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Order Statistics and Actuarial Measures from Powered Inverse Rayleigh Distribution

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Abstract. Nashaat [25] introduced the powered inverse Rayleigh (*PIR*) distribution. It provides a better fit other than (inverse Rayleigh, Rayleigh, and Weibull) distributions. The moments of order statistics and recurrence relations for the single and double moments have been established. The computation of the means and variances are enumerated. These computations can be truly interesting and applied in numerous domains of study. Moreover, cumulative entropy (C.E.) and actuarial measures (A.M.) are also calculated to address the uncertainty in portfolio optimization. The usages of C. E. and A.M. are widespread in many real-word applications specifically in physical sciences and insurance science.

2020 Mathematics Subject Classifications: 60E05, 62E15, 62G30, 62F10

Key Words and Phrases: Order statistics, Entropy, Single and double moments, Recurrence relations, Actuarial measures

1. Introduction

Trayer [30] discussed inverse Rayleigh (IR) distribution. Voda [31] documented its wide ranges of applicability in several areas of applied and allied sciences. Since then, IR distribution is steadily growing and drawing attention by several researchers via different modifications. The basic idea of the new model is to get the more accurate result of complex data. Some notable works are listed below.

Helbaway and Monem [1] and Sindhua et al. [29] estimated the parameters of IR distribution for complete and censored samples using the Bayesian approach. Merrovci [23] presented transmuted IR distribution. Khan [18] introduced modified IR distribution. Khan and King [21] introduced transmuted modified IR distribution. Haq [15] presented transmuted exponentiated IR distribution. Rao and Mbwambo [27] established exponentiated IR distribution. Khan [20] obtained moments properties of *PIR* distribution based on dual generalized order statistics. Mustafa and Khan [24] introduced the length-biased *PIR* distribution. Recently, Khan and Mustafa [19] presented *PIR* distribution using DUS

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transformation.
A random variable ($r.v.$) $X : (\Omega \rightarrow (0, \infty))$ is said to have PIR distribution, if its probability density function (PDF) is given by

$$f(x, \alpha, \theta) = \frac{2\alpha}{\theta x^{2\alpha+1}} e^{-\frac{1}{\theta x^{2\alpha}}}, x > 0, \alpha, \theta > 0 \tag{1}$$

The cumulative distribution function (CDF) of (1) is

$$F(x) = e^{-\frac{1}{\theta x^{2\alpha}}}, x > 0, \alpha, \theta > 0. \tag{2}$$

Hazard rate function:

$$H_{PIRD}(x) = \frac{2\alpha}{\theta x^{2\alpha+1}} \tag{3}$$

Reliability function:

$$S_{PIRD}(x) = 1 - e^{-\frac{1}{\theta x^{2\alpha}}}. \tag{4}$$

The following functional relationship exists from (1)and (2).

$$f(x) = \frac{2\alpha}{\theta x^{2\alpha+1}} F(x). \tag{5}$$

The PIR distribution has a tremendous application in finance, stock market and biological sciences. Note that the PIR distribution involves inverse Rayleigh, Rayleigh, and Weibull distributions as a sub class.
The moments of order statistics (O.S.) have been enumerated quite significantly for some probability models: Joshi [17], David [10], Mohie El-Din et al. [13], and Arnold et al.[4]. Goodness of fit tests, Hegazy et al. [16] and Glen et al. [14]. David and Nagaraja [11] and Arnold et al. [5] have documented and explored the characterization of distribution using O.S.The application of moments of O.S. can be especially noticed in areas including reliability theory and quality control processes.
In this manuscript, moments of O.S. are derived from PIR distribution. The tabulation of descriptive measures based on smallest, largest O.S. and C.E. are tabulated for some fixed parameters in Section 2. Moreover, recurrence relations based on single and double moments are extracted in Section 3. Actuarial measures are reported in Section 4. and Section 5 is reported conclusion.

