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Higher Order Bivariate Bell-Based Apostol-Frobenius-Type Poly-Genocchi Polynomials with Parameters a and b

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Abstract. In this paper, we unveil a novel category of Frobenius-Genocchi polynomials, grounded in the Bell numbers and Apostol-type functions. Our exploration delves into a comprehensive examination of these polynomials, elucidating various properties. Employing diverse analytical methods and leveraging generating functions for Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order, we derive explicit and implicit summation formulas, complemented by their symmetric identities.

2020 Mathematics Subject Classifications: 05A15, 11B68, 11B73, 26C05, 33B10

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1. Introduction

Similar to the Bernoulli and Euler numbers [1], the Genocchi numbers, denoted as G_n, are established by means of the subsequent generating function:

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}, \quad |t| < \pi.$$

Some novel identities involving these numbers can be found in [5, 19, 27]. These numbers have undergone diverse generalizations, often achieved by combining them with the principles of well-known polynomials. A specific example is the integration with exponential

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polynomials, leading to the formation of Genocchi polynomials and higher-order Genocchi polynomials (see [18]), outlined as follows:

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}, \quad |t| < \pi, \tag{1.1}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}. \tag{1.2}$$

Mixing with the Apostol polynomials yields the Apostol-Genocchi polynomials, and Apostol-Genocchi polynomials of higher order, which are respectively defined as follows:

$$\sum_{n=0}^{\infty} G_n(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t+1}e^{xt}, \tag{1.3}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt}, \tag{1.4}$$

where $|t| < \pi$ when $\lambda = 1$ and $|t| < \log(-\lambda)$ when $\lambda \neq 1, \lambda \in \mathbb{C}$. Also, mixing with Frobenius polynomials yields the so-called Frobenius-Genocchi polynomials, which are given by

$$\sum_{n=0}^{\infty} G_n^F(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t-u}e^{xt}, \tag{1.5}$$

and further gives

$$\sum_{n=0}^{\infty} G_n^F(x;u,\lambda)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t-u}e^{xt}, \tag{1.6}$$

the Apostol-Frobenius-Genocchi polynomials by mixing with Apostol-Genocchi polynomials (see [6, 15–17, 22, 23, 28, 30, 32, 33]). Further generalization and other variation of Frobenius-Genocchi polynomials, specifically, the generalized Apostol-Frobenius-Genocchi polynomials and Frobenius-Euler-Genocchi polynomials, are introduced in [34] and [3] respectively, and defined as follows:

$$\sum_{n=0}^{\infty} H_n^r(x;u,a,b,c,\lambda,)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^r c^{xt}, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \mathbb{A}_n^r(x;u)\frac{t^n}{n!} = \frac{(1-u)t^r}{e^t-u}e^{xt}. \tag{1.8}$$

It is worth-mentioning that (1.7) is parallel to the generalized Apostol type Frobenius-Euler polynomials of Kurt and Simsek [24]. Moreover, mixing the Genocchi numbers with the concept of polylogarithm $\mathrm{Li}_k(z)$ [9]

$$\mathrm{Li}_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k}, k \in \mathbb{Z}, \tag{1.9}$$

yields the poly-Genocchi polynomials, which are defined as follows

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{x^n}{n!} = \frac{2Li_k(1-e^t)}{e^t+1}e^{xt}, \tag{1.10}$$

such that when $k=1$, $Li_1(1-e^t) = ln(1-(1-e^t)) = ln(e^t) = t$ and so (1.10) gives (1.1). Furthermore, with a slight modification of the generating function, another generalization, denoted by $G_{n,2}^{(k)}(x)$, was defined by Kim et al. [31] as follows

$$\sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{x^n}{n!} = \frac{Li_k(1-e^{-2t})}{e^t+1}e^{xt}. \tag{1.11}$$

These polynomials are called modified poly-Genocchi polynomials. Note that, when $k=1$, equations (1.10) and (1.11) give the Genocchi polynomials in (1.1). That is,

$$G_n^{(1)}(x) = G_{n,2}^{(1)}(x) = G_n(x).$$

Kim et. al [31] obtained several properties of these polynomials.

