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Differentiating Odd Dominating Sets in Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple and undirected graph. A dominating set $S \subseteq V(G)$ is called a *differentiating odd dominating set* if for every vertex $v \in V(G)$, $|N[v] \cap S| \equiv 1(mod\ 2)$ and $N_G[u] \cap S \neq N_G[v] \cap S$ for every two distinct vertices u and v in $V(G)$. The minimum cardinality of a differentiating odd dominating set of G, denoted by $\gamma_D^o(G)$, is called the *differentiating odd domination number*. In this paper, we discuss differentiating odd dominating sets and give bounds or exact values of the differentiating odd domination numbers of some graphs. We give necessary and sufficient conditions for some graphs to admit a differentiating odd dominating set. Moreover, we characterize the differentiating odd dominating sets in graphs resulting from join, corona, and lexicographic product of some graphs and determine the differentiating odd domination numbers of these graphs.

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1. Introduction

Domination is one of the most explored areas in Graph Theory. Indeed, numerous variations of domination have been introduced and investigated from various perspectives and approaches (see [1], [2], [5], [7], [10], [13], [14], and [18]). One prominent area of research in this domain is the investigation of differentiating-dominating sets in graphs, which are alternatively referred to as identifying codes in certain contexts. This research has roots dating back to 1998 when Karpovsky, Chakrabarty, and Levitin introduced identifying codes (see [4]) and have been investigated further by Frick et al in 2008 (see [6]). Furthermore, in the study of Canoy and Malacas [12], they characterized the differentiating-dominating sets in the join, corona, and lexicographic product of

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graphs and determined the bounds or the exact differentiating-domination numbers of the aforementioned graphs. Other studies related to the topic can be found in [8], [9], [11], [15], [16], and [17].

In 1989, Sutner introduced the concept of odd dominating set under the name "odd-parity cover" (see [19]). Specifically, he showed that every graph contains odd dominating set in the context of cellular automata (see [19]). However, this parameter has been studied very little. Previous studies on parity domination mainly focused on algorithmic problems and even dominating sets [3]. In this paper, we introduce the concept of differentiating odd dominating sets in graphs. Note that the concept of differentiating-dominating set may be used to model problems which involve protection in a given network where the goal is to specifically determine the exact location of an intruder (e.g. burglar or fire). When used in this case as a protection strategy, an element of a differentiating-dominating set may refer to a monitoring device or location (vertex) where a monitoring device is positioned or placed. When, in addition, the number of these locations or monitors adjacent to a location (with or with no monitoring device) is required to be odd for every location, then the concept of odd dominating set is also imposed.

2. Terminologies and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The open neighborhood of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. Vertex v is a *leaf* if $deg_G(v) = 1$ and the vertex $u \in (V(G) \cap N_G(v))$ is called a *support vertex*. $\mathcal{L}(G)$ and $\mathcal{S}(G)$ denote the sets consisting of all leaves and support vertices in G, respectively. A graph G of order $n \geq 3$ is *point distinguishing* if for any two distinct vertices u and v of G, $N_G[u] \neq N_G[v]$. It is *totally point determining* if for any two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$.

A set $S \subseteq V(G)$ is a *dominating set* (respectively, *total dominating set)* in G if $N_G[S] = V(G)$ (respectively, $N_G(S) = V(G)$). The smallest cardinality of a dominating set in G, denoted by $\gamma(G)$, is called the *domination number* in G. A dominating set in G with cardinality $\gamma(G)$ is called a γ-set of G. A set of vertices S is called an *odd dominating set* (respectively, *even dominating set*) if for every vertex $v \in V(G)$, $|N_G[v] \cap S| \equiv 1 (mod\, 2)$ (respectively, $|N_G[v] \cap S| \equiv 0 (mod\, 2)$). The minimum cardinality of an odd dominating set is called the *odd domination number* in G (respectively, *even domination number*), denoted by $\gamma_{odd}(G)$ (respectively, $\gamma_{even}(G)$). Any odd dominating set with cardinality $\gamma_{odd}(G)$ is called a γ_{odd}-set.

A set $S \subseteq V(G)$ is a *differentiating set* in a graph G if for every two distinct vertices u and v in G, $N_G[u] \cap S \neq N_G[v] \cap S$. It is a *strictly differentiating set* if it is differentiating and $N_G[u] \cap S \neq S$ for all $u \in V(G)$. A differentiating (respectively, strictly differentiating) subset S of $V(G)$ which is also dominating is called a *differentiating-dominating* (respectively, *strictly differentiating-dominating*) set in a graph G. The minimum cardinality of a differentiating-dominating (respectively, strictly differentiating-dominating) set in G, denoted by $\gamma_D(G)$ (respectively, $\gamma_{SD}(G)$), is called the *differentiating-domination* (respectively, *strictly differentiating-domination*) *number* in G. Any differentiating-dominating (respectively, strictly differentiating-dominating) set with cardinality $\gamma_D(G)$ (respectively, $\gamma_{SD}(G)$) is called a γ_D-set (respectively, γ_{SD}-set). A set $S \subseteq V(G)$ is a *differentiating odd dominating* set (respectively, *differentiating even dominating set*) if it is both differentiating and odd dominating (respectively, both differentiating and even dominating). A set $S \subseteq V(G)$ is a *strictly differentiating odd dominating set* (respectively, *strictly differentiating even dominating set*) if it is both strictly differentiating and odd dominating (respectively, both strictly differentiating and even dominating). The sets $DOD(G)$ and $DED(G)$ is the set of all differentiating odd dominating sets and the set of all differentiating even dominating sets, respectively, in G. The sets $SDOD(G)$ and $SDED(G)$ is the set of all strictly differentiating odd dominating sets and the set of all strictly differentiating even dominating sets, respectively, in G. The minimum cardinality of a differentiating odd dominating (respectively, differentiating even dominating) set in G, denoted by $\gamma_D^o(G)$ (respectively, $\gamma_D^e(G)$), is called the *differentiating odd domination number* (respectively, *differentiating even domination number* in G. The minimum cardinality of a strictly differentiating odd dominating (respectively, strictly differentiating even dominating) set in G, denoted by $\gamma_{SD}^o(G)$ (respectively, $\gamma_{SD}^e(G)$), is called the *strictly differentiating odd domination number* (respectively, *strictly differentiating even domination number*) in G. Any differentiating odd dominating (respectively, strictly differentiating odd dominating) set with cardinality $\gamma_D^o(G)$ (respectively, $\gamma_{SD}^o(G)$) is called a γ_D^o-set (respectively, γ_{SD}^o-set).

