EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2264-2275
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

$\mathfrak{D}$-Semiprime Rings

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Abstract. Let R be an associative and 2-torsion-free ring with an identity. in this work, we will generaliz the results of differentially prime rings in [18] by applying the hypotheses in a differentially semiprime rings. In particular, we have proved that if R is a $\mathfrak{D}$-semiprime ring, then either R is a commutative ring or $\mathfrak{D}$ is a semiprime ring.

2020 Mathematics Subject Classifications: 16W25, 16N60

Key Words and Phrases: Derivation, semiprime ring, δ-semiprime ring, δ-ideal

1. Introduction

Let R be an associative ring with an identity element. We say that R is 2-torsion-free if for any $r \in R$ and an integer n, the condition $2r = 0$ holds if and only if $r = 0$. $Z(R)$ is the center of R. $\mathfrak{D}$ is the set of all derivations in R and $\mathfrak{d}$ is a non-empty subset of $\mathfrak{D}$. An additive subgroup A is said to be a Lie ideal of R if $[r, a] \in A$, for all $r \in R$ and $a \in A$. A Lie ideal A of R is called $\mathfrak{d}$- ideal if $\delta(a) \in A$, for all $a \in A$ and $\delta \in \mathfrak{d}$. $annT = \{x \in R \mid xT = Tx = 0\}$ is the annihilator of T. If $a \in R$, then $\partial_a(x) = [x, a] = ax - xa$ is an inner derivation of R induced by $a \in R$, where $\partial_a \in \mathfrak{D}$. $I_{\mathfrak{D}} = \{\partial_a \mid a \in R\}$ is an ideal of a ring $\mathfrak{D}$, see [13].

A ring R is called a $\mathfrak{d}$-prime (differentilly prime) if for each $\mathfrak{d}$-ideals A and B of R with $AB = 0$, implies that $A = 0$ or $B = 0$. A ring R is said to be $\mathfrak{d}$-semiprime (differentilly semiprime) if for every $\mathfrak{d}$-ideal I of R, the condition $I^2 = 0$, implies that $I = 0$. $C(R)$ is the commutator ideal of R and $charR$ is the characteristic of a ring R. By $Z_0(R)$ we denote the ideal of R generated by its central ideals.

The properties of differentially prime rings were studied by Herstein [7, 8] and also in his book [9]. After that, many authors have proved some results about this concept, such

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5210

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as Bergen and Herstein [3], Hirano [11], Hongan and Trzepizur [12], Beidar and Mikhalev [2], Chebotar and Lee [6] and could be seen in Lee and Liu [23].

Al Khalaf and others, see [18, 19, 27, 28], have demonstrated the differentially prime rings, simple rings, differentially δ- prime rings and reverse derivation on δ- prime rings. Furthermore, they discussed the differentially semiprime and semiprime gamma rings, see that in [20, 21].

Many authors have investigated Lie rings of differentially semiprime rings as [14], [24] and Jordan in [15–17] and Nowicki [25].

The commutative rings with semiprime Lie rings were studied by Passman [26] and Bresar [4].

Finally, all other definitions and facts are standard, which were be found in [1, 13] and also in [10].

2. Preliminaries

For any associative Lie ring R, the commutator $[R, R]$ is a subgroup of R, which is an additive subgroup generated by all $[s, t]$ with $s, t \in R$.

For what we will prove, we need some lemmas.

Lemma 1. *The following conditions are equivalent:*

(1) *R is $\mathfrak{d}$- semiprime ring,*

(2) *For any $\mathfrak{d}$-ideals A and B of R, the implication*

$$AB = 0 \Rightarrow A \cap B = 0$$

is true.

(3) *If $a \in R$, such that*

$$aR\delta_1^{m_1}...\delta_n^{m_n}(a) = 0,$$

for any integers $n \geq 1, m_i \geq 0$ and any derivation $\delta_i \in \mathfrak{d}$, where $i = 1, ..., n$, then $a = 0$.

proof. A simple modification of Proposition 2 from [22].