The higher order Apostol-Type poly-Genocchi polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ and Apostol-Frobenius-Type poly-Genocchi polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)$ and are respectively defined by (see [11, 12])

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} c^{xt}, \tag{1.12}$$

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,u,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} c^{xt}. \tag{1.13}$$

Further extension and variation of these polynomials can be found in [13, 14].

The Bell polynomials, represented as $B_n(x)$, are defined as polynomials with coefficients corresponding to the Stirling numbers of the second kind. To be more precise,

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k, \tag{1.14}$$

where $S(n,k)$ denotes the Stirling numbers of the second kind. These numbers adhere to the following exponential generating function

$$\sum_{n=0}^{\infty} S(n,j)\frac{t^n}{n!} = \frac{(e^t-1)^j}{j!}, \tag{1.15}$$

(see [10, 26]). By mixing the concept of Bell polynomials with partially degenerate Bernoulli polynomials of the first kind defined by the generating function

$$\frac{\log(1+\lambda t)^{1/\lambda}}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!},$$

the partially degenerate Bell-Bernoulli polynomials of the first kind are defined in [20] by the generating function

$$\frac{\log(1+\lambda t)^{1/\lambda}}{e^t-1}e^{xt+y(e^t-1)} = \sum_{n=0}^{\infty} {}_{Bel}B_{n,\lambda}(x,y)\frac{t^n}{n!}.$$

Further generalization is introduced in [20] by incorporating the concept of Dirichlet character with conductor d.

Also, by mixing the concept of Bell polynomials, Alam et al. [2] developed generating functions for new families of special polynomials, including two parametric types of Bell-based Bernoulli and Euler polynomials, defined as follows:

$$\sum_{n=0}^{\infty} {}_{Bell}\mathbb{B}_n^r(\xi+i\eta,x;u,\lambda)\frac{t^n}{n!} = \left(\frac{1}{e^t-1}\right)^r e^{(\xi+i\eta)t}e^{\zeta(e^x-1)},$$
$$\sum_{n=0}^{\infty} {}_{Bell}\mathbb{H}_n^r(\xi+i\eta,x;u,\lambda)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^r e^{(\xi+i\eta)t}e^{\zeta(e^x-1)}.$$

They investigated fundamental properties of these generating functions and used them, along with certain identities, to present relations among trigonometric functions, two parametric types of Bell-based Bernoulli and Euler polynomials, and Stirling numbers. They also derived computational formulae for these polynomials. By applying a partial derivative operator to these generating functions, they obtained various derivative formulae and finite combinatorial sums involving the aforementioned polynomials and numbers.

In separate papers, Alam et al. [4] and Ayed et al. [8] introduced a novel class of Bell-based Apostol-type Frobenius-Euler polynomials and Bell-based Apostol-type Frobenius-Genocchi polynomials, respectively. These are defined as follows:

$$\sum_{n=0}^{\infty} {}_{Bell}\mathbb{H}_n^r(x;u,\lambda)\frac{t^n}{n!} = \left(\frac{1-u}{\lambda e^t-u}\right)^r e^{\zeta(e^x-1)}, \tag{1.16}$$
$$\sum_{n=0}^{\infty} {}_{Bell}\mathbb{G}_n^r(\xi+i\eta,x;u,\lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t-u}\right)^r e^{(\xi+i\eta)t}e^{\zeta(e^x-1)}. \tag{1.17}$$

Their research explored various properties of these polynomials and numbers, deriving summation formulas in terms of Apostol-type Bernoulli, Euler, and Genocchi polynomials [4]. They established numerous identities using diverse analytical methods and the generating function technique, and introduced parametric variations that unveiled specific polynomial identities [4]. Additionally, they investigated various formulas and properties, including differentiation rules, addition formulas, relations, and summation formulas. Moreover, they identified the first few zero values of the Apostol-type Frobenius-Genocchi polynomials and provided graphical representations of these zero values [7, 8]. It is noteworthy that an alternative method for introducing Bell-based Frobenius-Euler polynomials has been established in [21]. This variation, known as Bell-based Frobenius-type Eulerian polynomials, is defined as follows:

$$\sum_{n=0}^{\infty} {}_{Bell}\mathbb{A}_n^r(\xi,\zeta|u)\frac{t^n}{n!} = \left(\frac{1-u}{e^{t(u-1)}-u}\right)^r e^{\xi t}e^{\zeta(e^x-1)}$$

In line with the polynomial exploration in [12], it is equally compelling to investigate Bell-based Apostol-Frobenius-type poly-Genocchi polynomials.