3. Results

Remark 1. *Every differentiating odd dominating set in a connected graph G is an odd dominating set.*

Remark 2. *Every differentiating odd dominating set in a connected graph G is a differentiating-dominating set.*

Theorem 1. *Let G be a graph. Then G admits a differentiating-dominating set if and only if it is point distinguishing.*

Proof. Suppose G admits a differentiating set, say S. Suppose G is not point distinguishing. Then there exist distinct vertices $a, b \in V(G)$ such that $N_G[a] = N_G[b]$. This implies that $N_G[a] \cap S = N_G[b] \cap S$, a contradiction. Thus, G is point distinguishing.

For the converse, suppose that G is point distinguishing. Then $S = V(G)$ is a differentiating-dominating set, showing that G has a differentiating-dominating set. □

Lemma 1. *Let G be a connected graph of order m and let S be a differentiating odd dominating set in G. Then $m \leq 2^{|S|-1}$. In particular, $m \leq 2^{\gamma_D^o(G)-1}$, i.e., $\gamma_D^o(G) \geq \frac{ln(m)+ln(2)}{ln(2)}$.*

Proof. Let S be a differentiating odd dominating set in G and let $k = |S|$. Let $\mathcal{D} = \{Q \subseteq S : |Q| \text{ is odd}\}$. Then $|\mathcal{D}| = 2^{k-1}$. Since S is differentiating odd dominating, $m \leq 2^{k-1}$. If S is a γ_D^o-set, then $m \leq 2^{\gamma_D^o(G)-1}$. This proves the assertion. □

Theorem 2. *Let G be a point distinguishing connected graph.*

(i) If G has a support vertex v with $|N_G(v)| = 2$, then G does not admit a differentiating odd dominating set.

(ii) If S is a differentiating odd dominating set in G and $v \in V(G)$ with $|N_G(v) \cap \mathcal{L}(G)| \geq 2$, then $N_G(v) \cap \mathcal{L}(G) \subseteq S$ and $v \notin S$.

Proof. Let $w \in \mathcal{L}(G) \cap N_G(v)$ and let $z \in N_G(v) \setminus \{w\}$. Suppose G has a differentiating odd dominating set S. If $w \in S$, then $v \notin S$ because S is an odd dominating set. Since S is a differentiating set, $N_G[w] \cap S = \{w\} \neq N_G[v] \cap S$. This forces $z \in S$. However, the assumption would imply that $N_G[v] \cap S = \{w, z\}$, contradicting the fact that S is an odd dominating set. Thus, $w \notin S$. Consequently, $v \in S$. Since S is odd dominating, $z \notin S$. It follows that $N_G[w] \cap S = \{v\} = N_G[v] \cap S$, contrary to the assumption that S is a differentiating set. Therefore, G has no differentiating odd dominating set, showing that (i) holds.

Next, suppose that S is a differentiating odd dominating set and $v \in V(G)$ with $|N_G(v) \cap \mathcal{L}(G)| \geq 2$. Suppose $v \in S$. Since S is odd dominating, $(N_G(v) \cap \mathcal{L}(G)) \cap S = \varnothing$. Let $x, y \in N_G(v) \cap \mathcal{L}(G)$ where $x \neq y$. Then $N_G[x] \cap S = N_G[y] \cap S = \{v\}$, contrary to the assumption that S is a differentiating set. Therefore, $v \notin S$. Since S is a dominating set, $N_G(v) \cap \mathcal{L}(G) \subseteq S$. Thus, (ii) holds. □

The next results follow from the preceding ones.

Corollary 1. *For $n \geq 2$, K_n and P_n do not admit a differentiating odd dominating set.*

Corollary 2. *Let $S_n = K_{1,n}$ be a star of order $n+1$ where $n \geq 3$. Then S_n admits a differentiating odd dominating set if and only if n is odd. Moreover, if n is odd, then $S = V(S_n) \setminus \{v_0\}$ where $deg_G(v_0) = n$, is the only differentiating odd dominating set in S_n. In particular, $\gamma_D^o(S_n) = n$.*

Proof. Let $V(S_n) = \{v_0, v_1, \cdots, v_n\}$, where $deg_G(v_0) = n$. Suppose S_n admits a differentiating odd dominating set, say S. By Theorem 2(ii), $S = V(S_n) \setminus \{v_0\}$. Since S is odd dominating, $|N_{S_n}[v] \cap S| = |S| = n$ is odd.

For the converse, suppose that n is odd. Then $S = V(S_n) \setminus \{v_0\}$ is a differentiating odd dominating set in S_n.

Note that if n is odd, then $S = V(S_n) \setminus \{v_0\}$ is the only differentiating odd dominating set in S_n by Theorem 2(ii). Hence, $\gamma_D^o(S_n) = n$. □

Corollary 3. *Let G be any non-trivial connected graph of order n and let m be a positive odd integer with $m \geq 3$. Then there exists a connected graph H obtained from G such that $\gamma_D^o(H) = mn$. Moreover, if every vertex in G has even degree, then $\gamma_{odd}(H) = n$. In particular, the difference $\gamma_D^o(G) - \gamma_{odd}(G)$ can be made arbitrarily large.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ and let $V(\overline{K}_m) = \{x_1, x_2, \cdots, x_m\}$. Let H be the graph obtained from G by adding the edges $v_i x_j$ for each $i \in \{1, 2, \cdots, n\}$ and for each $j \in \{1, 2, \cdots, m\}$. By Theorem 2($ii$), $S = V(H) \setminus V(G)$ is a γ_D^o-set in H. Thus, $\gamma_D^o(H) = mn$. Cleary, $\gamma_{odd}(H) = n$.

Suppose now that $|N_G(v)|$ is even for every $v \in V(G)$. Since $V(G)$ is a minimum dominating set in H and $|N_H[x] \cap V(G)|$ is odd for every $x \in V(H)$, $V(G)$ is a γ_{odd}-set in H. Thus, $\gamma_{odd}(H) = n$. □

Suppose $\gamma_D^o(G) = 1$, say $S = \{v\}$ is a γ_D^o-set in G. If there exists $w \in V(G) \setminus \{v\}$, then $N_G[v] \cap S = N_G[w] \cap S = \{v\}$, contrary to the fact that S is a differentiating set. Thus, $G = K_1$. We state this formally.

