to Example 4.*

The examples given below disprove the reverse of theorem 4:

Example 5. *Let $X = \{x\}$, let $\Omega = \{\omega_1, \omega_2\}$, and let $\boldsymbol{\Sigma} = \{\Sigma_{\omega_1}, \Sigma_{\omega_2}\}$, where $\Sigma_{\omega_1} = \Sigma_{\omega_2} = \{\emptyset, X\}$. One can check that each Σ_{ω_i} is trivially a regular space. On the other hand, the soft topology $\mathcal{T}(\boldsymbol{\Sigma}) = \{\widetilde{\Phi}, (F_1, \Omega), (F_2, \Omega), \widetilde{X}\}$ is not soft regular, where $(F_1, \Omega) = \{(\omega_1, \emptyset), (\omega_2, X)\}$ and $(F_2, \Omega) = \{(\omega_1, X), (\omega_2, \emptyset)\}$. Indeed, $x \notin (F_i, \Omega)^c$ for each i, but no soft open sets in $\mathcal{T}(\boldsymbol{\Sigma})$ can separate them.*

Example 6. *Consider* $(\mathbb{R}, \mathcal{U})$, *the natural topology on* $\mathbb{R}$. *Let* $\Omega = \{\omega_1, \omega_2\}$. *It is well known that the natural topology is* T_3 *space. We will prove that* $\mathcal{T}(\mathcal{U})$ *is not* T_3, *where* $\mathcal{T}(\mathcal{U}) = T(\mathcal{B}(\bar{\beta}))$ *which is defined in Lemma 5. Pick the point* 0 *and* $(F, \Omega) = \{(\omega_1, [-1, 1]), (\omega_2, [2, 3])\}$. *Hence,* $0 \notin (F, \Omega)$ *and* (F, Ω) *is a closed soft set. Let* (W_1, Ω) *and* (W_2, Ω) *be any two basic open soft sets such that* $0 \in (W_1, \Omega)$ *and* $(F, \Omega) \widetilde{\subseteq} (W_2, \Omega)$. *Then,* $(W_1, \Omega) = \{(\omega_1, (a, b)), (\omega_2, (c, d)\}$ *such that* $0 \in (a, b)$, $0 \in (c, d)$ *and* $(W_2, \Omega) = \{(\omega_1, (r, t)), (\omega_2, (s, k)\}$. *Since* $0 \in F(\omega_1) = [-1, 1] \subset W_2(\omega_1)$, *then* $0 \in (r, t)$ *which implies that* $(W_1, \Omega) \widetilde{\bigcap} (W_2, \Omega) \neq \widetilde{\Phi}$.

Theorem 5. *Let* $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ *be a family of crisp topologies on* X. *Then* Σ_ω *is a normal space for each* $\omega \in \Omega$ *if and only if* $\mathcal{T}(\boldsymbol{\Sigma})$ *is a soft normal space.*

Proof. Let Σ_ω be normal for each $\omega \in \Omega$. Suppose $(A, \Omega), (B, \Omega)$ are disjoint soft closed sets in $\mathcal{T}(\boldsymbol{\Sigma})$. Then $(A, \Omega) = \{(\omega, A(\omega)) : A^c(\omega) \in \Sigma_\omega, \omega \in \Omega\}$ and $(B, \Omega) = \{(\omega, B(\omega)) : B^c(\omega) \in \Sigma_\omega, \omega \in \Omega\}$. Therefore $\widetilde{\Phi} = (A, \Omega) \widetilde{\bigcap} (B, \Omega) = \{(\omega, A(\omega) \cap B(\omega)) : A^c(\omega), B^c(\omega) \in \Sigma_\omega, \omega \in \Omega\}$. We obtain that $A(\omega) \cap B(\omega) = \emptyset$. Since Σ_ω is a normal space for each $\omega \in \Omega$, there exist open sets $G(\omega), H(\omega)$ such that $A(\omega) \subseteq G(\omega)$, $B(\omega) \subseteq H(\omega)$ and $G(\omega) \cap H(\omega) = \emptyset$. Set $(G, \Omega) = \{(\omega, G(\omega)) : \omega \in \Omega\}$ and $(H, \Omega) = \{(\omega, H(\omega)) : \omega \in \Omega\}$. Then $(G, \Omega), (H, \Omega) \in \mathcal{T}(\boldsymbol{\Sigma})$ such that $(A, \Omega) \widetilde{\subseteq} (G, \Omega)$ and $(B, \Omega) \widetilde{\subseteq} (H, \Omega)$. Furthermore, $(G, \Omega) \widetilde{\bigcap} (H, \Omega) = \{(\omega, G(\omega) \cap H(\omega)) : \omega \in \Omega\} = \widetilde{\Phi}$. This shows that $\mathcal{T}(\boldsymbol{\Sigma})$ is soft normal.