2. Higher Order Bivariate Bell-Based Apostol-Frobenius-Type Poly-Genocchi Polynomials

In this section, we introduce higher-order bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials, aligning them with the Bell-based Apostol-type Frobenius-Euler polynomials as defined by Alam et al. [4] and the generalized Apostol-Frobenius-Type poly-Genocchi polynomials by Khan [34]. The following definition formally presents the higher-order Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials.

Definition 2.1. The bivariate Bell-based Apostol-Frobenius-type Poly-Genocchi polynomials of higher order with parameters a and b, denoted by ${}_BG_n^{(r)}(x,y;u,\lambda,a,b)$ are defined by

$$\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^r e^{xt+y(e^t-1)}. \tag{2.1}$$

When $x=0$, the Bell-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order ${}_BG_{n,k}^{(r)}(y;u,\lambda)$ are defined by

$$\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(y;u,\lambda,a,b)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t-ua^{-t}}\right)^r e^{y(e^t-1)} \tag{2.2}$$

Remark 2.2. *Using the fact that* $\mathrm{Li}_1(z) = -\ln(1-z)$, *we get*

$$Li_1(1-(ab)^{-(1-u)t}) = -\ln(1-(1-(ab)^{-(1-u)t})) = (1-u)t\ln ab.$$

Hence, when $k=1$, *the Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order* ${}_BG_{n,k}^{(r)}(y;u,\lambda)$ *in* (2.2) *can further be reduced to*

$$\sum_{n=0}^{\infty} {}_BG_n^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} = \left(\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}\right)^r e^{xt+y(e^t-1)}, \tag{2.3}$$

the higher order bivariate Bell-based Apostol-Frobenius-Type Genocchi numbers with parameters a *and* b.

Remark 2.3. *When $y = 1$, the Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order ${}_BG_{n,k}^{(r)}(y; u, \lambda)$ in (2.2) can further be reduced to*

$$\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(1; u, \lambda, a, b)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{e^t-1},$$

the Bell-based Apostol-Frobenius-Type poly-Genocchi numbers of higher order.

Remark 2.4. *The Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order ${}_BG_{n,k}^{(r)}(y; u, \lambda)$ in (2.2) can further be reduced as follows:*

(i) When $k = 1, a = 1, b = e$,

$$\sum_{n=0}^{\infty} {}_BG_n^{(r)}(x, y; u, \lambda)\frac{t^n}{n!} = \left(\frac{(1-u)t}{\lambda e^t - u}\right)^r e^{xt+y(e^t-1)}, \tag{2.4}$$

the Higher Order Bivariate Bell-based Apostol-Frobenius-Type Genocchi polynomials, where ${}_BG_n(x, y; u, \lambda) = {}_BG_n^{(1)}(x, y; u, \lambda, 1, e)$.

(ii) When $r = 1$, (2.4) gives

$$\sum_{n=0}^{\infty} {}_BG_n(x, y; u, \lambda)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u}\, e^{xt+y(e^t-1)}, \tag{2.5}$$

the Bivariate Bell-based Apostol-Frobenius-Type Genocchi polynomials, where ${}_BG_n(x, y; u, \lambda) = {}_BG_n^{(1)}(x, y; u, \lambda, 1, e)$.

Remark 2.5. *When $r = 0$, the Bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order ${}_BG_{n,k}^{(r)}(y; u, \lambda)$ in (2.1) can be reduced to*

$$\begin{aligned}\sum_{n=0}^{\infty} {}_BG_n^{(0)}(x, y; u, \lambda)\frac{t^n}{n!} &= e^{xt+y(e^t-1)},\\ B_n(x, y) &= {}_BG_n^{(0)}(x, y; u, \lambda) \end{aligned} \tag{2.6}$$

the Bivariate Bell Polynomial.

The following theorem contains the first identity for the bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order expressed in terms Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order and the Bell polyomials.