Remark 3. *Let G be a graph. Then $\gamma_D^o(G) = 1$ if and only if $G = K_1$.*

Remark 4. *There exists no connected graph G with $\gamma_D^o(G) = 2$.*

To see this, suppose that such a connected graph G exists. Then $|V(G)| = 2$ according to Lemma 1. Hence, $G = K_2$. This, however, is not possible by Corollary 1.

Theorem 3. *Let G be a connected graph of order $n \geq 4$. If G admits a differentiating odd dominating set, then $\max\{\gamma_D(G), \gamma_{odd}(G), 3\} \leq \gamma_D^o(G) \leq n - |\mathcal{S}(G)|$. Moreover, $\gamma_D^o(G) = 3$ if and only if $G = K_{1,3}$.*

Proof. By Remarks 1, 2, 3, and 4, $\max\{\gamma_D(G), \gamma_{odd}(G), 3\} \leq \gamma_D^o(G)$. Next, let S be a γ_D^o-set in G. Let $v \in \mathcal{S}(G)$ and let $x_v \in \mathcal{L}(G) \cap N_G(v)$ be fixed. Since S is an odd dominating set, $v \in S$ or $x_v \in S$ but not both. Let

$$D_G = \{w \in V(G) \setminus S : w = v \in \mathcal{S}(G) \text{ or } w = x_v\}.$$

Then $|D_G| = |\mathcal{S}(G)|$ and $S \subseteq V(G) \setminus D_G$. It follows that $\gamma_D^o(G) = |S| \leq n - |\mathcal{S}(G)|$.

For the second part, suppose that $\gamma_D^o(G) = 3$. From Lemma 1, it follows that $n = 4$. It can easily be verified that among the connected graphs of order 4, only $K_{1,3}$ satisfies the given property. Thus, $G = K_{1,3}$.

The converse is easy. □

Let $S_p = K_{1,p}$ and $S_q = K_{1,q}$ be stars with central vertices (support vertices) v_0 and w_0, respectively. Then the *double star* $S_{p,q}$ is the graph obtained from S_p and S_q by adding the edge $v_0 w_0$.

Corollary 4. *Let T_n be a tree of $n \geq 4$. If T_n has a differentiating odd dominating set, then $\gamma_D^o(T_n) \leq n - |\mathcal{S}(T_n)|$ with equality holding if $|N_{T_n}(v) \cap \mathcal{L}(T_n)|$ is odd and at least 3 for every $v \in \mathcal{S}(T_n)$. In particular, if $T_n = S_{p,q}$ (a double star), where $p \geq 3$ and $q \geq 3$ and are odd, then $\gamma_D^o(T_n) = n - 2 = p + q$.*

Proof. Suppose S is a γ_D^o-set in T_n. By Theorem 3, $\gamma_D^o(T_n) \leq n - |\mathcal{S}(T_n)|$. Next, suppose that $|N_{T_n}(v) \cap \mathcal{L}(T_n)|$ is odd and at least 3 for every $v \in \mathcal{S}(T_n)$. By theorem 2(ii), it follows that $N_{T_n}(v) \cap \mathcal{L}(T_n) \subseteq S$ and $v \notin S$ for every $v \in \mathcal{S}(T_n)$. Therefore, $S = N_{T_n}(v) \cap \mathcal{L}(T_n)$ and $\gamma_D^o(T_n) = n - |\mathcal{S}(T_n)|$. From this, it follows that $\gamma_D^o(T_n) = p + q$ when $T_n = S_{p,q}$. □

Theorem 4. *$\gamma_D^o(C_n) = n$ for $n \geq 4$.*

Proof. Let $C_n = [v_1, v_2, ..., v_n, v_1]$ and let S be a γ_D^o-set in C_n. Suppose $S \neq V(C_n)$. Then there exists $v \in V(C_n) \setminus S$. Without loss of generality, we may assume that $v = v_1$. Since S is odd dominating, $v_2 \in S$ or $v_n \in S$ but not both. Assume that $v_2 \in S$. Then $v_n \notin S$. Since $|N_{C_n}[v_2] \cap S|$ must be odd, $v_3 \notin S$. This implies that $N_{C_n}[v_1] \cap S = N_{C_n}[v_2] \cap S = \{v_2\}$, contrary to the assumption that S is a differentiating set. Therefore, $S = V(C_n)$ and $\gamma_D^o(C_n) = n$. □

Theorem 5. *[6] Let C_n be the cycle on n vertices. For $n \geq 3$, $\gamma_D(C_{2n}) = n$.*

Corollary 5. *Let n be a positive integer such that $n \geq 3$. Then there exists a connected graph G such that $\gamma_D^o(G) - \gamma_D(G) = n$. In other words, the difference $\gamma_D^o - \gamma_D$ can be made arbitrarily large.*

Proof. Let $G = C_{2n}$. By Theorem 5, $\gamma_D(C_{2n}) = n$ and by Theorem 4, $\gamma_D^o(C_{2n}) = 2n$. Therefore, $\gamma_D^o(G) - \gamma_D(G) = n$. □

Theorem 6. *Let $G = K_{n_1,n_2,...,n_k}$ be the complete k-partite graph with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, where $k \geq 2$. Then G admits a differentiating odd dominating set if and only if $\sum_{j \neq t} n_j$ is even for every $t \in \{1, 2, \ldots, k\}$. Moreover, in this case, $\gamma_D^o(G) = \sum_{j=1}^k n_j$.*

Proof. Let $S_{n_1}, S_{n_2}, \ldots, S_{n_k}$ be the partite sets in G. Suppose G admits a differentiating odd dominating set S. Let $j \in \{1, 2, \ldots, k\}$ and let $v \in S_{n_j}$. Suppose $v \notin S$. Note that since S is odd dominating, $|N_G[v] \cap S| = |N_G(v) \cap S|$ is odd. Pick any $w \in S_{n_j} \setminus \{v\}$. Since S is differentiating and $N_G(w) \cap S = N_G(v) \cap S$, it follows that $w \in S$ and $N_G[w] \cap S = \{w\} \cup (N_G(v) \cap S)$. Since $|N_G(v) \cap S|$ is odd, $|N_G[w] \cap S|$ is even, contrary to the assumption that S is an odd dominating set. Therefore, $S_{n_j} \subseteq S$ for each $j \in \{1, 2, \ldots, k\}$, i.e., $S = V(G)$. Now, let $t \in \{1, 2, \ldots, k\}$ and let $a \in S_{n_t}$. Then $N_G[a] \cap S = N_G[a] = \{a\} \cup (\cup_{j \neq t} S_{n_j})$. Since S is odd dominating, $|N_G[a]| = 1 + \sum_{j \neq t} |S_{n_j}| = 1 + \sum_{j \neq t} n_j$ is odd. This implies that $\sum_{j \neq t} n_j$ is even.