Lemma 2. *[1] Let A be a Lie $\mathfrak{d}$-ideal of a $\mathfrak{d}$-semiprime ring R of $charR \neq 2$. If $[A, A] \subseteq Z(R)$, then $A \subseteq Z(R)$.*

Lemma 3. *[1] Let R be a 2-torsion- free $\mathfrak{d}$-semiprime ring and A a nonzero Lie $\mathfrak{d}$-ideal of R and an associative subring. Then $A \subseteq Z(R)$ or A contains a non-central associative $\mathfrak{d}$-ideal of R.*

Lemma 4. *[20] If R is a $\mathfrak{D}$-semiprime ring, Φ an ideal of D. Then*

$$[\Phi, I_{\mathfrak{D}}] = 0 \Leftrightarrow \Phi \cap I_{\mathfrak{D}} = 0.$$

3. Lie ideals in $\mathfrak{D}$-semiprime rings

Lemma 5. *Let R be a 2-torsion-free $\mathfrak{d}$-semiprime ring, U its nonzero Lie $\mathfrak{d}$-ideal and an associative subring, where $a \in R$. If*

$$[\delta_1^{m_1} \cdots \delta_k^{m_k}(a), [a, U]] = 0,$$

for any integers $m_i \geq 0$, $k \geq 1$ and derivations $\delta_i \in \mathfrak{d}$, where $i = 1, \ldots, k$, then $a \in Z(R)$.

proof. Let $X_a = \{[\delta_1^{m_1} \cdots \delta_k^{m_k}(a), x]\}, x, a \in R, \delta_i \in \mathfrak{d}, m_i \geq 0$ and $x, y \in R$. From

$$[b, xy] = [b, x]y + x[b, y], b \in X_a, \tag{1}$$

we get $a[b, xy] = 0$, then $ax[b, y] = 0$. Hence $ayx[b, y] = 0$ and $yax[b, y] = 0$. Thus, we deduce that

$$(R[a, y]R)^2 = 0, a \in R. \tag{2}$$

In addition

$$0 = d(a[b, x]) = d(a)[b, x].$$

Multiply the identity (1) from the left by $d(a)$, then we get $d(a)x[b, y] = 0$. Therefore,

$$0 = \delta(ax[d(b), y] = \delta(a)x[d(b), y],$$

and by the similar argument, we have

$$\delta_1^{m_1} \cdots \delta_k^{m_k}(a)x[\delta_1^{m_1} \cdots \delta_k^{m_k}(a), y] = 0,$$

for any integers $k \geq 1, m_i \geq 0$ and derivations $\delta_i \in \mathfrak{d}$, where $i = 1, ..., k$. As in the proof of the condition (2), we deduce that $(R[\delta_1^{m_1} \cdots \delta_k^{m_k}(a), y]R)^2 = 0$. Then,

$$I = \sum_{k=1}^{\infty} \sum_{\delta_i \in \mathfrak{d}} R[\delta_1^{m_1} \cdots \delta_k^{m_k}(a), y]R, y \in R$$

is a sum of nilpotent ideals, therefore it will be a nil ideal as well. Since I is a $\mathfrak{d}$- ideal, we get $I = o$, hence $a \in Z(R)$

By the same way, we prove the following Lemma

Lemma 6. *Let R be a 2-torsion-free $\mathfrak{d}$-semiprime ring, U its Lie $\mathfrak{d}$-ideal. If*

$$a \in C_R([\delta_1^{s_1} \cdots \delta_l^{s_l}(a), U]),$$

for any integers $s_i \geq 0$, $l \geq 1$ and derivations $\delta_i \in \mathfrak{d}$, where $i = 1, \ldots, l$. Then $a \in C_R(U)$.

proof. Let $u, v \in U$, $s_i \geq 0$, $l \geq 1$ be any integers and $\varphi, \delta_i \in \mathfrak{d}$ be any derivation, where $i = 1, \ldots, l$. Since

$$\varphi(\delta_1^{s_1} \cdots \delta_l^{s_l}(a)[a, x]) = \varphi([a, x]\delta_1^{s_1} \cdots \delta_l^{s_l}(a)),$$

we have that,

$$\delta_1^{s_1} \cdots \delta_l^{s_l}(a) \in C_R([\delta_1^{s_1} \cdots \delta_l^{s_l}(a), x]).$$