Theorem 2.6. *The bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order with parameters a and b, ${}_BG_{n,k}^{(r)}(x, y; u, \lambda, a, b)$, are equal to*

$${}_BG_{n,k}^{(r)}(x, y; u, \lambda, a, b) = \sum_{k=0}^{n}\binom{n}{k} G_k^{(r)}(x; u, \lambda, a, b) B_{n-k}(y). \tag{2.7}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)} \\
&= \left\{\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt}\right\} e^{y(e^t-1)} \\
&= \left(\sum_{n=0}^{\infty} G_{n,k}^{(r)}(x;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(y)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k} G_k^{(r)}(x;u,\lambda,a,b)B_{n-k}(y)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired identity in (2.7).

The next theorem expresses the bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order as polynomial in x with Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order as the coefficients.

Theorem 2.7. *The bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order ${}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)$ are equal to*

$$ {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b) = \sum_{j=0}^{n}\binom{n}{k} {}_BG_{n-j,k}^{(r)}(y;u,\lambda,a,b)x^j. \tag{2.8}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{y(e^t-1)}e^{xt} \\
&= \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j} {}_BG_{j,k}^{(r)}(y;u,\lambda,a,b)x^{n-j}\right\}\frac{t^n}{n!}
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields

$$ {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b) = \sum_{j=0}^{n}\binom{n}{j} {}_BG_{j,k}^{(r)}(y;u,\lambda,a,b)x^{n-j},$$

which is equivalent to the desired identity in (2.8).

The next theorem contains the addition formula for bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order.

Theorem 2.8. *The bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order* ${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ *are equal to*

$$ {}_BG^{(r)}_{n,k}(x+y,z;u,\lambda,a,b) = \sum_{j=0}^{n}\binom{n}{j} G^{(r)}_{j,k}(x;u,\lambda,a,b)B_{n-j}(y,z). \tag{2.9}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_BG^{(r)}_{n,k}(x+y,z;u,\lambda,a,b)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{(x+y)t+z(e^t-1)} \\
&= \left\{\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt}\right\} e^{yt+z(e^t-1)} \\
&= \left(\sum_{n=0}^{\infty} G^{(r)}_{n,k}(x;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(y,z)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{k} G^{(r)}_{j,k}(x;u,\lambda,a,b)B_{n-j}(y,z)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired identity in (2.9).

3. Implicit Summation Formula

Within this section, we will derive different summation formulas for ${}_BG^{(r)}_n(x+y,z;u,\lambda)$, establishing implicit connections among the variables by considering them as arguments. The subsequent theorem encapsulates a particular expression of these summation formulas.

Theorem 3.1. *The bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order* ${}_BG^{(r)}_n(x,y;u,\lambda)$ *satisfy the following summation formula:*

$$\begin{aligned}
&{}_BG^{(r_1+r_2)}_{n,k}(x_1+x_2,y_2+y_2;u,\lambda,a,b) \\
&\quad = \sum_{j=0}^{n}\binom{n}{j} {}_BG^{(r_1)}_{j,k}(x_1,y_1;u,\lambda,a,b)\,{}_BG^{(r_2)}_{n-j,k}(x_2,y_2;u,\lambda).
\end{aligned} \tag{3.1}$$

Proof. Note that we can express the right hand side of (2.1) as follows:

$$\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{r_1+r_2} e^{(x_1+x_2)t+(y_1+y_2)(e^t-1)}$$

$$= \left\{ \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}} \right)^{r_1} e^{x_1 t + y_1(e^t-1)} \right\}$$
$$\left\{ \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}} \right)^{r_2} e^{x_2 t + y_2(e^t-1)} \right\}.$$

Applying (2.1) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r_1+r_2)}\left(x_1+x_2, y_2+y_2; u, \lambda, a, b\right) \frac{t^n}{n!} \\
&= \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r_1)}(x_1, y_1; u, \lambda, a, b) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r_2)}(x_2, y_2; u, \lambda, a, b) \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} {}_BG_{j,k}^{(r_1)}(x_1, y_1; u, \lambda, a, b) {}_BG_{n-j,k}^{(r_2)}(x_2, y_2; u, \lambda, a, b) \binom{n}{j}.
\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain

$$\begin{aligned}
&{}_BG_{n,k}^{(r_1+r_2)}\left(x_1+x_2, y_2+y_2; u, \lambda, a, b\right) \\
&\quad = \sum_{j=0}^{n} \binom{n}{k} {}_BG_{j,k}^{(r_1)}(x_1, y_1; u, \lambda, a, b) {}_BG_{n-j,k}^{(r_2)}(x_2, y_2; u, \lambda),
\end{aligned}$$

which is exactly the desired summation formula in (3.5).