For the converse, suppose that $\sum_{j \neq t} n_j$ is even for every $t \in \{1, 2, \ldots, k\}$. Let $D = V(G)$ and let $x, y \in V(G)$ with $x \neq y$. Suppose first that $x, y \in S_r$ for some $r \in \{1, 2, \ldots, k\}$. Since $y \notin N_G[x]$, $N_G[x] \cap D = N_G[x] \neq N_G[y] = N_G[y] \cap D$. Next, suppose that $x \in S_p$ and $y \in S_q$ for $p \neq q$, where $p, q \in \{1, 2, \ldots, k\}$. Since $V(S_q) \setminus \{y\} \subseteq N_G[x] \setminus N_G[y]$, $N_G[x] \cap D = N_G[x] \neq N_G[y] = N_G[y] \cap D$. Hence, D is a differentiating set. Next, let $w \in V(G)$ and let $w \in S_t$. Then, by assumption,

$|N_G[w] \cap D| = |N_G[w]| = 1 + \sum_{j \neq t} n_j$ is odd. Therefore, $D = V(G)$ is a differentiating odd dominating set in G.

Whenever the given property is satisfied, we find that $S = V(G)$ is the only differentiating odd dominating set in G. Thus, $\gamma_D^o(G) = |V(G)| = \sum_{j=1}^{k} n_j$. □

The next result is immediate from Theorem 6.

Corollary 6. *Let $K_{m,n}$ be a complete bipartite graph such that $m \geq 2$ and $n \geq 2$. Then $K_{m,n}$ admits a differentiating odd dominating set if and only if m and n are both even. Moreover, $\gamma_D^o(K_{m,n}) = m + n$.*

Theorem 7. *Let $G_1, G_2, \cdots, G_k$ be the components of G. Then G admits a differentiating odd dominating set if and only if G_j admits a differentiating odd dominating set for each $j \in \{1, 2, \ldots, k\}$. In this case,*

$$\gamma_D^o(G) = \sum_{j=1}^{k} \gamma_D^o(G_j).$$

Proof. Suppose G admits a differentiating odd dominating set, say S. Let $S_j = S \cap V(G_j)$ for each $j \in \{1, 2, \ldots, k\}$. Since S is dominating, S_j is dominating in G_j for each $j \in \{1, 2, \ldots, k\}$. Next, let $j \in \{1, 2, \ldots, k\}$ and let $u, v, w \in V(G_j)$, where $u \neq v$. Since S is differentiating odd dominating,

$$N_{G_j}[u] \cap S_j = N_G[u] \cap S \neq N_G[v] \cap S = N_{G_j}[v] \cap S_j$$

and $|N_{G_j}[w] \cap S_j|$ is odd. This implies that S_j is a differentiating odd dominating set in G_j.

For the converse, suppose that each component G_j admits a differentiating odd dominating set, say D_j. Then, clearly, $S' = \cup_{j=1}^{k} D_j$ is a differentiating odd dominating set in G.

Now, let S_0 be a γ_D^o-set in G. Then $S_j' = S_0 \cap V(G_j)$ is a differentiating odd dominating set in G_j for each $j \in \{1, 2, \ldots, k\}$ and $S_0 = \cup_{j=1}^{k} S_j'$. Hence,

$$\gamma_D^o(G) = |S_0| = \sum_{j=1}^{k} |S_j'| \geq \sum_{j=1}^{k} \gamma_D^o(G_j).$$

On the other hand, if D_j' is a γ_D^o-set in G_j for each $j \in \{1, 2, \ldots, k\}$, then $S_0' = \cup_{j=1}^{k} D_j'$ is a differentiating odd dominating set in G. It follows that

$$\gamma_D^o(G) \leq |S_0'| = \sum_{j=1}^{k} |D_j'| = \sum_{j=1}^{k} \gamma_D^o(G_j).$$

This proves the desired equality. □

Corollary 7. *Let G be a graph. Then each of the following holds:*

(*i*) *If* $G = \overline{K}_n$, *then* $\gamma_D^o(G) = n$.

(*ii*) $\gamma_D^o(G) = 2$ *if and only if* $G = \overline{K}_2$.

(*iii*) $\gamma_D^o(G) = 3$ *if and only if* $G \in \{\overline{K}_3, K_{1,3}\}$.

The ***join*** $G+H$ of two graphs G and H is the graph with $V(G+H) = V(G) \cup V(H)$ (disjoint union) and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 8. *Let G be a non-trivial point distinguishing graph and let $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a differentiating odd dominating set in $K_1 + G$ if and only if one of the following holds:*

(*i*) $S = \{v\} \cup S_G$ *where* $S_G = V(G) \cap S \neq \varnothing$ *satisfies the following:*

  (*a*) $|S_G|$ *is even and* S_G *is a strictly differentiating set in* G

  (*b*) $|N_G[u] \cap S_G|$ *is even for all* $u \in V(G)$

(*ii*) $S \subseteq V(G)$, $|S|$ *is odd, and* S *is a strictly differentiating odd dominating set in* G.

Proof. Let S be a differentiating odd dominating set in $K_1 + G$. Let $V(K_1) = \{v\}$ and let $S_G = V(G) \cap S$. Consider the following cases:

Case 1: $v \in S$.
Then $S = \{v\} \cup S_G$. Since S is differentiating in $K_1 + G$, $S_G \neq \varnothing$. Since S is odd dominating in $K_1 + G$, $|N_{K_1+G}[v] \cap S| = |\{v\}| + |S_G|$ is odd. It follows that $|S_G|$ is even. Suppose S_G is not differentiating in G. Then there exist $x, y \in V(G)$ and $x \neq y$ such that $N_G[x] \cap S_G = N_G[y] \cap S_G$. It follows that

$$N_{K_1+G}[x] \cap S = \{v\} \cup (N_G[x] \cap S_G) = N_{K_1+G}[y] \cap S,$$

a contradiction to the fact that S is differentiating in $K_1 + G$. Therefore, S_G is differentiating in G. Furthermore, suppose S_G is not strictly differentiating in G. Then there exists $w \in V(G)$ such that $N_G[w] \cap S_G = S_G$. It follows that

$$N_{K_1+G}[w] \cap S = \{v\} \cup S_G = N_{K_1+G}[v] \cap S,$$

a contradiction to the fact that S is differentiating in $K_1 + G$. Thus, S_G is strictly differentiating in G. This proves that (a) holds.