Conversely, for each $\omega \in \Omega$, we let $C(\omega), D(\omega)$ be disjoint closed sets in Σ_ω. By Lemma 8, $(C, \Omega) = \{(\omega, C(\omega)) : C^c(\omega) \in \Sigma_\omega, \omega \in \Omega\}$ and $(D, \Omega) = \{(\omega, D(\omega)) : D^c(\omega) \in \Sigma_\omega, \omega \in \Omega\}$ are soft closed sets in $\mathcal{T}(\boldsymbol{\Sigma})$ and $(C, \Omega) \widetilde{\bigcap} (D, \Omega) = \{(\omega, C(\omega) \cap D(\omega)) : \omega \in \Omega\} = \widetilde{\Phi}$. Since $\mathcal{T}(\boldsymbol{\Sigma})$ is soft normal, then there exist disjoint soft open sets $(U, \Omega), (V, \Omega)$ such that $(C, \Omega) \widetilde{\subseteq} (U, \Omega)$ and $(D, \Omega) \widetilde{\subseteq} (V, \Omega)$. This implies that $C(\omega) \subseteq U(\omega)$, $D(\omega) \subseteq V(\omega)$ and $U(\omega) \cap V(\omega) = \emptyset$ for each $\omega \in \Omega$. Thus, Σ_ω is normal for each $\omega \in \Omega$.

We shall remark that if (X, Σ, Ω) is a soft normal space, then Σ_ω need not be a normal space for each $\omega \in \Omega$.

Example 7. *Let* $X = \{x_1, x_2, x_3\}$ *and* $\Omega = \{\omega_1, \omega_2\}$. *Suppose*

$$\Sigma = \{\widetilde{\Phi}, (H_1, \Omega), (H_2, \Omega), \cdots, (H_9, \Omega), \widetilde{X}\},$$

where
$(H_1, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_3\})\}$,
$(H_2, \Omega) = \{(\omega_1, \{x_3\}), (\omega_2, \emptyset)\}$,
$(H_3, \Omega) = \{(\omega_1, \{x_2, x_3\}), (\omega_2, \emptyset)\}$,
$(H_4, \Omega) = \{(\omega_1, X), (\omega_2, \emptyset)\}$,
$(H_5, \Omega) = \{(\omega_1, \{x_1, x_3\}), (\omega_2, \emptyset)\}$,
$(H_6, \Omega) = \{(\omega_1, \{x_3\}), (\omega_2, \{x_3\})\}$,
$(H_7, \Omega) = \{(\omega_1, \{x_2, x_3\}), (\omega_2, \{x_3\})\}$,
$(H_8, \Omega) = \{(\omega_1, X), (\omega_2, \{x_3\})\}$, *and*

$(H_9, \Omega) = \{(\omega_1, \{x_1, x_3\}), (\omega_2, \{x_3\})\}.$

Then Σ is a soft normal space as each pair of non-null soft closed sets intersects each other. On the other hand, $\Sigma_{\omega_1} = \{X, \emptyset, \{x_3\}, \{x_2, x_3\}, \{x_1, x_3\}\}$ is not normal. Indeed, $x_3 \notin \{x_1\}$ and $\{x_1\}$ is a closed set in Σ_{ω_1}. Let $U, V \in \Sigma_{\omega_1}$ such that $x_3 \in U$ and $\{x_1\} \subseteq V$. Hence, by the definition of Σ_{ω_1}, we know that $\{x_3\} \subseteq U$ and $\{x_1, x_3\} \subseteq V$. Therefore, $U \cap V \neq \emptyset$.

The following example demonstrates that $\mathcal{T}(\boldsymbol{\Sigma})$ needs not to be soft normal if Σ_ω is not normal for some $\omega \in \Omega$.