Remark 3.2. *When $r_1 = r, r_2 = 0, x_1 = x, x_2 = 1, y_1 = y,\ y_2 = 0$, the summation formula in* (3.5) *reduces to*

$$\begin{aligned}
{}_BG_{n,k}^{(r)}(x+1, y; u, \lambda, a, b) &= \sum_{j=0}^{n} \binom{n}{j} {}_BG_{j,k}^{(r)}(x, y; u, \lambda, a, b) B_{n-k}(1, 0) \\
&= \sum_{j=0}^{n} \binom{n}{j} {}_BG_{j,k}^{(r)}(x, y; u, \lambda, a, b). \qquad (3.2)
\end{aligned}$$

On the other hand, when $y = 1$, (2.9) *gives*

$${}_BG_{n,k}^{(r)}(x+1, z; u, \lambda, a, b) = \sum_{j=0}^{n} \binom{n}{j} G_{j,k}^{(r)}(x; u, \lambda, a, b) B_{n-j}(1, z). \qquad (3.3)$$

Replacing z with y in (3.3) *and subtract it from* (3.2) *yields*

$$\sum_{j=0}^{n} \binom{n}{j} G_{j,k}^{(r)}(x; u, \lambda, a, b) B_{n-j}(1, z) = \sum_{j=0}^{n} \binom{n}{j} {}_BG_{j,k}^{(r)}(x, y; u, \lambda, a, b).$$

We recall the following series manipulation formula:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^n}{m!}. \tag{3.4}$$

Applying (3.4) obtains

$$\begin{aligned}
&\left(\frac{Li_k(1-(ab)^{-(1-u)(t+v)})}{\lambda b^{t+v}-ua^{-(t+v)}}\right)^r e^{y(e^{t+v}-1)}\\
&\qquad = e^{-x(t+v)}\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{(t+v)^n}{n!}\\
&\qquad = e^{-x(t+v)}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} {}_BG_{j+l,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^j}{j!}\frac{v^l}{l!}.
\end{aligned}$$

Replacing x with z yields

$$\begin{aligned}
&\left(\frac{Li_k(1-(ab)^{-(1-u)(t+v)})}{\lambda b^{t+v}-ua^{-(t+v)}}\right)^r e^{y(e^{t+v}-1)}\\
&\qquad = e^{-z(t+v)}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} {}_BG_{j+l,k}^{(r)}(z,y;u,\lambda,a,b)\frac{t^j}{j!}\frac{v^l}{l!}.
\end{aligned}$$

Thus, by using (3.4) again, we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} {}_BG_{j+l,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^j}{j!}\frac{v^l}{l!}\\
&\quad = e^{(x-z)(t+v)}\sum_{j,l\geq 0} {}_BG_{j+l,k}^{(r)}(z,y;u,\lambda,a,b)\frac{t^j}{j!}\frac{v^l}{l!}\\
&\quad = \left(\sum_{N=0}^{\infty}(x-z)^N\frac{(t+v)^N}{N!}\right)\left(\sum_{j,l\geq 0} {}_BG_{j+l,k}^{(r)}(z,y;u,\lambda,a,b)\frac{t^j}{j!}\frac{v^l}{l!}\right)\\
&\quad = \left(\sum_{n,m\geq 0}(x-z)^{n+m}\frac{t^n}{n!}\frac{v^m}{m!}\right)\left(\sum_{j,l\geq 0} {}_BG_{j+l,k}^{(r)}(z,y;u,\lambda,a,b)\frac{t^j}{j!}\frac{v^l}{l!}\right)\\
&\quad = \sum_{j,l\geq 0}\left\{\sum_{n,m=0}^{j,l}\binom{j}{n}\binom{l}{m}(x-z)^{n+m}{}_BG_{j+l,k}^{(r)}(z,y;u,\lambda,a,b)\right\}\frac{t^j}{j!}\frac{v^l}{l!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^j}{j!}\frac{v^l}{l!}$ completes the proof of the following theorem.