Now, let $u \in V(G)$. Since S is odd dominating, $|N_{K_1+G}[u] \cap S| = |\{v\}| + |N_G[u] \cap S_G|$ is odd. Thus, $|N_G[u] \cap S_G|$ is even. This shows that (b) holds. Hence, (i) holds.

Case 2: $v \notin S$.
Then $S \subseteq V(G)$. Since S is odd dominating, $|N_{K_1+G}[v] \cap S| = |S|$ is odd. Since S is odd dominating and $v \notin S$, it follows that S is odd dominating in G. Moreover, as in Case 1, S is strictly differentiating in G. Therefore, (ii) holds.

Conversely, suppose (i) holds. Let x, y be distinct vertices in $V(K_1 + G)$. If $x, y \in V(G)$, then $N_G[x] \cap S_G \neq N[y] \cap S_G$ by (a). It follows that

$$N_{K_1+G}[x] \cap S = \{v\} \cup [N_G[x] \cap S_G] \neq \{v\} \cup [N_G[y] \cap S_G] = N_{K_1+G}[y] \cap S.$$

Suppose $x = v$. Then $N_G[y] \cap S_G \neq S_G$ because S_G is strictly differentiating. Since $N_{K_1+G}[x] \cap S = S_G \cup \{v\}$, then $N_{K_1+G}[x] \cap S \neq N_{K_1+G}[y] \cap S$. Since $|S_G|$ is even and (b) holds, it follows that $|N_{K_1+G}[z] \cap S|$ is odd for all $z \in V(K_1 + G)$. Therefore, S is a differentiating odd dominating set in $K_1 + G$.

Next, suppose (ii) holds. Since S is strictly differentiating-dominating set in G, S is differentiating-dominating in $K_1 + G$. Let $w \in V(K_1 + G)$. If $w = v$, then $|N_{K_1+G}[w] \cap S| = |S|$ is odd, by assumption. If $w \in V(G)$, then $|N_{K_1+G}[w] \cap S| = |N_G[w] \cap S|$ is odd because S is odd dominating in G. Therefore, S is a differentiating odd dominating set in $K_1 + G$. $\square$

The graphs G and $K_1 + G$ in Figure 1 illustrate the graphs described in Thereom 8(i).

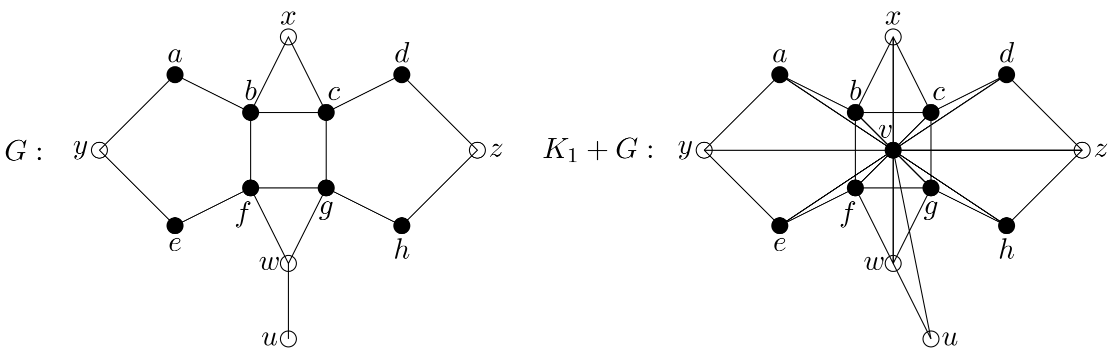

Figure 1: Graphs G and $K_1 + G$ illustrating Theorem 8 (i)

In the next results, we use the following parameters for any graph G' admitting the given set:

$$\gamma_D^{eo}(G') = \min\{|S| : |S| \text{ is even and } S \in DOD(G')\}$$
$$\gamma_{SD}^{eo}(G') = \min\{|S| : |S| \text{ is even and } S \in SDOD(G')\}$$
$$\gamma_{SD}^{oo}(G') = \min\{|S| : |S| \text{ is odd and } S \in SDOD(G')\}$$
$$\gamma_{SD}^{ee}(G') = \min\{|S| : |S| \text{ is even and } S \in SDED(G')\}$$

Corollary 8. *Let G be a point distinguishing graph.*

(*i*) *If G admits a strictly differentiating odd dominating set with odd cardinality, then $\gamma_D^o(K_1+G) \leq \gamma_{SD}^{oo}(G)$ and equality holds if G does not admit a differentiating even dominating set.*

(*ii*) *If G admits a strictly differentiating even dominating set with even cardinality, then $\gamma_D^o(K_1+G) \leq \gamma_{SD}^{ee}(G)+1$ and equality holds if G does not admit a differentiating odd dominating set with odd cardinality.*

Corollary 9. *All fans $F_n = K_1+P_n$ of order $n+1$ have no differentiating odd dominating set for all n.*

Proof. Clearly, F_1 and F_2 do not admit a differentiating odd dominating set. Suppose now that $n \geq 3$. Let $K_1 = \langle v\rangle$ and $G = P_n = [v_1, v_2, \ldots, v_n]$. Suppose F_n has a differentiating odd dominating set, say S. Suppose $S = \{v\} \cup S_G$, where S_G satisfies (a) and (b) in Theorem 8. If $v_1 \in S_G$, then $v_2 \in S_G$ by property (b). Hence, by the same property (b), $v_3 \notin S_G$. This implies that $N_G[v_1]\cap S_G = N_G[v_2]\cap S_G = \{v_1, v_2\}$, contradicting property (a). This forces $v_1 \notin S_G$. By property (b), $v_2, v_3 \notin S_G$. Thus, $N_G[v_1] \cap S_G = N_G[v_2] \cap S_G = \varnothing$, contradicting (a). Therefore, $S \subseteq V(P_n)$ and satisfies (ii) in Theorem 8. If $v_1 \in S$, then $v_2, v_3 \notin S$ because S is odd dominating in G. This would imply that S_G is not differentiating in G, contradicting (ii). Hence, $v_1 \notin S$. Since S is odd dominating in G, $v_2 \in S$ and $v_3 \notin S$. This implies that $N_G[v_1] \cap S = N_G[v_2] \cap S = \{v_2\}$, contrary to the assumption that S is differentiating in G. Therefore, such S does not exist, i.e., F_n does not admit a differentiating odd dominating set. □