2. Order Statistics

In quality control processes and reliability theory, the O.S. contributes an important feature in forecasting the time to fail of a specific item by reviewing few early failures, Dey et al [12].
A sequence of $r.v's.$ are arranged in their magnitude of ascending order referred to O.S. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the O.S. Then the PDF of k^{th} O.S. $X_{k:n}$ for $1 \leq k \leq n$ is reported by David and Nagaraja [11].

$$f_k(x) = C_{k:n}[F(x)]^{k-1}[1 - F(x)]^{n-k} f(x), -\infty < x < \infty \tag{6}$$

where

$$C_{k:n} = \frac{n!}{(k-1)!(n-k)!}$$

The PDF of k^{th} O.S. is written as follows.

$$f_k(x) = C_{k:n}\left[e^{-\frac{1}{\theta x^{2\alpha}}}\right]^{k-1}\left[1-e^{-\frac{1}{\theta x^{2\alpha}}}\right]^{n-k}\frac{2\alpha}{\theta x^{2\alpha+1}}e^{-\frac{1}{\theta x^{2\alpha}}}. \tag{7}$$

Using binomial expansion of $\left[1-e^{-\frac{1}{\theta x^{2\alpha}}}\right]^{n-k}$, we have

$$f_k(x) = C_{k:n}\sum_{t=0}^{n-k}\binom{n-k}{t}(-1)^t e^{-\frac{(k+t)}{\theta x^{2\alpha}}}\frac{2\alpha}{\theta x^{2\alpha+1}}. \tag{8}$$

For $k = 1$, we obtain the PDF of k^{th} smallest O.S. as:

$$f_1(x) = n\sum_{t=0}^{n-1}\binom{n-1}{t}(-1)^t e^{-\frac{(t+1)}{\theta x^{2\alpha}}}\frac{2\alpha}{\theta x^{2\alpha+1}}. \tag{9}$$

Similarly, $k = n$, we obtain the PDF of k^{th} largest O.S. as:

$$f_n(x) = ne^{-\frac{n}{\theta x^{2\alpha}}}\frac{2\alpha}{\theta x^{2\alpha+1}}. \tag{10}$$

2.1. Moments of k^{th} O. S.

In this subsection, we derive the moments of O.S. when parent population is PIR distribution.

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be a random sample (R.S.) of size n from PIR distribution and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ mark the corresponding the O.S. Then i^{th} moments of the k^{th} O.S. for $i = 1, 2, \ldots$ denoted by $\mu_{k:n}^{(i)}$ is given by*

$$\mu_{k:n}^{(i)} = C_{k:n}\sum_{t=0}^{n-k}\binom{n-r}{t}(-1)^t(k+t)^{\frac{i}{2\alpha}-1}\left(\frac{1}{\theta}\right)^{\frac{i}{2\alpha}}\gamma\left(1-\frac{i}{2\alpha}\right),\ i = 1, 2, 3, 4. \tag{11}$$

Proof: We know that

$$\mu_{k:n}^{(i)} = \int_{-\infty}^{\infty} x^i f_k(x)dx$$

$$= C_{k:n}\sum_{t=0}^{n-k}\binom{n-k}{t}(-1)^t\int_0^{\infty} x^i e^{-\frac{(k+t)}{\theta x^{2\alpha}}}\frac{2\alpha}{\theta x^{2\alpha+1}}dx. \tag{12}$$

Letting $u = \frac{(k+t)}{\theta x^{2\alpha}}$ in (12), yields (11).

Remark 1. *The k^{th} moments for smallest O.S. from* (11) *is as follows.*

$$\mu_1^{(i)} = n\sum_{t=0}^{n-1}\begin{pmatrix} n-1 \\ t \end{pmatrix}(-1)^t(t+1)^{\frac{i}{2\alpha}-1}\left(\frac{1}{\theta}\right)^{\frac{i}{2\alpha}}\Gamma\left(1-\frac{i}{2\alpha}\right)$$

First and second order moments of k^{th} smallest O.S. can be obtained at $i = 1, 2$.

$$\mu_1^{(1)} = n\sum_{t=0}^{n-1}\begin{pmatrix} n-1 \\ t \end{pmatrix}(-1)^t(t+1)^{\frac{1}{2\alpha}-1}\left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}\Gamma\left(1-\frac{1}{2\alpha}\right)$$

and

$$\mu_1^{(2)} = n\sum_{t=0}^{n-1}\begin{pmatrix} n-1 \\ t \end{pmatrix}(-1)^t(t+1)^{\frac{1}{\alpha}-1}\left(\frac{1}{\theta}\right)^{\frac{1}{\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)$$

Therefore, the variance for k^{th} smallest O.S. can be obtained as follows.