Then from

$$\delta_1^{s_1} \cdots \delta_l^{s_l}(a)[\delta_1^{s_1} \cdots \delta_l^{s_l}(a), uv] = [\delta_1^{s_1} \cdots \delta_l^{s_l}(a), uv]\delta_1^{s_1} \cdots \delta_l^{s_l}(a)$$

it holds that

$$[\delta_1^{s_1} \cdots \delta_l^{s_l}(a), u][\delta_1^{s_1} \cdots \delta_l^{s_l}(a), v] = 0,$$

what forces

$$[\delta_1^{s_1} \cdots \delta_l^{s_l}(a), u]t[\delta_1^{s_1} \cdots \delta_l^{s_l}(a), v] = 0,$$

where $t \in R$. Thus, the sum of nilpotent idal of R is $\mathfrak{d}$-ideal. Then $a \in C(R)$.

Now, we will extend result given by [8, Theorem 3].

Proposition 1. *Let R be a 2-torsion-free $\mathfrak{d}$-semiprime ring, W its associative $\mathfrak{d}$-subring and U its Lie $\mathfrak{d}$-ideal. If*

$$[W, U] \subseteq W,$$

then $[W, U] = 0$ or W contains a non-zero associative $\mathfrak{d}$-ideal of R.

proof. Let $x, y, r \in R$, $t_1, t \in U \cap W$ and $v, w, w_1, s, a, b \in W$. Assume that

$$[W, U] \neq 0.$$

By Lemma 2, $[U, U] \neq 0$. Since the subring $\overline{U}$ of R generated by U satisfies that $\delta(\overline{U}) \subseteq \overline{U}$, for all $\delta \in \mathfrak{d}$, then, as in the proof of [8, Theorem 3], we can obtain that

$$R[a, b]RzR \subseteq \overline{U}zR \subseteq W,$$

where $z = [s, t][t, w]$. Thus, the sum of nilpotent idal of R is $\mathfrak{d}$-ideal of R contained in W. Otherwise

$$[a, b]RzR = 0,$$

and consequently

$$A = \sum_{\substack{s, w \in W \\ t \in U \cap W}} RzR$$

is a $\mathfrak{d}$-ideal such that

$$[a, b] \in ann_l A.$$

But A is non-zero and $A \cap ann_l A = 0$, implies that

$$[a, b] = 0.$$

Inasmuch $b = z \in \overline{U}$, we have

$$[z, R] \subseteq \overline{U} \text{ and } \delta_1^{m_1} \cdots \delta_k^{m_k}(z) \in \overline{U},$$

for any integers $k \geq 1$, $m_i \geq 0$ and derivations $\delta_i \in \mathfrak{d}$ $(i = 1, \ldots, k)$, we conclude that

$$[\delta_1^{m_1} \cdots \delta_k^{m_k}(z), R] \subseteq \overline{U}$$

what gives that

$$z \in C_R([\delta_1^{m_1} \cdots \delta_k^{m_k}(z), R]).$$

By Lemma 5, $z \in Z(R)$. Then

$$B = \sum_{\substack{s, w \in W \\ t \in U \cap W}} [s, t][t, w]R \subseteq W$$

is a $\mathfrak{d}$-ideal of R. Therefore, $B = 0$ and, as a consequence, $z = 0$. This means that

$$[s, t][t, w] = 0. \tag{3}$$

Replace w by vw in the identity (3). Then $[s, t]v[t, w] = 0$ and consequently

$$[s, t]W[t, w] = 0. \tag{4}$$

Linearize the identity (3) on t and put $s = w = a$; then

$$[a, t_1][a, t] + [a, t][a, t_1] = 0. \tag{5}$$

Since $x := [[a, t_1], r] \in U$ and

$$2[a, t_1]r[a, t_1] = [x, [a, t_1]] \in W,$$

we see that, using (5)

$$2[a, t_1]R[a, t_1] \subseteq W,$$

and, in view of the identity (4),

$$[s, t][a, t_1]R[a, t_1][w, t] = 0. \tag{6}$$

In the identity (6), put $s = a = w$; we get

$$[a, t][a, t_1]R[a, t][a, t_1] = 0.$$

This means that

$$(R[a, t][a, t_1]R)^2 = 0.$$

Since

$$C = \sum_{\substack{a \in W \\ t, t_1 \in U \cap W}} R[a, t][a, t_1]R$$

is a $\mathfrak{d}$-ideal, which is a sum of nilpotent ideals, we deduce that $C = 0$ and so