Corollary 10. *The wheel $W_n = K_1 + C_n$ admits a differentiating odd dominating set if and only if n is odd and $n \neq 3$. Moreover, if n is odd and $n \geq 5$, then $\gamma_D^o(W_n) = n$.*

Proof. Suppose W_n admits a differentiating odd dominating set, say S. Since $W_3 = K_4$ is not point distinguishing, $n \neq 3$. Let $K_1 = \langle v\rangle$ and $G = C_n = [v_1, v_2, \ldots, v_n, v_1]$. Suppose $S = \{v\} \cup S_G$, where S_G satisfies (a) and (b) in Theorem 8(i). If $v_1 \in S_G$, then $v_2 \in S_G$ or $v_n \in S_G$ but not both by (b). We may assume that $v_2 \in S_G$. Then $v_n \notin S_G$. Again, by property (b), $v_3 \notin S_G$. It follows that $N_G[v_1] \cap S_G = N_G[v_2] \cap S_G = \{v_1, v_2\}$, contradicting property (a). Consequently, $v_1 \notin S_G$. By property (b), $v_2, v_3, v_n \in S_G$ and $v_4 \notin S_G$ (hence, $n \neq 4$). This implies that $N_G[v_2] \cap S_G = N_G[v_3] \cap S_G = \{v_2, v_3\}$, contradicting property (a). Therefore, $v \notin S$. Thus, $S \subseteq V(G)$ and satisfies (ii) of Theorem 8. Suppose $S \neq V(G)$. We may assume that $v_1 \in V(G) \setminus S$. Since S is odd dominating, $v_2 \in S$ or $v_n \in S$ but not both. Assume that $v_2 \in S$. Then $v_n \notin S$. Again, since S is odd dominating, $v_3 \notin S$. Therefore, $N_G[v_1] \cap S = N_G[v_2] \cap S = \{v_2\}$, contrary to the fact that S is differentiating in G. Thus, $S = V(G)$. Since S is odd dominating and $v \notin S$, $|N_{W_n}[v] \cap S| = |V(C_n)| = n$ is odd.

For the converse suppose that $n \geq 5$ and is odd. By Theorem 8, $S_0 = V(C_n)$ is a differentiating odd dominating set in W_n.

As seen earlier, if n is odd and $n \geq 5$, then $V(C_n)$ is the only differentiating odd dominating set in W_n. Accordingly, $\gamma_D^o(W_n) = n$. □

The next result is due to Canoy and Malacas [12].

Theorem 9. *[12] Let G and H be non-trivial graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G+H)$ is a differentiating-dominating set in $G+H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating sets in G and H, respectively, and either S_G or S_H is strictly differentiating.*

Theorem 10. *Let G and H be non-complete graphs of order $m \geq 4$ and $n \geq 4$, respectively. Then $S \subseteq V(G+H)$ is a differentiating odd dominating set in $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are differentiating sets in G and H, respectively, either S_G or S_H is strictly differentiating, and one of the following statements holds:*

(i) $|S_G|$ and $|S_H|$ are even, and S_G and S_H are both odd dominating sets in G and H, respectively.

(ii) $|S_G|$ and $|S_H|$ are odd, and S_G and S_H are both even dominating sets in G and H, respectively.

(iii) $|S_G|$ is odd, $|S_H|$ is even, S_G is odd dominating in G, S_H is even dominating set in H.

(iv) $|S_G|$ is even, $|S_H|$ is odd, S_G is even dominating in G, and S_H is odd dominating sets in H.

Proof. Let $S \subseteq V(G+H)$ be a differentiating odd dominating set in $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. By Theorem 9, S_G and S_H are differentiating-dominating sets in G and H, respectively, and either S_G or S_H is stricly differentiating. Now, since S is odd dominating in $G+H$, $|N_{G+H}[x] \cap S| = |N_G[x] \cap S_G| + |S_H|$ and $|N_{G+H}[y] \cap S| = |N_H[y] \cap S_H| + |S_G|$ are odd for every $x \in V(G)$ and for every $y \in V(H)$. Hence, if $|S_G|$ is even (or $|S_H|$ is even), then $|N_H[y] \cap S_H|$ is odd (resp. $|N_G[x] \cap S_G|$ is odd). This implies that S_H is odd dominating (resp. S_G is odd dominating). If $|S_G|$ is odd (or $|S_H|$ is odd), then $|N_H[y] \cap S_H|$ is even (resp. $|N_G[x] \cap S_G|$ is even). Hence, S_H is even dominating (resp. S_G is even dominating). Therefore, (i), or (ii), or (iii), or (iv) holds.

For the converse, suppose that $S = S_G \cup S_H$, where S_G and S_H are differentiating-dominating sets in G and H, respectively, and either S_G or S_H is strictly differentiating. Then S is a differentiating set in $G+H$. Now let $p \in V(G+H)$. Then $|N_{G+H}[p] \cap S| = |N_G[p] \cap S_G| + |S_H|$ if $p \in V(G)$ and $|N_{G+H}[p] \cap S| = |N_H[p] \cap S_H| + |S_G|$ if $p \in V(H)$. Hence, if one of (i), (ii), (iii), and (iv) holds, then S is an odd dominating set in $G+H$. □

Corollary 11. *Let G and H be a non-complete graphs of order $m \geq 4$ and $n \geq 4$, respectively, such that $G+H$ admits differentiating odd dominating set. If both G and H do not have an even dominating set and both G and H admit a strictly differentiating set then*

$$\gamma_D^o(G+H) = \min\{\gamma_D^{eo}(G) + \gamma_{SD}^{eo}(H), \gamma_D^{eo}(H) + \gamma_{SD}^{eo}(G)\},$$

where we set $\gamma_{SD}^{eo}(G') = +\infty$ *whenever* $SDOD(G') = \varnothing$, *where* $G' \in \{G, H\}$.