$$Var(X_1) = \mu_1^{(2)} - [\mu_1^{(1)}]^2$$

Table 1: Values of $\mu_{1:n}^{(i)}$ for smallest O.S. when $n = 4$

			$\mu_{1:n}^{(1)}$		
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	0.898	0.782	0.721	0.680	0.651
3.0	0.913	0.814	0.76	0.725	0.698
3.5	0.925	0.837	0.79	0.758	0.735
4.0	0.933	0.856	0.814	0.785	0.763
			$\mu_{1:n}^{(2)}$		
2.5	0.818	0.620	0.527	0.470	0.430
3.0	0.842	0.669	0.584	0.531	0.493
3.5	0.861	0.706	0.629	0.580	0.544
4.0	0.876	0.737	0.666	0.619	0.586
			$\mu_{1:n}^{(3)}$		
2.5	0.756	0.499	0.391	0.329	0.288
3.0	0.785	0.555	0.453	0.393	0.351
3.5	0.808	0.600	0.505	0.446	0.405
4.0	0.827	0.638	0.548	0.492	0.452
			$\mu_{1:n}^{(4)}$		
2.5	0.711	0.408	0.295	0.235	0.196
3.0	0.740	0.466	0.356	0.294	0.253
3.5	0.764	0.514	0.408	0.346	0.305
4.0	0.785	0.555	0.453	0.393	0.351

Table 1 reveals that for fixed value of θ moments are increasing when α is increasing.

Table 2: Values of mean, variance, C.V, skewness, and kurtosis for the smallest O.S. when $n = 4$

	$Mean = \mu_{1:n}^{(1)}$				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	0.898	0.782	0.721	0.680	0.651
3.0	0.913	0.814	0.76	0.725	0.698
3.5	0.925	0.837	0.79	0.758	0.735
4.0	0.933	0.856	0.814	0.785	0.763
	Variance				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	0.0117	0.0089	0.0076	0.0067	0.0062
3.0	0.0083	0.0066	0.0058	0.0052	0.0049
3.5	0.0062	0.0051	0.0046	0.0042	0.0039
4.0	0.0048	0.0041	0.0037	0.0034	0.0032
	C.V.				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	12.057	12.057	12.057	12.057	12.057
3.0	9.995	9.995	9.995	9.995	9.995
3.5	8.536	8.536	8.536	8.536	8.536
4.0	7.450	7.450	7.450	7.450	7.450
	Skewness				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	0.809	0.809	0.809	0.809	0.809
3.0	0.734	0.734	0.734	0.734	0.734
3.5	0.682	0.682	0.682	0.682	0.682
4.0	0.643	0.643	0.643	0.643	0.643
	Kurtosis				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	4.426	4.426	4.426	4.426	4.426
3.0	4.178	4.178	4.178	4.178	4.178
3.5	4.021	4.021	4.021	4.021	4.021
4.0	3.914	3.914	3.914	3.914	3.914

Table 2 exhibits that variances, skewness and kurtosis are decreasing when α is increasing except momentsat fixed value of θ.

Remark 2. *The k^{th} moments for largest O.S. from* (11) *is as follows.*

$$\mu_n^{(i)} = n^{\frac{i}{2\alpha}} \left(\frac{1}{\theta}\right)^{\frac{i}{2\alpha}} \Gamma\left(1 - \frac{i}{2\alpha}\right).$$

Table 3: Values of $\mu_{n:n}^{(i)}$ for largest O. S. when $n = 4$.

α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
			$\mu_{n:n}^{(1)}$		
2.5	1.536	1.337	1.233	1.164	1.113
3.0	1.422	1.267	1.184	1.129	1.088
3.5	1.348	1.221	1.152	1.106	1.071
4.0	1.296	1.188	1.13	1.09	1.06
			$\mu_{n:n}^{2}$		
2.5	2.593	1.965	1.671	1.489	1.362
3.0	2.15	1.706	1.49	1.354	1.257
3.5	1.896	1.555	1.385	1.276	1.197
4.0	1.733	1.457	1.317	1.225	1.159
			$\mu_{n:n}^{3}$		
2.5	5.096	3.362	2.636	2.218	1.94
3.0	3.545	2.507	2.047	1.772	1.585
3.5	2.823	2.098	1.763	1.559	1.416
4.0	2.413	1.86	1.598	1.435	1.319
			$\mu_{n:n}^{4}$		
2.5	13.917	7.993	5.779	4.591	3.84
3.0	6.751	4.253	3.245	2.679	2.309
3.5	4.565	3.072	2.437	2.068	1.820
4.0	3.545	2.507	2.047	1.772	1.585

Table 3 shows that for fixed value of θ moments are decreasing when α is increasing.