$$[a,t][a,t_1] = 0. \tag{7}$$

We linearize the identity (7) on a to get

$$[a,t][b,t_1] + [b,t][a,t_1] = 0.$$

Using the previous relation in the identity (6) with $w = b$, we obtain

$$[s,t][a,t_1]R[b,t_1][a,t] = 0. \tag{8}$$

By linearization the identity (3) for t, we have

$$[s,t][t_1,w] + [s,t_1][t,w] = 0.$$

In view of it, from the identity (8), by replacing b instead of s and w by a, it follows that

$$[s,t][a,t_1]R[s,t][a,t_1] = 0.$$

Then

$$D = \sum_{\substack{a,s \in W \\ t,t_1 \in U \cap W}} R[s,t][a,t_1]R$$

is a $\mathfrak{d}$-ideal. Then $D = 0$ and

$$[s,t][a,t_1] = 0.$$

Denote $[W,[U,W]]$ by W_1. Then W_1 is a Lie $\mathfrak{d}$-ideal of R and

$$[s,t]W_1 = 0.$$

Furthermore, $[U,W_1] \subseteq W_1$, $[s,t]UW_1 = 0$ and $[s,t]\overline{U}W_1 = 0$. From the equation $R[a,b]R \subseteq \overline{U}$, we deduce that

$$[s,t]R[a,b]RW_1 = 0.$$

Assume that $p,q \in U \cap W$, then we get

$$[p,q]R[p,q]RW_1 = 0.$$

Therefore,

$$(R[s,t]R)^3 = 0.$$

Then

$$E = \sum_{p,q \in U \cap W} R[p,q]R$$

is a nil $\mathfrak{d}$-ideal of R, hence $[p,q] = 0$. As a consequence, $t \in [U,W]$ is commuting with $[U,[U,W]]$. By Lemma 3, $t \in C_R(U)$. Then

$$t \in C_R([\delta_1^{m_1} \cdots \delta_k^{m_k}(t), U]),$$

for any integers $k \geq 1$, $m_i \geq 0$ and derivations $\delta_i \in \mathfrak{d}$, where $i = 1, \dots, k$ and $t \in Z(R)$. This means that $[U, W] \subseteq Z(R)$.

Since $u \in U$ is commuting with

$$[\delta_1^{m_1} \cdots \delta_k^{m_k}(u), w],$$

for any $w \in W$, we deduce that

$$[U, W] = 0,$$

is a contradiction.

Lemma 7. *Let R be a 2-torsion-free $\mathfrak{d}$-semiprime ring, U its Lie $\mathfrak{d}$-ideal. If $A \subseteq U$ and satisfies that $\delta(A) \subseteq A$ for all $\delta \in \mathfrak{d}$ and it is an additive subgroup such that $[U, A] \subseteq A$ and $[A, A] \subseteq Z(R)$, then $[A, U] = 0$.*

proof. Let $u \in U$ and $x \in R$. If $[A, A] = 0$, then $[a, u] \in A \cap C_R(a)$. By Lemma 5, $[A, U] = 0$. Therefore, we assume that $0 \neq [a, b] \in Z(R)$ for some $a, b \in A$. As in the proof of [8, Lemma 4], we can obtain that

$$[a, b]^4 = 0.$$

Since $[A, A] \subseteq Z(R)$, we deduce that

$$I = \sum_{a,b \in A} [a, b]R$$

is a nil $\mathfrak{d}$-ideal. Then $I = 0$, which is a contradiction.

Corollary 1. *Let R be a 2-torsion-free $\mathfrak{d}$-semiprime ring, U its Lie $\mathfrak{d}$-ideal and V satisfies that $\delta(V) \subseteq V$ for all $\delta \in \mathfrak{d}$ and it is an additive subgroup of R such that $[V, U] \subseteq V$. Then either $[V, U] = 0$ or there exists a $\mathfrak{d}$-ideal M of R such that*

$$0 \neq [M, R] \subseteq V$$

(in particular, in the second case, V contains a non-zero Lie $\mathfrak{d}$-ideal of R).