Theorem 3.3. *The bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order ${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ satisfy the following summation formula:*

$$ {}_BG^{(r)}_{j+l,k}(x,y;u,\lambda,a,b) = \sum_{n,m=0}^{j,l} \binom{k}{n}\binom{l}{m}(x-z)^{n+m}\,{}_BG_{k+l-n-m}(z,y;u,\lambda,a,b). \quad (3.5) $$

The next theorem gives the difference when the variable x in ${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ is shifted by 1.

Theorem 3.4. *For $n \geq 1$, the difference ${}_BG^{(r)}_{n,k}(x+1,y;u,\lambda) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ equals*

$$ {}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) = \sum_{j=0}^{n-1}\binom{n}{k}{}_BG^{(r)}_{j,k}(x,y;u,\lambda,a,b). \quad (3.6) $$

Proof. Using Definition 2.1, we have

$$ \begin{aligned} &\sum_{n=0}^{\infty} {}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!} \\ &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{(x+1)t+y(e^t-1)} - \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)} \\ &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)}(e^t-1) \\ &= \left(\sum_{n=0}^{\infty} {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0}\frac{t^{n+1}}{(n+1)!}\right) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n+1}{j}{}_BG^{(r)}_{j,k}(x,y;u,\lambda,a,b)\right\}\frac{t^n}{n!}. \end{aligned} $$

Thus, we have

$$ \begin{aligned} &\sum_{n=0}^{\infty}\left({}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n+1}{j}{}_BG^{(r)}_{j,k}(x,y;u,\lambda,a,b)\right\}\frac{t^{n+1}}{(n+1)!} \\ &= \sum_{n=1}^{\infty}\left\{\sum_{j=0}^{n-1}\binom{n}{j}{}_BG^{(r)}_{j,k}(x,y;u,\lambda,a,b)\right\}\frac{t^n}{n!}. \end{aligned} $$

This immediately gives

$$ {}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) = \sum_{j=0}^{n-1}\binom{n}{j}{}_BG^{(r)}_{j,k}(x,y;u,\lambda,a,b). $$

4. Connection with Second Kind Stirling Numbers and Bivariate Bell Polynomials

In this section, we derive some formulas connecting ${}_BG_n^{(r)}(x,y;u,\lambda)$ with Stirling numbers of the second kind given in (1.15) and bivariate Bell polynomials in (2.6).

Theorem 4.1. *The bivariate Bell-based Apostol-Frobenius-Type poly-Genocchi polynomials of higher order ${}_BG_n^{(r)}(x,y;u,\lambda)$ satisfy the following summation formula*

$$ {}_BG_{n,k}^{(r)}(x,y;u,\lambda) = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}(x)_j S(i,j)\,{}_BG_{n-i,k}^{(r)}(y;u,\lambda). \tag{4.1} $$

Proof. Using Definition 2.1, we have

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{y(e^t-1)}(1+e^t-1)^x \\ &= \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\binom{x}{n}(e^t-1)^n\right) \\ &= \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}(x)_j\frac{(e^t-1)^j}{j!}\right) \\ &= \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}(x)_j\sum_{n=0}^{\infty}S(n,j)\frac{t^n}{n!}\right) \\ &= \left(\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\infty}(x)_jS(n,j)\right\}\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\left\{\sum_{j=0}^{\infty}(x)_jS(i,j)\,{}_BG_{n-i,k}^{(r)}(y;u,\lambda,a,b)\right\}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{\infty}(x)_jS(i,j)\,{}_BG_{n-i,k}^{(r)}(y;u,\lambda,a,b)\right\}\frac{t^n}{n!} \end{aligned} $$

$$ \begin{aligned} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b) &= \sum_{i=0}^{n}\sum_{j=0}^{\infty}\binom{n}{i}(x)_jS(i,j)\,{}_BG_{n-i,k}^{(r)}(y;u,\lambda,a,b) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}(x)_jS(i,j)\,{}_BG_{n-i,k}^{(r)}(y;u,\lambda,a,b). \end{aligned} $$

The subsequent theorem is another relation for ${}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)$ in connection with Stirling numbers of the second kind.