Table 4: Values of mean, variance, C.V, skewness, and kurtosis for the largest O. S. when $n = 4$

	$Mean = \mu_{n:n}^{(1)}$				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	1.536	1.337	1.233	1.164	1.113
3.0	1.422	1.267	1.184	1.129	1.088
3.5	1.348	1.221	1.152	1.106	1.071
4.0	1.296	1.188	1.13	1.09	1.06
	Variance				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	0.2329	0.1765	0.1501	0.1338	0.1223
3.0	0.1269	0.1007	0.088	0.08	0.0742
3.5	0.0792	0.0649	0.0578	0.0533	0.05
4.0	0.0538	0.0453	0.0409	0.0381	0.036
	C.V.				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	31.414	31.414	31.414	31.414	31.414
3.0	25.051	25.051	25.051	25.051	25.051
3.5	20.873	20.873	20.873	20.873	20.873
4.0	17.907	17.907	17.907	17.907	17.907
	Skewness				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	3.535	3.535	3.535	3.535	3.535
3.0	2.806	2.806	2.806	2.806	2.806
3.5	2.425	2.425	2.425	2.425	2.425
4.0	2.189	2.189	2.189	2.189	2.189
	Kurtosis				
α	$\theta = 1$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$
2.5	48.092	48.092	48.092	48.092	48.092
3.0	24.678	24.678	24.678	24.678	24.678
3.5	17.534	17.534	17.534	17.534	17.534
4.0	14.166	14.166	14.166	14.166	14.166

The behavior of Table 4 is that descriptive measures are decreasing when α is increasing at the fixed value of θ.

2.2. The Joint *PDF* of k^{th} and l^{th} O.S.

The joint PDF of $X_{k:n}$ and $X_{l:n}$ is given by (Arnold et al. [4]) for $1 \leq k \leq l \leq n$.

$$f_{k,l}(x, y) = C_{k,l:n}[F(x)]^{k-1}[F(y) - F(x)]^{l-k-1}[1 - F(y)]^{n-l} f(x)f(y), -\infty < x < y < \infty. \tag{13}$$

Equation (13) can be re-written using binomial expansion.

$$f_{k,l}(x,y) = C_{k,l:n} \sum_{t=0}^{n-l} \sum_{z=0}^{l-k-1} \binom{n-s}{t} \binom{l-k-1}{z} (-1)^{t+z} [F(x)]^{k+z} [F(y)]^{l-k+t-z} \times \frac{f(x)}{F(x)} \frac{f(y)}{F(y)}.$$

Therefore, the joint PDF of $X_{k,l:n}$ from PIR distribution.

$$f_{k,l}(x,y) = \frac{4\alpha^2}{\theta^2} C_{k,l:n} \sum_{t=0}^{n-l} \sum_{z=0}^{l-k-1} \binom{n-l}{t} \binom{l-k-1}{z} (-1)^{t+z} e^{-\left(\frac{k+t}{\theta x^{2\alpha}} + \frac{l-k+t-z}{\theta y^{2\alpha}}\right)} \times \frac{1}{x^{2\alpha+1}} . \frac{1}{y^{2\alpha+1}}.$$

2.3. Cumulative Entropy

There are several types of entropies that exist in literature. Each one is employed for a specific situation. The cumulative entropy (C.E.) is the most prominent version of the entropy reported by Crescenzo and Longobardi [9] in (14).

$$CE(X) = -\int_0^\infty F(x) \ln F(x) dx \tag{14}$$

The C.E. for (1) is.

$$CE(X) = \frac{1}{2\alpha} \left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}} \Gamma\left(1 - \frac{1}{2\alpha}\right)$$

Table 5: The nature of C.E. for PIR distribution.