proof. Clearly that $A = [V, U] \subseteq V \cap U$, where $\delta(A) \subseteq A$ for all $\delta \in \mathfrak{d}$ and $[A, U] \subseteq [V, U]$. Then

$$T = \{x \in R \mid [x, R] \subseteq U\}$$

is a $\mathfrak{d}$-subring of R. Let T_0 be a subring of T generated by $[A, A]$. Then T_0 satisfies that $\delta(T) \subseteq T$ for all $\delta \in \mathfrak{d}$. Inasmuch

$$[[A, A], U] \subseteq [A, A],$$

we have $[T_0, U] \subseteq T_0$. By Lemma 3, $[T_0, U] = 0$ or T_0 contains a non-zero $\mathfrak{d}$-ideal of R.

$a)$ If $[T_0, U] = 0$, then using the fact that $[A, A] \subseteq T_0$ we have

$$[[A, A], U] = 0.$$

Since $[A, A]$ satisfies $\delta([A, A]) \subseteq [A, A]$ for all $\delta \in \mathfrak{d}$, we conclude that for $a \in [A, A]$ we have

$$[\delta_1^{m_1} \cdots \delta_k^{m_k}(a), [a, R]] = 0.$$

By Lemma 5, a will be from $Z(R)$. This means that $[A, A] \subseteq Z(R)$. By Lemma 7, $[A, U] = 0$. Hence by Lemma 3 $A \subseteq Z(R)$ and $[v, u] \in A$ for any $v \in V$ and $u \in U$. Then

$$[\delta_1^{m_1} \cdots \delta_k^{m_k}(v), [v, u]] = 0$$

for any integers $k \geq 1$, $m_i \geq 0$ and derivations $\delta_i \in \mathfrak{d}$ where $i = 1, \ldots, k$, and by Lemma 5,

$$v \in C_R(U)$$

what forces that

$$[V, U] = 0.$$

b) Assume that T_0 contains a non-zero $\mathfrak{d}$-ideal M of R. Then $[M, R] \neq 0$ or $[M, R] = 0$. In the last case

$$MC(R) = 0$$

and $MT_0 = 0$. As a consequence, $M^2 = 0$. By the $\mathfrak{d}$-semiprimeness of R, $M = 0$, which is a contradiction.

Now we extended [5, Theorem 1] in the next proposition

Proposition 2. *Let R be a 2-torsion-free δ-semiprime ring, U its δ-ideal, where $0 \neq \delta \in \mathfrak{D}$. If $\delta^2(U) = 0$, then $\delta(U) \subseteq Z(R)$.*

proof. Let $a, b, u, v \in U$ and $x, r \in R$. From

$$0 = \delta^2([u, v]) = 2[\delta(u), \delta(v)],$$

we deduced U is commutative. Since

$$u[u, r] = u(ur - ru) = u(ur) - (ur)u = [u, ur] \in U$$

it follows that

$$0 = \delta^2(u[u, r]) = 2\delta(u)\delta([u, r])$$

and therefore,

$$\delta(u)\delta([u, r]) = 0.$$

Multiplying

$$[\delta(u), rx] = [\delta(u), r]x + r[\delta(u), x],$$

by $\delta(u)$ on left we get

$$\delta(u)r[\delta(u), x] = 0.$$

Since

$$\delta(u)xr[\delta(u), x] = 0, \;\; x\delta(u)r[\delta(u), x] = 0,$$

we obtain that

$$[\delta(u), x]R[\delta(u), x] = 0.$$

This means that

$$I_{ux} = R[\delta(u), x]R$$

is a nilpotent ideal. Inasmuch

$$I = \sum_{u \in U, x \in R} I_{ux}$$

is a nil δ-ideal, hence we conclude that $I = 0$. This means that $\delta(U) \subseteq Z(R)$.

Now we will investigate the inverse problem and prove the main result.

Theorem 1. *Let R be a 2-torsion-free ring. If R is a $\mathfrak{D}$-semiprime ring then one of the following holds:*

(1) *R is a commutative ring,*

(2) *$\mathfrak{D}$ is a semiprime ring.*

proof. Assume that R is not commutative, then $C(R) \neq 0$. By the $\mathfrak{D}$-semiprimeness of R, $C(R)^2 \neq 0$. Suppose that B is a non-zero ideal of $\mathfrak{D}$, where $[B, B] = 0$. Let $J = B \cap I_{\mathfrak{D}}$ and $x, y, r, t \in R$.