Theorem 4.2. *The higher order Bivariate Bell-based Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b satisfy the relation,*

$$ {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b) = \sum_{j=0}^{n} \binom{n}{j} (-1)^r {}_BG_{n-j}^{(r)}(x,y;u,\lambda,a,b) d_j \tag{4.2}$$

where

$$\begin{aligned} d_j &= \sum_{n_1+n_2+\ldots+n_r=j} \prod_{i=1}^{r} c_{n_i} \binom{j}{n_1, n_2, \ldots, n_r} \\ c_j &= \sum_{m=0}^{j} (-1)^{m+j+1} \frac{((1-u)\ln ab)^j m! S(j+1,m+1)}{(j+1)(m+1)^{k-1}}. \end{aligned}$$

Proof. Now, (2.1) can be written as

$$\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} = \frac{e^{xt+y(e^t-1)}}{(\lambda b^t - ua^{-t})^r} \left(\sum_{m=1}^{\infty} \frac{(1-e^{-(1-u)t\ln ab})^m}{m^k} \right)^r$$

$$\begin{aligned} &= \frac{e^{xt+y(e^t-1)}}{(\lambda b^t - ua^{-t})^r} \left(\sum_{m=0}^{\infty} \frac{(1-e^{-(1-u)t\ln ab})^{m+1}}{(m+1)^k} \right)^r \\ &= \frac{e^{xt+y(e^t-1)}}{(\lambda b^t - ua^{-t})^r} \left(\sum_{m=0}^{\infty} \frac{m!}{(m+1)^{k-1}} \frac{(1-e^{-(1-u)t\ln ab})^{m+1}}{(m+1)!} \right)^r \\ &= \frac{e^{xt+y(e^t-1)}}{(\lambda b^t - ua^{-t})^r} \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+1} m!}{(m+1)^{k-1}} \sum_{j=m+1}^{\infty} S(j,m+1) \frac{(-(1-u)t\ln ab)^j}{j!} \right)^r \\ &= (-1)^r e^{xt+y(e^t-1)} \left(\frac{(1-u)t\ln ab}{\lambda b^t - ua^{-t}} \right)^r \left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!} \right)^r, \end{aligned}$$

where

$$c_j = \sum_{m=0}^{j} (-1)^{m+j+1} \frac{((1-u)\ln ab)^j m! S(j+1,m+1)}{(j+1)(m+1)^{k-1}}.$$

Note that the power series $\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^r$ can be expressed as

$$\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!} \right)^r = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!},$$

where

$$d_n = \sum_{n_1+n_2+\ldots+n_r=n} \prod_{i=1}^{r} c_{n_i} \binom{n}{n_1, n_2, \ldots, n_r},$$

(see [10]). It follows that

$$\sum_{n=0}^{\infty} {}_BG_{n,k}^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!} = (-1)^r\left(\sum_{n=0}^{\infty} {}_BG_n^{(r)}(x,y;u,\lambda,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} d_n\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}(-1)^r {}_BG_{n-j}^{(r)}(x,y;u,\lambda,a,b)d_j\right\}\frac{t^n}{n!}$$

Comparing the coefficients completes the proof of the theorem.

The next theorem contains a relation that expresses the bivariate Bell polynomials in terms of bivariate Bell-based Apostol-Frobenius-type poly-Genocchi polynomials.

Theorem 4.3. *The following relation holds*

$$B_n(x,y) = \frac{\lambda\ {}_BG_{n+1}(x+1,y;u,\lambda) - u\ {}_BG_{n+1}(x,y;u,\lambda)}{(1-u)(n+1)}. \tag{4.3}$$