	θ									
α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
1.0	2.22	1.57	1.29	1.11	0.99	0.91	0.85	0.79	0.74	0.71
1.5	0.567	0.45	0.40	0.36	0.33	0.31	0.30	0.28	0.27	0.26
2.0	0.369	0.31	0.28	0.26	0.25	0.24	0.23	0.22	0.21	0.21
2.5	0.265	0.23	0.212	0.20	0.19	0.18	0.18	0.17	0.17	0.17
3.0	0.213	0.19	0.18	0.17	0.16	0.16	0.15	0.15	0.15	0.14

The value of C.E. is decreasing when α is increasing for the fixed value of θ.

3. Recurrence Relations based on O.S.

The recurrence relations based on O.S. and its applications have been well documented via Balakrishnan and Malik [7], Balakrishnan et al. [8], Arnold and Balakrishnan [3], Ali and Khan [2] and Samual and Thomas [28] in detail. To calculate the moments of O.S. is a tedious task for some distribution. For this fact, recursive computation approaches are repeatedly desired.

Theorem 2. *As stated in Theorem 1, we have the following single moments relation:*

$$\mu_{k:n}^{(i)} = \frac{(n-k+1)}{\theta(i-2\alpha)} \left[\mu_{k:n}^{(i-2\alpha)} - \mu_{k-1:n}^{(i-2\alpha)}\right] \tag{15}$$

Proof: We know that

$$\mu_{k:n}^{(i)} = \int_{-\infty}^{\infty} x^i f_k(x)dx$$

$$\mu_{k:n}^{(i)} = C_{k:n} \int_0^{\infty} x^i [F(x)]^{k-1} [1-F(x)]^{n-k} f(x)dx. \tag{16}$$

Using (5) in (16), we have

$$\mu_{k:n}^{(i)} = C_{k:n} \frac{1}{\theta} \int_0^{\infty} x^{i-2\alpha} [F(x)]^{k-1} [1-F(x)]^{n-k+1} dx. \tag{17}$$

Integrating (17) by parts and simplifying yields (15).

Theorem 3. *For $1 \leq k \leq l \leq n, n \in N$, we have the following product moment relations.*

$$\mu_{k,l:n}^{(i_1,i_2)} = \frac{(n-l+1)}{\theta(i_2-2\alpha)} \left[\mu_{k,l:n}^{(i_1,i_2-2\alpha)} - \mu_{k,l-1:n}^{(i_1,i_2-2\alpha)}\right] \tag{18}$$

Proof: We start from (13),

$$\mu_{k,l:n}^{(i_1,i_2)} = C_{k,l:n} \int_0^{\infty} \int_x^{\infty} x^{i_1} y^{i_2} f_{k,l}(x,y)dydx \tag{19}$$

or,

$$\mu_{k,l:n}^{(i_1,i_2)} = C_{k,l:n} \int_0^{\infty} x^{i_1} [F(x)]^{k-1} f(x) W_x dx \tag{20}$$

where

$$W_x = \int_x^{\infty} y^{i_2} [F(y)-F(x)]^{l-k-1} [1-F(y)]^{n-l} f(y)dy$$

or,

$$W_x = \frac{1}{\theta} \int_x^{\infty} y^{i_2-2\alpha} [F(y)-F(x)]^{l-k-1} [1-F(y)]^{n-l+1} dy.$$

Now integrating the above equation by parts, we get,

$$W_x = \frac{1}{\theta} \left\{ \frac{(n-l+1)}{(i_2-2\alpha)} \int_x^{\infty} y^{i_2-2\alpha} [F(y)-F(x)]^{l-k-1} [1-F(y)]^{n-l} f(y)dy \right.$$
$$\left. - \frac{l-k-1}{i_2-2\alpha} \int_x^{\infty} y^{i_2-2\alpha} [F(y)-F(x)]^{l-k-2} [1-F(y)]^{n-l+1} f(y)dy \right\}.$$

Putting the values of W_x in (20), directly yields (18).

4. Actuarial Measures (A.M.)

The A.M. plays a leading role in insurance science via uncertainty. Due to its usefulness in portfolio optimization, the interested readers refer to Panjer [26], Artzrner [6] and Landsman [22]. We derive some important risks as follows.