Example 8. *Let $X = \{x_1, x_2, x_3\}$ and $\Omega = \{\omega_1, \omega_2\}$. Take $\Sigma_{\omega_1} = \{\emptyset, X\}$ and $\Sigma_{\omega_2} = \{\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, X\}$. One can easily verify that Σ_{ω_1} is normal. However, Σ_{ω_2} is not normal since $\{x_2\}, \{x_3\}$ are disjoint closed sets in Σ_{ω_2} that cannot be separated by two disjoint open sets. The soft topology:*

$$\mathcal{T}(\boldsymbol{\Sigma}) = \big\{\widetilde{\Phi}, (F_1, \Omega), (F_2, \Omega), (F_3, \Omega), (F_4, \Omega), (F_5, \Omega), (F_6, \Omega), (F_7, \Omega), (F_8, \Omega), \widetilde{X}\big\},$$

where
$(F_1, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_1\})\},$
$(F_2, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_1, x_2\})\},$
$(F_3, \Omega) = \{(\omega_1, \emptyset), (\omega_2, \{x_1, x_3\})\},$
$(F_4, \Omega) = \{(\omega_1, \emptyset), (\omega_2, X)\},$
$(F_5, \Omega) = \{(\omega_1, X), (\omega_2, \emptyset)\},$
$(F_6, \Omega) = \{(\omega_1, X), (\omega_2, \{x_1\})\},$
$(F_7, \Omega) = \{(\omega_1, X), (\omega_2, \{x_1, x_2\})\},$ *and*
$(F_8, \Omega) = \{(\omega_1, X), (\omega_2, \{x_1, x_3\})\},$

is not a soft normal space. Indeed, since $\{(\omega_1, \emptyset), (\omega_2, \{x_3\})\}$ and $\{(\omega_1, X), (\omega_2, \{x_2\})\}$ are disjoint soft closed sets in $\mathcal{T}(\boldsymbol{\Sigma})$ but there are no disjoint soft open sets can separate them.

Remark 2. *Notice that we can provide a more general proof sketch that proves the above claim, which is another proof to the part one of Theorem 5. If $\boldsymbol{\Sigma} = \{\Sigma_\omega : \omega \in \Omega\}$ is not a normal topology for some $\bar{\omega} \in \Omega$, then there are closed sets A, B in $\Sigma_{\bar{\omega}}$ which cannot be separated by any open sets. The soft sets $(F_A^{\bar{\omega}}, \Omega), (F_{\bar{\omega}}^B, \Omega)$ are closed and disjoint in $\mathcal{T}(\boldsymbol{\Sigma})$. If there exist soft open sets $(G, \Omega), (H, \Omega)$ in $\mathcal{T}(\boldsymbol{\Sigma})$ such that $(F_A^{\bar{\omega}}, \Omega) \widetilde{\subseteq} (G, \Omega),$ $(F_{\bar{\omega}}^B, \Omega) \widetilde{\subseteq} (H, \Omega)$ and $(G, \Omega) \widetilde{\bigcap} (H, \Omega) = \widetilde{\Phi}$, then $A \subseteq G(\bar{\omega}), B \subseteq H(\bar{\omega})$ and $G(\bar{\omega}) \cap H(\bar{\omega}) = \emptyset$, which is a contradiction.*

6. Conclusion

This study develops a methodical understanding of the connections between a system of crisp topologies and the soft topology produced by it. The procedure is carried out with the

help of two formulas. If we start from an original soft topology, then it produces a system of crisp topologies. With our formulas, we can generate two different new soft topologies. We have discussed the relationships between these soft topologies. Moreover, we show that the resulting soft topology via Formula 1 is always finer than the original one, while the soft topology generated by Formula 2 is incomparable. We see that two different original soft topologies may generate a single soft topology by either of the formulas. Furthermore, we study the preservation of separation axioms between the system of crisp topologies and the soft topology generated by it. More precisely, we show that Hausdorffness and normality behave better in transforming to soft topologies and conversely. On the other hand, other separation axioms act differently. If one of the crisp topologies is respectively T_0, T_1, then it guarantees that the resulting soft topology is soft T_0, T_1. The converse may not be true. All of the crisp topologies are regular when the soft topology generated by them is a soft regular.

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