(a) If $J = 0$, then, for any $d \in B$,

$$\partial_{d(x)} = [d, \partial_x] = 0$$

that is $d(x) \in Z(R)$. Then, for any $z \in C_R(x)$, we obtain that

$$d([x, y]) = [d(x), y] + [x, d(y)] = 0,$$

$$d(z)[x, y] = d(z[x, y]) = d([x, zy]) = 0 = d([x, y]z) = [x, y]d(z)$$

and

$$rd(z)t = rtd(z) + r[d(z), t] = rtd(z).$$

Assume that $x \notin Z(R)$. The ideal A_x generated by all $d(z)$, where $d \in \mathfrak{D}$ and $z \in C_R(x)$, is a $\mathfrak{D}$-ideal of R. If $A_x \neq 0$, then, using the non commutativity of a ring R and the definition of the annihilator, we see that $annA_x$ is a non-zero $\mathfrak{D}$-ideal, which is a contradiction. Hence, $d(C_R(x)) = 0$. If $d(Z(R)) = 0$, then $d(R) = 0$ and so $d = 0$. Therefore, we assume that $d(Z(R)) \neq 0$. If $a \in Z(R)$, then $aC_R(x) \subseteq C_R(x)$ and then

$$d(C_R(x)a) = C_R(x)d(a) = 0 = d(aC_R(x)) = d(a)C_R(x)$$

and consequently

$$d(R) \subseteq annC_R(x).$$

In view of Lemma 4, $d(R) \subseteq Z(R)$ for $d \in B$. Let $\mathfrak{D}_B(R)$ by the ideal of R generated by all $d(R)$, where $d \in B$. Then $u \in C_R(u) \subseteq annD_B(R)$ for any $u \in \mathfrak{D}_B(R)$ and so $u \in \mathfrak{D}_B(R) \cap ann\mathfrak{D}_B(R) = 0$. This means that $\mathfrak{D}_B(R) = 0$, which leads to a contradiction.

(b) Assume that $J \neq 0$. Then $I = \{t \in R \mid \partial_t \in J\}$ is a non-zero $\mathfrak{D}$-ideal of R and

$$\partial_{[t_1,t_2]} = [\partial_{t_1}, \partial_{t_2}] \in [J, J] = 0$$

for any $t_i \in J$ and, as a consequence,

$$[I, I] \subseteq Z(R). \tag{9}$$

Let

$$T(I) = \{w \in R \mid [w, R] \subseteq I\}.$$

Then $I \subseteq T(I)$ and $T(B)$ is an associative subring and a Lie ideal of R (see [7, Lemma 3]). Since

$$[I, T(I)] \subseteq I,$$

we deduce that

$$0 = \partial_{t_1}([t_1, t_2^2]) = 2\partial_{t_1}(t_2)^2.$$

From this and the condition in the equation (9) it holds that

$$\partial_{t_1}(t_2) \in \mathbb{P}(R) \cap Z(R).$$

Then

$$\sum_{t_1,t_2 \in I} [t_1, t_2]R$$

is a nil $\mathfrak{D}$-ideal of R, which is a contradiction.

4. Conclusion

Through this work, firstly, we found some properties of Lie ideals in $\mathfrak{D}$-semiprime ring, then we demonstrated when a commutator of a composite derivation for an element and Lie ideal in a $\mathfrak{D}$-semiprime ring equal to zero, implies the element belongs to center of this ring. Also, we investigated the relationship between an element of Lie $\mathfrak{d}$- ideal and the center (the commutator ideal) of a ring. After that, we showed that for any an ideal contained in Lie $\mathfrak{d}$-ideal of a $\mathfrak{d}$-semiprime ring, their commutator must be contained in the ideal itself. Furthermore, we related between the commutator of $\mathfrak{d}$- ideal and every associative subgroup of it under specific conditions. In addition, if any δ-ideal of a δ-semiprime ring satisfies $\delta^2(U) = 0$, this gives $\delta(U)$ is contained in the center of ring. Finally, we proved that every a 2-torsion-free $\mathfrak{D}$-semiprime with an identity ring is either commutative or a $\mathfrak{D}$ is a semiprime ring.

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