Example 1. Let $G = S_{p,q}$ and $H = S_{r,t}$ (double stars), where $p, q, r,$ and t are odd numbers greater than 2. By Corollary 4, $\gamma_D^o(G) = \gamma_D^{eo}(G) = \gamma_{SD}^{eo}(G) = p + q$ and $\gamma_D^o(H) = \gamma_D^{eo}(H) = \gamma_{SD}^{eo}(H) = r + t$. By Corollary 11, $\gamma_D^o(G + H) = p + q + r + t$.

The ***corona*** $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, we denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle v \rangle + H^v$, where $v \in V(G)$.

Theorem 11. *[12] Let G (not necessarily point distinguishing) and let H be non-trivial connected graphs. Then* $C \subseteq V(G \circ H)$ *is a differentiating-dominating set in* $G \circ H$ *if and only if for every* $v \in V(G)$, *one of the following is true:*

(i) $v \in C$, $N_G(v) \cap C \neq \varnothing$, *and* $C \cap V(H^v)$ *is a differentiating set in* H^v;

(ii) $v \in C$, $N_G(v) \cap C = \varnothing$, *and* $C \cap V(H^v)$ *is a strictly differentiating set in* H^v;

(iii) $v \notin C$, $N_G(v) \cap C \neq \varnothing$, *and* $C_1 = V(H^v) \cap C$ *is a differentiating-dominating set in* H^v; *or*

(iv) $v \notin C$, $N_G(v) \cap C = \varnothing$, *and* $C_1 = V(H^v) \cap C$ *is a strictly differentiating-dominating set in* H^v.

Theorem 12. *Let G be a non-trivial connected graph and let H be any non-trivial graph such that* $G \circ H$ *admits a differentiating odd dominating set. Then* $S \subseteq V(G \circ H)$ *is a differentiating odd dominating set in* $G \circ H$ *if and only if* $S = S_G \cup [\cup_{v \in V(G)} S_v]$, *where* $S_G \subseteq V(G)$ *and* $S_v \subseteq V(H^v)$ *for each* $v \in V(G)$, *and satisfies the following conditions:*

(i) *For each* $v \in S_G$ *with* $|N_G(v) \cap S_G| \neq 0$, S_v *is differentiating even dominating in* H^v, *and either*

  (a) $|N_G(v) \cap S_G|$ *and* $|S_v|$ *are odd or*

  (b) $|N_G(v) \cap S_G|$ *and* $|S_v|$ *are even.*

(ii) *For each* $w \in V(G) \backslash S_G$ *with* $|N_G(w) \cap S_G| \neq 0$, S_w *is differentiating odd dominating in* H^w, *and either*

  (c) $|N_G(w) \cap S_G|$ *is even and* $|S_w|$ *is odd or*

  (d) $|N_G(w) \cap S_G|$ *is odd and* $|S_w|$ *is even.*

(iii) *For each* $v \in V(G)$ *with* $|N_G(v) \cap S_G| = 0$, *it holds that*

  (e) S_v *is a strictly differentiating even dominating set in* H^v *and* $|S_v|$ *is even if* $v \in S_G$ *and* (*f*) *S_v is strictly differentiating odd dominating and $|S_v|$ is odd if $v \in V(G) \setminus S_G$.*

Proof. Suppose S is a differentiating odd dominating set in $G \circ H$. Let $S_G = S \cap V(G)$ and let $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = S_G \cup [\cup_{v \in V(G)} S_v]$. Let $v \in S_G$ such that $|N_G(v) \cap S_G| \neq 0$. By Theorem 11, S_v is a differentiating set in H^v. Suppose first that $|N_G(v) \cap S_G|$ is odd. Since S is odd dominating in $G \circ H$, $|N_{G \circ H}[p] \cap S| = |N_{H^v}[p] \cap S_v| + |\{v\}|$ is odd for every $p \in V(H^v)$. This implies that $|N_{H^v}[p] \cap S_v|$ is even for every $p \in V(H^v)$. Thus, S_v is even dominating in H^v. Moreover, because $|N_{G \circ H}[v] \cap S| = |N_G(v) \cap S_G| + (|\{v\}| + |S_v|)$ is also odd, $|S_v|$ is odd. This shows that $(i)(a)$ holds. Similarly, $(i)(b)$ also holds. Therefore, (i) holds.

Next, let $w \in V(G) \setminus S_G$ with $|N_G(w) \cap S_G| \neq 0$. Again, by Theorem 11, S_w is a differentiating set in H^w. Suppose $|N_G(w) \cap S_G|$ is even. Since S is odd dominating in $G \circ H$, $|N_{G \circ H}[q] \cap S| = |N_{H^w}[q] \cap S_w|$ is odd for every $q \in V(H^w)$. It follows that S_w is differentiating odd dominating in H^v. Since $|N_{G \circ H}[w] \cap S| = |N_G(w) \cap S_G| + |S_w|$ is odd and $|N_G(w) \cap S_G|$ is even, $|S_w|$ is odd. This shows that $(ii)(c)$ holds. Similar arguments will show that $(ii)(d)$ holds. Thus, (ii) holds.

Finally, let $v \in V(G)$ such that $|N_G(v) \cap S_G| = 0$. Suppose first that $v \in S_G$. Then, by Theorem 8(i), $|S_v|$ is even and S_v is a strictly differentiating even dominating set in H^v. If $v \in V(G) \setminus S_G$, then S_v is odd and S_v is strictly differentiating odd dominating in H^v by Theorem 8(ii). Thus, (iii) holds.