Proof. Using equation (2.6), we have

$$\sum_{n=0}^{\infty} B_n(x,y)\frac{t^n}{n!} = \left(\frac{\lambda e^t-u}{(1-u)t}\right)\left(\frac{(1-u)\,t}{\lambda e^t-u}\,e^{xt+y(e^t-1)}\right)$$
$$= \frac{1}{(1-u)t}\left(\lambda\left(\frac{(1-u)\,t}{\lambda e^t-u}e^{(x+1)t+y(e^t-1)}\right) - u\left(\frac{(1-u)\,t}{\lambda e^t-u}e^{xt+y(e^t-1)}\right)\right)$$
$$= \frac{1}{(1-u)}\left(\lambda\sum_{n=0}^{\infty} {}_BG_n(x+1,y;u,\lambda)\frac{t^{n-1}}{n!} - u\sum_{n=0}^{\infty} {}_BG_n^{(r)}(x,y;u,\lambda)\frac{t^{n-1}}{n!}\right)$$
$$= \frac{1}{1-u}\left(\lambda\sum_{n=-1}^{\infty} {}_BG_n(x+1,y;u,\lambda)\frac{t^n}{(n+1)!} - u\sum_{n=-1}^{\infty} {}_BG_n^{(r)}(x,y;u,\lambda)\frac{t^n}{(n+1)!}\right)$$
$$= \sum_{n=-1}^{\infty}\left(\frac{\lambda\ {}_BG_{n+1}(x+1,y;u,\lambda) - u\ {}_BG_{n+1}(x,y;u,\lambda)}{(1-u)(n+1)}\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields

$$B_n(x,y) = \frac{\lambda\ {}_BG_{n+1}(x+1,y;u,\lambda) - u\ {}_BG_{n+1}(x,y;u,\lambda)}{(1-u)(n+1)}.$$

5. Derivative Formulas

The derivative formulas for special polynomials play a crucial role in various areas of mathematics, physics, engineering, and other scientific disciplines. For instance, derivative formulas allow for the analysis of the behavior and properties of special polynomials in terms of their rates of change. This is fundamental in calculus and mathematical analysis for understanding functions and their behavior. They are also essential for manipulating generating functions, which represent sequences of coefficients of special polynomials. These functions are widely used in combinatorics, number theory, and discrete mathematics for counting and enumerative purposes.

The following theorem contains the derivative formula for ${}_BG^{(r)}_{n,k}(x,y;u,\lambda)$ with respect to the variable x.

Theorem 5.1. *The following derivative formula holds*

$$\frac{\partial}{\partial x}{}_BG^{(r)}_{n,k}(x,y;u,\lambda) = n{}_BG^{(r)}_{n-1,k}(x,y;u,\lambda). \tag{5.1}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\sum_{n=0}^{\infty}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!} &= \frac{\partial}{\partial x}\left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)} \\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)}\ t \\
&= t\sum_{n=0}^{\infty}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty}n{}_BG^{(r)}_{n-1,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields

$$\frac{\partial}{\partial x}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) = n{}_BG^{(r)}_{n-1,k}(x,y;u,\lambda,a,b).$$

Remark 5.2. *This relation shows that ${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ is an Apell Polynomial (see [25, 29]). Belonging to the category of Appell polynomials, the polynomials ${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ are expected to demonstrate the following characteristics:*

$$\begin{aligned}
{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) &= \sum_{j=0}^{n}\binom{n}{j}c_jx^{n-j} \\
{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) &= \left(\sum_{j=0}^{n}\frac{c_j}{j!}D^j\right)x^n
\end{aligned}$$

for some scalar $c_k \neq 0$. Clearly, using (2.8), $c_j = {}_BG^{(r)}_{j,k}(y;u,\lambda,a,b)$. Hence,

$${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) = \left(\sum_{j=0}^{n}\frac{{}_BG^{(r)}_{j,k}(y;u,\lambda,a,b)}{j!}D^j\right)x^n. \tag{5.2}$$

The next theorem contains the derivative formula for ${}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)$ with respect to the variable y.

Theorem 5.3. *The following derivative formula holds*

$$\frac{\partial}{\partial y}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) = {}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b). \tag{5.3}$$

Proof. Using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\partial}{\partial y}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\frac{t^n}{n!}\\
&= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)}(e^t-1)\\
&= \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{(x+1)t+y(e^t-1)}\\
&\quad - \left(\frac{Li_k(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^r e^{xt+y(e^t-1)}\\
&= \sum_{n=0}^{\infty}{}_BG^{(r)}_{n,k}(x+1,y;u,\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty}{}_BG^{(r)}_{n,k}(x,y;u,\lambda)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left\{{}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields

$$\frac{\partial}{\partial y}{}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b) = {}_BG^{(r)}_{n,k}(x+1,y;u,\lambda,a,b) - {}_BG^{(r)}_{n,k}(x,y;u,\lambda,a,b).$$

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