4.1. Value at Risk:

It is represented with a confidence level q (typically 90%, 95% or 99%). The VaR of $R.V.$ X is the q^{th} quantile of (2).

$$VaR_q(x) = F^{-1}(q). \tag{21}$$

Therefore, the $VaR_q(x)$ of PIR distribution is given by.

$$X_q = [-\theta ln\ (q)\ ]^{-\frac{1}{2\alpha}}. \tag{22}$$

The value of VaR is increasing at the different level of q for fixed values of α and θ.

4.2. Tail value at Risk:

The expected value of the loss, which is greater than the VaR is called Tail value at Risk ($TVaR$).
By definition

$$TVaR_q(x) = \frac{1}{1-q}\int_{VaR_q}^{\infty} xf(x)dx. \tag{23}$$

Therefore, the $TVaR_q(x)$ from (23) is.

$$TVaR_q(x) = \frac{(\theta)^{-\frac{1}{2\alpha}}}{1-q}\gamma\left(1-\frac{1}{2\alpha}, \frac{1}{\theta(VaR_q)^{2\alpha}}\right). \tag{24}$$

4.3. Tail Variance:

The variability of the risk along the tail of distribution is known as Tail Variance (TV). It is determined as.

$$TV_q(x) = E\left[X^2|X > x_q\right] - [TVaR_q]^2. \tag{25}$$

Therefore, the *TV(X)* of the *PIR* distribution is addressed in (26)

$$TV_q(x) = \frac{(\theta)^{-\frac{1}{\alpha}}}{1-q}\gamma\left(1-\frac{1}{\alpha}, \frac{1}{\theta(VaR_q)^{2\alpha}}\right) - \left[\frac{(\theta)^{-\frac{1}{2\alpha}}}{1-q}\gamma\left(1-\frac{1}{2\alpha}, \frac{1}{\theta(VaR_q)^{2\alpha}}\right)\right]^2 \tag{26}$$

where

$$E\left[X^2|X > x_q\right] = \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f\,(x)\,dx = \frac{(\theta)^{-\frac{1}{\alpha}}}{1-q}\gamma\left(1-\frac{1}{\alpha}, \frac{1}{\theta(VaR_q)^{2\alpha}}\right)$$

Table 6: $VaR_q(x)$ at $\theta = 1,\ 2, and\ 3$ at different level of q.

$\alpha\backslash q$	75%	80%	85%	90%	95%	99%
			$\theta = 1$			
0.5	3.45	4.55	6.25	9.09	20.0	100
1.0	1.86	2.13	2.50	3.02	4.47	10.0
1.5	1.51	1.66	1.84	2.09	2.71	4.64
2.0	1.36	1.46	1.58	1.74	2.11	3.16
2.5	1.28	1.35	1.44	1.55	1.82	2.51
3.0	1.23	1.28	1.36	1.44	1.65	2.15
			$\theta = 2$			
$\alpha\backslash q$	75%	80%	85%	90%	95%	99%
0.5	1.72	2.22	3.03	4.76	10.0	50.0
1.0	1.31	1.49	1.74	2.18	3.16	7.07
1.5	1.20	1.30	1.45	1.68	2.15	3.68
2.0	1.15	1.22	1.32	1.48	1.78	2.66
2.5	1.12	1.17	1.25	1.37	1.58	2.19
3.0	1.10	1.14	1.20	1.29	1.47	1.92
			$\theta = 3$			
$\alpha\backslash q$	75%	80%	85%	90%	95%	99%
0.5	1.16	1.49	2.04	3.13	6.67	33.3
1.0	1.08	1.22	1.43	1.77	2.58	5.78
1.5	1.05	1.14	1.27	1.46	1.88	3.22
2.0	1.04	1.11	1.20	1.32	1.61	2.40
2.5	1.03	1.08	1.15	1.26	1.46	2.02
3.0	1.02	1.07	1.13	1.21	1.37	1.79

4.4. Total Variance Premium:

The combination of TV_q and $TVaR_q$ is called the Total Variance Premium *(TVP)*. It is definedas follows.

$$TVP_q(X) = TVaR_q + \delta TV_q \tag{27}$$

where $0 < \delta < 1$.
Substituting the expressions (24) and (26) into (27) completes the proof.

5. Conclusion

The current study explores the moments of the O.S. from the *PIR* distribution. The numerical computations are reported based on the O.S. Cumulative entropy is evaluated. The expressions for single and double moments are setup. The actuarial measures are also tabulated.

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