For the converse, suppose that S satisfies (i), (ii), and (iii). By Theorem 11, S is a differentiating-dominating set in $G \circ H$. Let $x \in V(G \circ H) \setminus S$ and $v \in V(G)$ be such that $x \in V(v + H^v)$. Suppose $|N_G(x) \cap S_G| \neq 0$. If $v \in S_G$, then

$$|N_{G \circ H}[x] \cap S| = |N_G(x) \cap S_G| + (|S_v| + 1) \text{ if } x = v;$$

otherwise, $|N_{G \circ H}[x] \cap S| = |N_{H^v}[x] \cap S_v| + 1$. By (a) and (b), $|N_{G \circ H}[x] \cap S|$ is odd. Suppose $v \in V(G) \setminus S_G$. Then $|N_{G \circ H}[x] \cap S| = |S_v|$ if $x = v$; otherwise, $|N_{G \circ H}[x] \cap S| = |N_{H^v}[x] \cap S_v|$. By (c) and (d), $|N_{G \circ H}[x] \cap S|$ is odd. Lastly, suppose that $|N_G(x) \cap S_G| = 0$. Then parts of (e) and (f) would imply that $|N_{G \circ H}[x] \cap S|$ is odd. Therefore, S is a differentiating odd dominating set in $G \circ H$. □

Corollary 12. *Let G be a non-trivial connected graph of order n and let H be a graph that admits a strictly differentiating odd dominating set with odd cardinality. Then $\gamma_D^o(G \circ H) \leq |V(G)|\gamma_{SD}^{oo}(H)$ and equality holds if $H = \overline{K}_p$, where p is odd and at least 3.*

Proof. Let $S_v \subseteq V(H^v)$ be a strictly differentiating odd dominating set with odd cardinality such that $|S_v| = \gamma_{SD}^{oo}(H^v)$ for each $v \in V(G)$. Set $S = \cup_{v \in V(G)} S_v$. By Theorem 12, S is a differentiating odd dominating set in $G \circ H$. Thus, $\gamma_D^o(G \circ H) \leq |S| = n\gamma_{SD}^{oo}(H)$. If $H = \overline{K}_p$, then $\gamma_{SD}^o(H) = p$. Desired equality follows now from Theorem 2(ii). □

Corollary 13. *Let G be a non-trivial connected graph of order n and let H be a graph that admits a differentiating even dominating set with even cardinality. If G is r-regular, where r is a positive even integer, then $\gamma_D^o(G \circ H) \leq n + n\gamma_D^{ee}(H)$.*

The ***lexicographic product*** $G[H]$ also has $V(G[H]) = V(G) \times V(H)$ as its vertex set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$.

Observe that any subset C of $V(G) \times V(H)$ (infact, any set of ordered-pairs) can be written as $C = \cup_{v\in A}(\{v\} \times B_v)$, where $S \subseteq V(G)$ and $B_v \subseteq V(H)$ for each $v \in S$. Henceforth, we shall use this form to denote any subset C of $V(G)V(H)$.

Theorem 13. *[12] Let G (not necessarily point distinguishing) and H be non-trivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a differentiating-dominating set in $G[H]$ if and only if*

(i) $S = V(G)$;

(ii) *T_x is a differentiating set in H for every $x \in V(G)$;*

(iii) *T_x or T_y is strictly differentiating in H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$; and*

(iv) *T_x or T_y is (differentiating) dominating in H whenever x and y are distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$.*

Theorem 14. *Let G be a non-trivial connected graph and let H be a non-trivial point distinguishing connected graph. Then $S = \cup_{v\in A}(\{v\} \times B_v)$, where $A \subseteq V(G)$ and $B_v \subseteq V(H)$ for each $v \in A$, is a differentiating odd dominating set in $G[H]$ if and only if*

(i) *$A = V(G)$;*

(ii) *B_v is a differentiating set in H for every $v \in V(G)$;*

(iii) *B_v or B_u is strictly differentiating in H whenever u and v are adjacent vertices of G with $N_G[u] = N_G[v]$;*

(iv) *B_v or B_u is differentiating-dominating in H whenever u and v are distinct non-adjacent vertices of G with $N_G(u) = N_G(v)$; and*

(v) *For every $v \in V(G)$ and for every $p \in V(H)$, $|N_H[p] \cap B_v| + \sum_{w\in N_G(u)} |B_w|$ is odd.*

Proof. Suppose S is a differentiating odd dominating set in $G[H]$. Then (i), (ii), (iii) and (iv) hold by Theorem 13. Let $v \in V(G)$ and $p \in V(H)$. Then

$$N_{G[H]}[(v, p)] = [\{v\} \times (N_H[p] \cap B_v)] \cup [\cup_{w\in N_G(v)}(\{w\} \times B_w)].$$

Since S is odd dominating in $G[H]$, $|N_{G[H]}[(v, p)] \cap S| = |N_H[p] \cap B_v| + \sum_{w\in N_G(v)} |B_w|$ is odd, showing that (v) holds.

For the converse, suppose that S satifies the five conditions. Since (i), (ii), (iii), and (iv) hold, S is a differentiating-dominating set in $G[H]$ by Theorem 13. By (v), it follows that S is odd dominating. $\square$

Corollary 14. *Let G be non-trivial totally point determining graph and let H be a point distinguishing connected graph. If H admits a differentiating odd dominating set with even cardinality, then $\gamma_D^o(G[H]) \leq |V(G)|\gamma_D^{eo}(H)$.*

Proof. For each $v \in V(G)$, let B_v be a differentiating odd dominating set with even cardinality such that $|B_v| = \gamma_D^{eo}(H)$. Then $S = \cup_{v\in V(G)}(\{v\} \times B_v)$ satisfies the first four properties in Theorem 14. Now let $v \in V(G)$ and $p \in V(H)$. Since B_v is differentiating odd dominating in H, $|N_H[p] \cap B_v|$ is odd. Moreover, since $|B_w|$ is even for every $w \in V(G)$, it follows that $\sum_{w\in N_G(v)} |B_w|$ is even. Therefore, $|N_H[p] \cap B_v| + \sum_{w\in N_G(v)} |B_w|$ is odd, showing that property (v) in Theorem 14 is also satisfied. Accordingly, S is a differentiating odd dominating set in $G[H]$ and $\gamma_D^o(G[H]) \leq |S| = |V(G)|\gamma_D^{eo}(H)$. □

Conclusion

The concept of differentiating odd dominating set has been introduced and initially investigated in this study. The differentiating odd domination number of a graph on at least four vertices is at least equal to the maximum of the odd domination number, the differentiating-domination number of the graph and, 3, and at most equal to the difference of the order of the graph and the number of its support vertices. As shown in this study, some graphs do not admit this kind of dominating set. The newly defined concept and parameter have been investigated for the join, corona, and lexicographic products of some classes of graphs. It may be interesting and worthwhile to find necessary and sufficient conditions for a graph to admit a differentiating odd dominating set, study the complexity of the decision problem involving the parameter, and investigate the parameter for some other families of graphs.

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